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Coupled discrete solitonic equations of additive Bogoyavlensky and the periodic reduction

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Using the Hirota bilinear formalism we build the Hirota bilinear form and construct the multisoliton solutions for the coupled semidiscrete additive Bogoyavlensky system with branched dispersion, proving its complete integrability. The same result can be reached also applying the periodic reduction technique. Starting from a general completely integrable "diagonal" equation in two dimensions and performing periodic reduction, one can obtain coupled completely integrable equations. The idea is to consider that the independent discrete variable of the analysed equation is in fact diagonal in a two-dimensional lattice. Imposing periodic reduction on one such coordinate in the 2D-lattice, one can obtain coupled integrable systems with branched disperssion. We will exemplify the technique on the semidiscrete additive Bogoyavlensky equation (aB), which is an integrable semidiscrete generalized Volterra type equation.

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1. Introduction

Intensively studied since the seventies, the Volterra type systems, introduced by Hirota and Satsuma [1–3], are still a subject of interest today from different points of view [4–7]. One of them is given by the integrable extensions of the Volterra systems, known in literature as the Bogoyavlensky lattices [8–11]. Three such integrable extensions where recently investigated from the singularity analysis and bilinear integrability point of view in [12], more precisely: the additive Bogoyavlensky equation (aB), the first multiplicative Bogoyavlensky equation (mB2) and the second multiplicative Bogoyavlensky equation (mB2).

In this paper we go a step forward, analysing the first of the above equations, but extended into a multicomponent additive Bogoyavlensky system with any M coupled equations with branched dispersion relation. The investigation starts with the particular case known as the coupled Lotka Volterra system and continues with the most general case. For proving complete integrability of analysed systems we use the well known tool called the Hirota bilinear formalism [13–15], and for extending an integrable 2D-lattice to a coupled integrable system we apply the periodic reduction [16, 17]. The particular and central feature of such coupled systems is the structure of the phases of the components which are parametrized by the order M roots of unity, and the structure of the dispersion relation which has multiple branches and allows more freedom in the soliton interactions.

The paper is organized as follows: after a brief introduction, in Chapter 2 we discuss a generalisation of the additive Bogoyavlensky model to the multicomponent (matrix) case and find the multisoliton solutions for a M-component Lotka-Volterra system ($N = 1$), while in Chapter 3 we study the general case ($\forall\ N$) and construct the N-soliton solution, proving complete integrability. Also, in the same chapter, we show that starting from a two-dimensional additive Bogoyavlensky equation, with two discrete independent variables, one can obtain the same multisoliton solution through the periodic reduction tool [18]. In Chapter 4 we summarize our conclusions.

2. The coupled semidiscrete aB system

The coupled semidiscrete additive Bogoyavlensky system with branched dispersion:

$$\frac{d}{dt}Q_n(t) = Q_n\left(\sum_{j=1}^{N} E^j_{\sigma_1}Q_{n+j}(t)E^j_{\sigma_2} - \sum_{j=1}^{N} E^j_{\sigma_2}Q_{n-j}(t)E^j_{\sigma_1}\right), \tag{1}$$

where $Q_n(t) = Q(n,t)$ is a diagonal matrix of complex functions $u_\nu(n,t)$, $\nu = \overline{1, M}$:

$$Q_n(t) = \begin{pmatrix} u_1(n,t) & 0 & 0 & \ldots\ldots & 0 \\ 0 & u_2(n,t) & 0 & \ldots\ldots & 0 \\ 0 & 0 & u_3(n,t) & \ldots\ldots & 0 \\ \ldots. & \ldots. & \ldots. & \ldots.. & \ldots\ldots \\ 0 & 0 & 0 & \ldots\ldots & u_M(n,t) \end{pmatrix}$$

and E_{σ_1},E_{σ_2} are permutation matrices corresponding to the following permutations:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & . & . & . & . & M \\ 2 & 3 & . & . & . & . & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & . & . & . & . & M \\ M & 1 & 2 & . & . & . & M-1 \end{pmatrix}$$

can be written on components in the following form:

$$\begin{aligned}
\dot{u}_1 &= u_1\left(\overline{u_2}+\overset{(2)}{\overline{u_3}}+..+\overset{(N)}{\overline{u_{N+1}}}-\underline{u_M}-\underset{(2)}{\underline{u_{M-1}}}-..-\underset{(N)}{\underline{u_{M-N+1}}}\right) \\
\dot{u}_2 &= u_2\left(\overline{u_3}+\overset{(2)}{\overline{u_4}}+..+\overset{(N)}{\overline{u_{N+2}}}-\underline{u_1}-\underset{(2)}{\underline{u_M}}-..-\underset{(N)}{\underline{u_{M-N+2}}}\right) \\
\ldots &= \ldots\ldots\ldots\ldots\ldots \\
\dot{u}_{M-1} &= u_{M-1}\left(\overline{u_M}+\overset{(2)}{\overline{u_1}}+..+\overset{(N)}{\overline{u_{M+N-1}}}-\underline{u_{M-2}}-\underset{(2)}{\underline{u_{M-3}}}-..-\underset{(N)}{\underline{u_{M-N-1}}}\right) \\
\dot{u}_M &= u_M\left(\overline{u_1}+\overset{(2)}{\overline{u_2}}+..+\overset{(N)}{\overline{u_{M+N}}}-\underline{u_{M-1}}-\underset{(2)}{\underline{u_{M-2}}}-..-\underset{(N)}{\underline{u_{M-N}}}\right)
\end{aligned} \tag{2}$$

where we denoted:

$$u_\nu(n+1)=\overline{u_\nu},\quad u_\nu(n+2)=\overset{(2)}{\overline{u_\nu}},\quad u_\nu(n+N)=\overset{(N)}{\overline{u_\nu}},\quad \nu=\overline{1,M}$$

$$u_\nu(n-1)=\underline{u_\nu},\quad u_\nu(n-2)=\underset{(2)}{\underline{u_\nu}},\quad u_\nu(n-N)=\underset{(N)}{\underline{u_\nu}}.$$

For $M=1$ and $N=1$, system (2) reduces to the well known Lotka-Volterra equation [11]:

$$\dot{u}_1=u_1(\overline{u_1}-\underline{u_1}).$$

In order to prove complete integrability for the general case of coupled aB system (2) for any M, we are going to use the Hirota bilinear formalism. The existence of an infinite number of independent integrals in involution, computed from the Lax pair [19, 20], is a commune proof of complete integrability for a partial discrete equation. There are several other criteria and techniques used for investigating integrability such as: complexity growth, singularity confinement, cube consistency, Lie symmetry approach [21] and for constrained systems and gauge theories, the Becchi-Rouet-Stora-Tyutin (BRST) technique [22] can be applied. But we will not use these approaches on the integrability concept, but the Hirota alternative formulation, which requires the existence of general multisoliton solution. In other words, the proof of complete integrability in Hirota formalism is the construction of a solution describing multiple collisions of an arbitrary number of solitons having arbitrary parameters and phases, considering all branches of dispersion relations.

2.1 The coupled semidiscrete Lotka-Volterra system

For any M and $N=1$, on the components, system (2) becomes coupled semidiscrete Lotka-Volterra system [23, 24] and has the following expression:

$$\begin{aligned}
\dot{u}_1 &= u_1(\overline{u_2}-\underline{u_M}) \\
\dot{u}_2 &= u_2(\overline{u_3}-\underline{u_1}) \\
&\ldots\ldots \\
\dot{u}_{M-1} &= u_{M-1}(\overline{u_M}-\underline{u_{M-2}}) \\
\dot{u}_M &= u_M(\overline{u_1}-\underline{u_{M-1}})
\end{aligned} \tag{3}$$

where:

$$u_\nu = u_\nu(n,t), \quad \overline{u_\nu} = u_\nu(n+1,t), \quad \underline{u_\nu} = u_\nu(n-1,t), \quad \nu = \overline{1,M}.$$

In order to check integrability of (3), we apply the Hirota bilinear fomalism. Using the nonlinear substitution:

$$u_\nu(n,t) = 1 + \frac{\partial}{\partial t} \ln \frac{\overline{F_{\nu+1}}}{F_\nu}, \quad \nu = \overline{1,M} \tag{4}$$

where $F_\nu = F_\nu(n,t)$, $\overline{F_\nu} = F_\nu(n+1,t)$, we cast (3), which can be written in a compact manner as:

$$\dot{u}_\nu = u_\nu(\overline{u_{\nu+1}} - \underline{u_{\nu+M-1}}), \quad \nu = \overline{1,M}, \tag{5}$$

into the Hirota bilinear form:

$$\mathbf{D_t}\overline{F_{\nu+1}} \cdot F_\nu + \overline{F_{\nu+1}}F_\nu = \overset{(2)}{\overline{F_{\nu+2}}}\ \underline{F_{\nu+M-1}}, \tag{6}$$

where $\overset{(2)}{\overline{F_\nu}} = F_\nu(n+2,t)$, $\underline{F_\nu} = F_\nu(n-1,t)$, $F_\nu(n,t)$ is a complex function and D_t is the Hirota bilinear operator [15] defined as:

$$D_t^n a(t) \cdot b(t) = (\partial_t - \partial_{t'})^n a(t) b(t')|_{t=t'}. \tag{7}$$

In order to build the 1-soliton solutions (1-ss) for coupled Lotka-Volterra system (3), we consider the ansatz:

$$F_\nu = 1 + \epsilon_1^{\nu-1} e^{\eta_1}, \quad \nu = \overline{1,M}, \tag{8}$$

where $\eta_1 = k_1 n + \omega_1 t + \eta_1^{(0)}$, k_1 is the wave number, ω_1 is the angular frequency and $\eta_1^{(0)}$ an arbitrary phase. The dispersion has M possible branches of dispersion for the soliton:

$$\omega_1(k_1) = 2\left[\frac{\epsilon_1^2+1}{2\epsilon_1}\sinh k_1 + \frac{\epsilon_1^2-1}{2\epsilon_1}\cosh k_1\right], \ \epsilon_1 \in \left\{e^{l\frac{2\pi i}{M}}\right\}, l = \overline{1,M}.$$

The 2-ss has the form:

$$F_\nu = 1 + \epsilon_1^{\nu-1} e^{\eta_1} + \epsilon_2^{\nu-1} e^{\eta_2} + \epsilon_1^{\nu-1}\epsilon_2^{\nu-1} e^{\eta_1+\eta_2+A_{12}}, \quad \nu = \overline{1,M} \tag{9}$$

where:

$$\eta_j = k_j n + \omega_j t + \eta_j^{(0)}, \quad j = \overline{1,2},$$

and the interaction phase is given by:

$$e^{A12} = \frac{(e^{k_2}\epsilon_2 - e^{k_1}\epsilon_1)}{(e^{k_1+k_2}\epsilon_1\epsilon_2 - 1)^2},$$

with M possible branches of dispersion for each of the 2 solitons:

$$\omega_j(k_j) = 2\left[\frac{\epsilon_j^2+1}{2\epsilon_j}\sinh k_j + \frac{\epsilon_j^2-1}{2\epsilon_j}\cosh k_j\right], \ \epsilon_j \in \left\{e^{l\frac{2\pi i}{M}}\right\}, l = \overline{1,M}, j = \overline{1,2}.$$

The $\mathcal{N}$-soliton solution for the coupled Lotka-Volterra system with any number of equations, given in (3), has the following expressions for F_ν, $(\nu = \overline{1, M})$:

$$F_\nu(n,t) = \sum_{\mu_1,..,\mu_{\mathcal{N}}=\{0,1\}} \exp\left(\sum_{i=1}^{\mathcal{N}} \mu_i[\eta_i + (\nu-1)\ln(\epsilon_i)] + \sum_{1\le i<j}^{\mathcal{N}} \mu_i\mu_j A_{ij}\right), \tag{10}$$

where:

$$\eta_j = k_j n + \omega_j t + \eta_j^{(0)}, \quad j = \overline{1, \mathcal{N}},$$

and the interaction term has the form:

$$e^{Aij} = \frac{(e^{k_j}\epsilon_j - e^{k_i}\epsilon_i)}{(e^{k_i+k_j}\epsilon_i\epsilon_j - 1)^2}$$

with the M branches of dispersion for each of the $\mathcal{N}$ solitons (k_j is the wave number of the j-soliton):

$$\omega_j(k_j){=}2\left[\frac{\epsilon_j^2+1}{2\epsilon_j}\sinh k_j + \frac{\epsilon_j^2-1}{2\epsilon_j}\cosh k_j\right],\ \epsilon_j \in \left\{e^{l\frac{2\pi i}{M}}\right\}, l = \overline{1, M}, j = \overline{1, \mathcal{N}}.$$

The branches of dispersion are labelled by the index l. The parameter ϵ_j which characterizes the j-soliton ($j = \overline{1, \mathcal{N}}$) can have M values, i.e. the order M roots of unity.

3. The semidiscrete coupled additive Bogoyavlensky

The semidiscrete coupled additive Bogoyavlensky system given in (2) can be solved in the same manner as the coupled general Lotka-Volterra system (3). The difference is that for coupled aB system the parameter N can have any natural value, while for coupled Lotka-Volterra it has $N = 1$. There are several important differences that appear in the solution form.

Using the same nonlinear substitution:

$$u_\nu(n,t) = 1 + \frac{\partial}{\partial t}\ln\frac{\overline{F_{\nu+1}}}{F_\nu}, \quad \nu = \overline{1, M} \tag{11}$$

the coupled aB system, which can be written in a compact manner as:

$$\dot{u}_\nu = u_\nu(\sum_{j=1}^{N}\overline{u_{\nu+j}} - \sum_{j=1}^{N}\underline{u_{\nu+M-j}}), \quad \nu = \overline{1, M}, \tag{12}$$

becomes in the Hirota bilinear form:

$$\mathbf{D_t}\overline{F_{\nu+1}}\cdot F_\nu + \overline{F_{\nu+1}}F_\nu = \overset{(N+1)}{\overline{F_{\nu+1+N}}}\ \underset{(N)}{\underline{F_{\nu-N}}}\ , \tag{13}$$

where:

$$\overset{(N+1)}{\overline{F_\nu}} = F_\nu(n+N+1,t), \quad \underset{(N)}{\underline{F_\nu}} = F_\nu(n-N,t)$$

.

The general N-soliton solutions of coupled additive Bogoyavlensky system is:

$$F_\nu(n,t) = \sum_{\mu_1,..,\mu_N=\{0,1\}} \exp\left(\sum_{i=1}^{N} \mu_i[\eta_i + (\nu-1)\ln(\epsilon_i)] + \sum_{1\le i<j}^{N} \mu_i\mu_j A_{ij}\right), \quad \eta_j = k_j n + \omega_j t + \eta_j^{(0)} \tag{14}$$

with the dispersion relation and the interaction phase given by:

$$\omega_j = 2\frac{\sinh\frac{(k_j+\ln\epsilon_j)N}{2}\sinh\frac{(k_j+ln\epsilon_j)(N+1)}{2}}{\sinh\frac{k_j+\ln\epsilon_j}{2}}, \quad \epsilon_j\in\left\{e^{\nu\frac{2\pi i}{M}}\right\}, \quad j=\overline{1,N}, \quad \nu=\overline{1,M}$$

$$e^{A_{ij}} = \frac{-\cosh\frac{k_i-k_j+\ln\frac{\epsilon_i}{\epsilon_j}}{2} + \cosh\frac{(k_i-k_j+\ln\frac{\epsilon_i}{\epsilon_j})(1+2N)}{2} - (\omega_i-\omega_j)\sinh\frac{k_i-k_j+\ln\frac{\epsilon_i}{\epsilon_j}}{2}}{-\cosh\frac{k_i+k_j+\ln(\epsilon_i\epsilon_j)}{2} - \cosh\frac{(k_i+k_j+\ln(\epsilon_i\epsilon_j))(1+2N)}{2} + (\omega_i+\omega_j)\sinh\frac{k_i+k_j+\ln(\epsilon_i\epsilon_j)}{2}}.$$

In the particular case when $M = 1$ we obtain the solution presented in [12].

3.1 The semidiscrete aB 2D-lattice

In order to solve the coupled semidiscrete aB system (1):

$$\frac{d}{dt}Q_n(t) = Q_n(t)\left(\sum_{j=1}^{N} E^j_{\sigma_1}Q_{n+j}(t)E^j_{\sigma_2} - \sum_{j=1}^{N} E^j_{\sigma_2}Q_{n-j}(t)E^j_{\sigma_1}\right),$$

one could start from the completely integrable semidiscrete aB 2D-lattice (in two discrete dimensions):

$$\frac{d}{dt}Q_{n,m}(t) = Q_{n,m}(t)\left(\sum_{j=1}^{N} Q_{n+j,m+j}(t) - \sum_{j=1}^{N} Q_{n-j,m-j}(t)\right). \tag{15}$$

Considering $Q(n,m,t)$ to be a periodic function only with respect to m and imposing periodic reduction on such coordinate in the 2D-lattice, one could obtain a coupled systems of aB equations.

Now lets consider the periodic 2-reduction on the m direction (meaning that $Q_{n,m}(t) = Q(n,m,t)$ is a periodic function only with respect to m, with period 2). We omit writing the t dependency for simplicity. This means that:

$$Q(n,m) \equiv u_1(n), \;\; Q(n,m+1) \equiv u_2(n),$$

$$Q(n,m+2) \equiv u_1(n), \;\; Q(n,m-1) \equiv u_2(n).$$

Introducing this reduction in (15) and denoting:

$$u_1(n+1) = \overline{u_1}, \quad u_1(n+2) = \overset{(2)}{\overline{u_1}}, \quad u_1(n+N) = \overset{(N)}{\overline{u_1}}$$

$$u_1(n-1) = \underline{u_1}, \quad u_1(n-2) = \underset{(2)}{\underline{u_2}}, \quad u_1(n-N) = \underset{(N)}{\underline{u_1}}$$

we get precisely (for N even):

$$\begin{aligned}
\dot{u}_1 &= u_1\left(\overline{u_2}+\overset{(2)}{\overline{u_1}}+..+\overset{(N-1)}{\overline{u_2}}+\overset{(N)}{\overline{u_1}}-\underline{u_2}-\underset{(2)}{\underline{u_1}}-..-\underset{(N-1)}{\underline{u_2}}-\underset{(N)}{\underline{u_1}}\right)\\
\dot{u}_2 &= u_2\left(\overline{u_1}+\overset{(2)}{\overline{u_2}}+..+\overset{(N-1)}{\overline{u_1}}+\overset{(N)}{\overline{u_2}}-\underline{u_1}-\underset{(2)}{\underline{u_2}}-..-\underset{(N-1)}{\underline{u_1}}-\underset{(2)}{\underline{u_2}}\right).
\end{aligned}$$

For $N = 1$ we obtain coupled Lotka-Volterra system:

$$\begin{aligned}
\dot{u}_1 &= u_1\left(\overline{u_2}-\underline{u_2}\right)\\
\dot{u}_2 &= u_2\left(\overline{u_1}-\underline{u_1}\right).
\end{aligned}$$

In the same way, if we impose periodic-3 reduction:

$$Q(n,m)\equiv u_1(n),\ Q(n,m+1)\equiv u_2(n),\ Q(n,m+2)\equiv u_3(n),$$

$$Q(n,m+3)\equiv u_1(n),\ Q(n,m-1)\equiv u_3(n),$$

we get the system with the following three coupled equations (for N multiple of three):

$$\begin{aligned}
\dot{u}_1 &= u_1\left(\overline{u_2}+\overset{(2)}{\overline{u_3}}+\overset{(3)}{\overline{u_1}}+..+\overset{(N)}{\overline{u_1}}-\underline{u_3}-\underset{(2)}{\underline{u_2}}-\underset{(3)}{\underline{u_1}}-..-\underset{(N)}{\underline{u_1}}\right)\\
\dot{u}_2 &= u_2\left(\overline{u_3}+\overset{(2)}{\overline{u_1}}+\overset{(3)}{\overline{u_2}}+..+\overset{(N)}{\overline{u_2}}-\underline{u_1}-\underset{(2)}{\underline{u_3}}-\underset{(3)}{\underline{u_2}}-..-\underset{(N)}{\underline{u_2}}\right)\\
\dot{u}_3 &= u_3\left(\overline{u_1}+\overset{(2)}{\overline{u_2}}+\overset{(3)}{\overline{u_3}}+..+\overset{(N)}{\overline{u_3}}-\underline{u_2}-\underset{(2)}{\underline{u_1}}-\underset{(3)}{\underline{u_3}}-..-\underset{(N)}{\underline{u_3}}\right)
\end{aligned}$$

The coupled aB system comes out from the aB 2D-lattice equation (15) for any N, choosing a periodic M-reduction on m.

$$\begin{aligned}
\dot{u}_1 &= u_1\left(\overline{u_2}+\overset{(2)}{\overline{u_3}}+..+\overset{(N)}{\overline{u_{N+1}}}-\underline{u_M}-\underset{(2)}{\underline{u_{M-1}}}-..-\underset{(N)}{\underline{u_{M-N+1}}}\right) \qquad (16)\\
\dot{u}_2 &= u_2\left(\overline{u_3}+\overset{(2)}{\overline{u_4}}+..+\overset{(N)}{\overline{u_{N+2}}}-\underline{u_1}-\underset{(2)}{\underline{u_M}}-..-\underset{(N)}{\underline{u_{M-N+2}}}\right)\\
\ldots &= \ldots\ldots\ldots\ldots\\
\dot{u}_{M-1} &= u_{M-1}\left(\overline{u_M}+\overset{(2)}{\overline{u_1}}+..+\overset{(N)}{\overline{u_{M+N-1}}}-\underline{u_{M-2}}-\underset{(2)}{\underline{u_{M-3}}}-..-\underset{(N)}{\underline{u_{M-N-1}}}\right)\\
\dot{u}_M &= u_M\left(\overline{u_1}+\overset{(2)}{\overline{u_2}}+..+\overset{(N)}{\overline{u_{M+N}}}-\underline{u_{M-1}}-\underset{(2)}{\underline{u_{M-2}}}-..-\underset{(N)}{\underline{u_{M-N}}}\right)
\end{aligned}$$

3.2 The Hirota bilinear form and multisoliton solutions for aB 2D-lattice

Using the substitution[1] $Q_{n,m}(t) = 1 + \frac{\partial}{\partial t} \ln \frac{F_{n+1}^{m+1}}{F_n^m}$, we cast the aB 2D-lattice (15) into the Hirota bilinear form:

$$\mathbf{D_t} F_{n+1}^{m+1} \cdot F_n^m + F_{n+1}^{m+1} F_n^m = F_{n+1+N}^{m+1+N} F_{n-N}^{m-N}, \tag{17}$$

where F_n^m is a complex function and D_t is the Hirota bilinear operator.

The 1-soliton solution is:

$$u_n^m = 1 + \frac{\partial}{\partial t} \log \frac{F_{n+1}^{m+1}}{F_n^m} = 1 + \frac{\partial}{\partial t} \log \frac{1 + e^{k_1(n+1)+p_1(m+1)+\omega_1 t+\eta_1^{(0)}}}{1 + e^{k_1 n+p_1 m+\omega_1 t+\eta_1^{(0)}}}$$

where:

$$F_n^m = 1 + e^{k_1 n+p_1 m+\omega_1 t+\eta_1^{(0)}}, \quad (\forall) \quad k_1, p_1 \in \mathbb{C}$$

and the dispersion relation has the form:

$$\omega_1 = 2\frac{\sinh\frac{(k_1+p_1)N}{2} \sinh\frac{(k_1+p_1)(N+1)}{2}}{\sinh\frac{k_1+p_1}{2}}.$$

For the 2-soliton solution we obtain:

$$F_n^m = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}},$$

where:

$$\eta_j = k_j n + p_j m + \omega_j t + \eta_j^{(0)}, \quad j = 1, 2,$$

with the dispersion relation and interaction phase given by:

$$\omega_j = 2\frac{\sinh\frac{(k_j+p_j)N}{2} \sinh\frac{(k_j+p_j)(N+1)}{2}}{\sinh\frac{k_j+p_j}{2}},$$

$$e^{A_{12}} = \frac{-\cosh\frac{k_1+p_1-k_2-p_2}{2} + \cosh\frac{(k_1+p_1-k_2-p_2)(1+2N)}{2} - (\omega_1-\omega_2)\sinh\frac{k_1+p_1-k_2-p_2}{2}}{-\cosh\frac{k_1+p_1+k_2+p_2}{2} - \cosh\frac{(k_1+p_1+k_2+p_2)(1+2N)}{2} + (\omega_1+\omega_2)\sinh\frac{k_1+p_1+k_2+p_2}{2}}.$$

For the 3-soliton solution we obtain the form:

$$F_n^m = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_1+\eta_2+A_{12}} + e^{\eta_1+\eta_3+A_{13}} + e^{\eta_2+\eta_3+A_{23}} + e^{\sum_{i=1}^{3}\eta_i + \sum_{1\le i<j}^{3} A_{ij}},$$

with:

$$\eta_j = k_j n + p_j m + \omega_j t + \eta_j^{(0)}, \quad \omega_j = 2\frac{\sinh\frac{(k_j+p_j)N}{2} \sinh\frac{(k_j+p_j)(N+1)}{2}}{\sinh\frac{k_j+p_j}{2}}, \quad j = \overline{1,3}$$

$$e^{A_{ij}} = \frac{-\cosh\frac{k_i+p_i-k_j-p_j}{2} + \cosh\frac{(k_i+p_i-k_j-p_j)(1+2N)}{2} - (\omega_i-\omega_j)\sinh\frac{k_i+p_i-k_j-p_j}{2}}{-\cosh\frac{k_i+p_i+k_j+p_j}{2} - \cosh\frac{(k_i+p_i+k_j+p_j)(1+2N)}{2} + (\omega_i+\omega_j)\sinh\frac{k_i+p_i+k_j+p_j}{2}}.$$

[1] In this notation, m is not an exponent

The $\mathcal{N}$-soliton solution has the following expression for F_n^m:

$$F_n^m(t) = \sum_{\mu_1,\mu_N=\{0,1\}} \exp\left(\sum_{i=1}^{N} \mu_i\eta_i + \sum_{1\le i<j}^{N} \mu_i\mu_j A_{ij}\right), \tag{18}$$

where:

$$\eta_j = k_j n + p_j m + \omega_j t + \eta_j^{(0)}, \quad j = \overline{1,\mathcal{N}}$$

$$\omega_j = 2\frac{\sinh\frac{(k_j+p_j)N}{2}\sinh\frac{(k_j+p_j)(N+1)}{2}}{\sinh\frac{k_j+p_j}{2}},$$

$$e^{A_{ij}} = \frac{-\cosh\frac{k_i+p_i-k_j-p_j}{2} + \cosh\frac{(k_i+p_i-k_j-p_j)(1+2N)}{2} - (\omega_i-\omega_j)\sinh\frac{k_i+p_i-k_j-p_j}{2}}{-\cosh\frac{k_i+p_i+k_j+p_j}{2} - \cosh\frac{(k_i+p_i+k_j+p_j)(1+2N)}{2} + (\omega_i+\omega_j)\sinh\frac{k_i+p_i+k_j+p_j}{2}}.$$

3.3 The periodic reduction

Now, all the multi-soliton solutions for the coupled aB systems for any N are coming straightforward from the aB 2D-lattice (15) and one can easily see this by looking at the two bilinear forms:

- for aB (n,m,t) 2D-lattice:

$$\mathbf{D_t} F_{n+1}^{m+1}\cdot F_n^m + F_{n+1}^{m+1}F_n^m = F_{n+1+N}^{m+1+N}F_{n-N}^{m-N}, \tag{19}$$

- for coupled aB (n,t) system:

$$\mathbf{D_t}\overline{F_{\nu+1}}\cdot F_\nu + \overline{F_{\nu+1}}F_\nu = \overset{(N+1)}{\overline{F_{\nu+1+N}}}\;\underset{(N)}{\underline{F_{\nu-N}}}\,. \tag{20}$$

The systems are the same, considering that the second index, m, of $Q_{n,m}(t) = 1+\frac{\partial}{\partial t}log\frac{F_{n+1}^{m+1}}{F_n^m}$ in (19) becomes $\nu = \overline{1,M}$ in (20), parameter which indiciates the soliton solutions $u_\nu(t) = 1+\frac{\partial}{\partial t}log\frac{\overline{F_{\nu+1}}}{F_\nu}$ for the M-component aB system.

For example, in the case $M = 2$, the m-dependence is dropped, p_j appearing in the definitions will be $-\pi$i , $+\pi$i making the dispersion relation to have two branches (allowing solitons to move either in the same direction or opposite to one another).

For $M = 3$, again the m-dependence is dropped, p_j will be $-2\pi\mathrm{i}/3, +2\pi\mathrm{i}/3, 2\pi\mathrm{i}$ (its exponentials are the cubic roots of unity), leading to the three branches of the dispersion relation.

For $\forall M$, dropping the m-dependence, $p_j \in \{\nu\frac{2\pi\mathrm{i}}{M}\}$, $\nu = 1, M$ (its exponentials are the M roots of unity), we have the M branches of dispersion. Applying the above correspondence to the multi-soliton solution of aB 2D-lattice (18), we rediscover the multi-soliton solution of coupled aB system (14).

Considering the above parallel, the periodic reduction proves again to be a very effective tool for deriving multi-soliton solution for multicomponent systems.

4. Conclusions

In this paper we studied the coupled additive Bogoyavlensky system with branched dispersion relations and as a particular case ($N = 1$) the coupled Lotka-Volterra system. The main motivation was to prove once again that the integrability survives in coupled systems. The main feature of such coupled systems is the structure of the dispersion relation (having multiple branches) and the structure of the phases of the components, parametrised by the order M roots of unity. The existence of many branches of the dispersion relation allows more freedom in solitons interaction. It was shown by Hirota bilinear formalism that the coupled aB system is integrable and moreover it was shown that with a periodic reduction of an integrable aB 2D-lattice equation the multi-solitons are easier to construct.

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