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Toward tensor renormalization group study of three-dimensional non-Abelian gauge theory

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We propose a new method for representing the path integral over gauge fields as a tensor network. We apply the tensor renormalization group with this method to three-dimensional pure $SU(2)$ gauge theory. Our result for the free energy in the weak/strong coupling regime agrees with that obtained by the weak/strong coupling expansion.

The 39th International Symposium on Lattice Field Theory,
8th-13th August, 2022,
Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, Germany

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PoS(LATTICE2022)021

1. Introduction

The tensor renormalization group (TRG) [1] serves as a new numerical method for lattice field theories. In the TRG, it is non-trivial to represent the path integral over continuous bosonic fields as a tensor network. For scalar fields, the Gaussian quadrature works well in two[2, 3] and four [4, 5] dimensions. For gauge fields, the character expansion[6] works well for the U(1) [7–11], SU(2)[12, 13], SU(N) and U(N) [14] gauge fields. The random sampling method is also applied successfully to SU(2) and SU(3) gauge theories [15].

However, the cost of calculation for the TRG is more sensitive to the dimensionality of space-time than the Monte Carlo method. Indeed, to our best knowledge, no non-Abelian gauge theories in more than two dimensions have been studied through the TRG.

Here we propose an efficient method for representing the path integral over gauge fields as a tensor network that enables the TRG study of higher-dimensional gauge theories Ref.[16]. We introduce a trial action with variational parameters and generate field configurations with the weight defined by the trial action. We construct a tensor network in which the labels of the field configurations are identified with the tensor indices and perform the TRG starting with this network. We repeat this procedure for various values of the variational parameters and finally fix the variational parameters such that the results are insensitive to them. We apply this method to three-dimensional pure SU(2) gauge theory. Our result for the free energy in the weak/strong coupling regime agrees with that obtained by the weak/strong coupling expansion.

This paper is organized as follows. In Sect. 2, we describe our method for representing the path integral over SU(N) gauge fields as a tensor network. In Sect. 3, we show the numerical results for three-dimensional pure SU(2) gauge theory obtained by using our method. Sect. 4 is devoted to conclusion.

2. Tensor network formulation

In this section we describe our method for representing three-dimensional pure SU(N) gauge theories on the lattice as a tensor network. The extension to higher dimensions is straightforward.

The partition function is

$$Z = \int \prod_{n,\mu} dU_{n,\mu} e^{-S} , \tag{1}$$

where n and (n,μ) ($\mu = 1,2,3$) denote the lattice sites and the links, respectively. $U_{n,\mu}$ are the link variables that are SU(N) matrix-valued. The $dU_{n,\mu}$ is the Haar measure, normalized to $\int dU_{n,\mu} = 1$. The plaquette action S is defined by

$$S = \frac{\beta}{N} \sum_{n,\mu>\nu} \mathrm{ReTr}(1 - U_{\mu\nu}(n)) \tag{2}$$

with $U_{\mu\nu}(n) = U_{n,\mu} U_{n+\hat{\mu},\nu} U^{\dagger}_{n+\hat{\nu},\mu} U^{\dagger}_{n,\nu}$. We introduce a trial action S_v with some variational parameters such that the partition function remains unchanged:

$$Z = \int \prod_{n,\mu} dU_{n,\mu} e^{-(S-S_v)-S_v} . \tag{3}$$

We assume that the trial action consists of single link actions $\tilde{S_v}(U(n,\mu))$ as

$$S_v = \sum_{n,\mu} \tilde{S}_v(U_{n,\mu}) \ , \tag{4}$$

and the partition function for the single link action

$$Z_v = \int dU e^{-\tilde{S_v}(U(n,\mu))} \tag{5}$$

is calculable by a certain method. The simplest form of the trial function is given by

$$\tilde{S}_v(U) = -\frac{H}{N}\mathrm{ReTr}U \ , \tag{6}$$

where H is the variational parameter. In the case of SU(2), Z_v with Eq.(6) is given by

$$Z_v = 2\frac{I_1(H)}{H} \ , \tag{7}$$

where I_1 is the modified Bessel function. In the numerical analysis in Sect. 3, we use the trial action given by Eq. (6). We can express Eq.(3) as

$$Z = Z_v^{3V}\langle e^{-(S-S_v)}\rangle_v \ , \tag{8}$$

where V is the number of lattice sites and $\langle\cdots\rangle_v$ is the statistical average with respect to the Boltzmann weight e^{-S_v}:

$$\langle\cdots\rangle_v = \frac{1}{Z_v^{3V}}\int \prod_{n,\mu} dU_{n,\mu}\cdots e^{-\sum n,\mu \tilde{S_v}(U_{n,\mu})} \ . \tag{9}$$

Based on the weight $e^{-\tilde{S}_v(U)}$, we generate a set of field configurations $G = \{U_1, U_2, \cdots, U_K\}$ (by e.g. the Monte Carlo method). Using the generated configurations, the integral for each link variable $U_{n,\mu}$ is approximated as

$$\int dU_{n,\mu} g(U_{n,\mu}, U_{n',\mu'}, \ldots) \approx \frac{1}{K}\sum_{i=1}^{K} g(U_i, U_{n',\mu'}, \ldots) \ . \tag{10}$$

The case $H = 0$ corresponds to the random sampling method[15]. In principle, the partition function is independent of $\tilde{S}_v$ if K is large enough. In practice, however, K cannot be so large due to the computational cost. Thus, we need to adjust the variational parameters. We determine their values such that the free energy is insensitive to their changes.

We construct a tensor that resides on the center of a plaquette:

$$A_{ijkl} = \exp\left[\frac{\beta}{N}\mathrm{Tr}\left(U_i U_j U_k^\dagger U_l^\dagger\right) - \frac{1}{4}\left(\tilde{S}_v(U_i) + \tilde{S}_v(U_j) + \tilde{S}_v(U_k) + \tilde{S}_v(U_l)\right)\right] \ . \tag{11}$$

To construct an initial tensor from A_{ijkl}, we introduce a tensor B_{ijkl} located on the link, following the exact blocking formula[6]:

$$B_{ijkl} = \delta_{ijkl} = \delta_{ij}\delta_{jk}\delta_{kl}\delta_{li} \ . \tag{12}$$

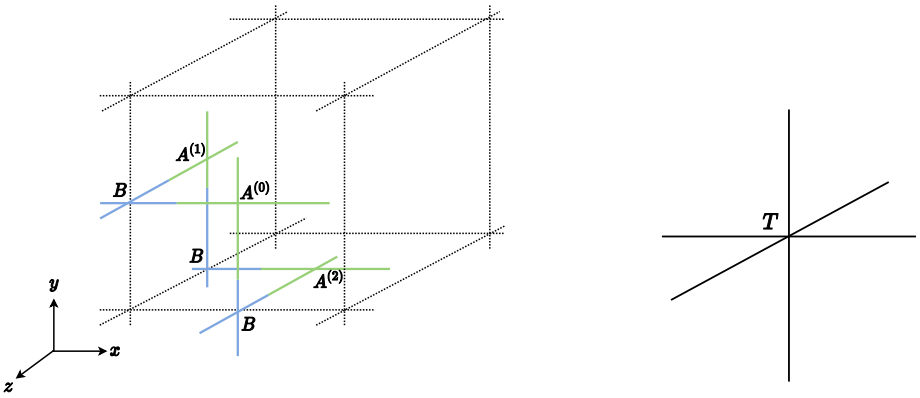

Figure 1: (Left) A tensors on plaquettes and B tensors on links. (Right) T tensors in the center of a cube, where $T = A^{(0)} \otimes A^{(1)} \otimes A^{(2)} \otimes B \otimes B$.

A graphical representation of A tensors and B tensors is given in Fig. 1. Using these tensors, the initial tensor T is constructed as

$$T = A^{(0)} \otimes A^{(1)} \otimes A^{(2)} \otimes B \otimes B \otimes B \, . \tag{13}$$

Here we generate three configuration sets $G^{(0)}$, $G^{(1)}$, and $G^{(2)}$ for $A^{(0)}$, $A^{(1)}$, and $A^{(2)}$, respectively. The $A^{(0)}$, $A^{(1)}$, and $A^{(2)}$ tensors resides on the (xy), (yz), and (zx) planes, respectively(see Fig. 1). The T tensor is a six-rank tensor located in the center of the cube, and the bond dimension is K^2. By introducing isometries, we truncate the bond dimension of the T tensor from K^2 to D. For details of the truncation, see the appendix in Ref.[16]. Thus, the tensor network representation of the three-dimensional SU(N) gauge theory is

$$Z(K) = \left(e^{-\beta} \frac{Z_v}{K} \right)^{3V} \mathrm{tTr} \otimes T \, , \tag{14}$$

where tTr is the trace over the tensor. Hereafter, we consider the case of SU(2) and adopt Eq.(6) as the trial action.

3. Numerical results

In this section we present the numerical results for three-dimensional pure SU(2) gauge theory on the lattice, which we obtain by using our method introduced in the previous section. The lattice size L is fixed to $L = 1024$, and the volume V is given by $V = L^3$. We adopt the trial action $\tilde{S}_v$ given in Eq.(6) with $N = 2$. We calculate the free energy (density) $F = (1/V) \log Z$ using the anisotropic TRG[17].

The procedure of calculation is as follows. First, we generate three sets of K field configurations $G^{(i)} = \{U_1^{(i)}, U_2^{(i)}, \cdots, U_K^{(i)}\}$ $(i = 0, 1, 2)$ with the Boltzmann weight be $e^{-\tilde{S}_v(U(n,\mu))}$ by the Monte Carlo method. Next, as explained in the previous section, $A^{(i)}$ tensors are constructed from $G^{(i)}$ to make the initial tensor T. Third, we truncate the bond dimension of the initial tensor T from K^2 to D by introducing the isometries. (For details, see the appendix of Ref.[16].) Finally, we apply the anisotropic TRG[17] with the bond dimension set to D and calculate the free energy.

In this calculation of free energy, there are systematic and statistical errors due to the introduction of finite parameters K and D. The statistical error, given as error bars in the figures below, is obtained by 10 independent trials. We fix the parameters as $D = 12$ and $K = 16$. See Ref.[16] about the D and K dependence in this calculation.

We show the result of the free energy in Fig.2. We search for the plateau for each value of β in the range $0 < H \leq 20$. Namely, we require that it should be insensitive to changes in H. The free energy is obtained from $F = F(H_*)$. H_* is the point in the plateau with the smallest statistical error. Note that H_* depends on β. The H-dependence of the free energy for $\beta = 1$ and $\beta = 50$ is shown in Fig. 3. We see that there is a plateau in the $H \leq 0.6$ region for $\beta = 1$ and in the $H \leq 16$ region for $\beta = 50$. We take $H_* = 0.001$ for $\beta \leq 7$ and an appropriate value of H_* in the region $H_* > 10$ for $\beta \geq 20$.

The strong coupling expansion of the free energy is given by

$$F(\beta) = -3\beta + \frac{3}{8}\beta^2 - \frac{3}{384}\beta^4 + O(\beta^6) \ . \tag{15}$$

The weak coupling expansion of the expectation value of a plaquette is given by $W_{1\times 1} = e^{-1/\beta}$ [18]. Thus we have

$$F(\beta) = -3\log\beta + C + O\left(\frac{1}{\beta}\right) \ , \tag{16}$$

where C is the integration constant. We determine C to be $C = -5.8426$ by fitting the data at $20 \leq \beta \leq 50$ to $-3\log\beta + C$. The TRG result in the strong/weak coupling regime agrees with that obtained from the strong/weak coupling expansion. However, no plateau is found in the intermediate region $7 \leq \beta \leq 19$. We expect this to be resolved if we increase K and/or improve the trial action. Our result suggests that the random sampling method [15] works well in the strong coupling regime for higher dimensional gauge theories. On the other hand, in the weak coupling regime, we need to adjust the variational parameter H to an appropriate finite value.

4. Conclusion

We proposed a method for the tensor network representation of the path integral of SU(N) gauge theory. The integral is approximated using the field configuration generated by the weight with the trial action. The labels of the field configurations are identified with the indices of the tensors. As the first application of the TRG to higher dimensional gauge theory, we studied three-dimensional pure SU(2) gauge theory by using this method. By tuning the variational parameters, the TRG can work with not so large truncation parameters K and D, at least except in the intermediate regime. We reproduced the results of the strong and weak coupling expansions for the free energy. On the other hand, any definite plateau in the intermediate coupling region cannot be found, and the free energy is not completely stable at the current value of K. We expect to resolve these problems by increase K and/or improving the trial action. We hope that our method will enable the TRG studies of higher dimensional gauge theories including the sign problem.

Acknowledgment

We would like to thank S. Akiyama, M. Fukuma, D. Kadoh and S. Takeda for discussions. The computation was carried out using the supercomputer "Flow" at Information Technology Center,

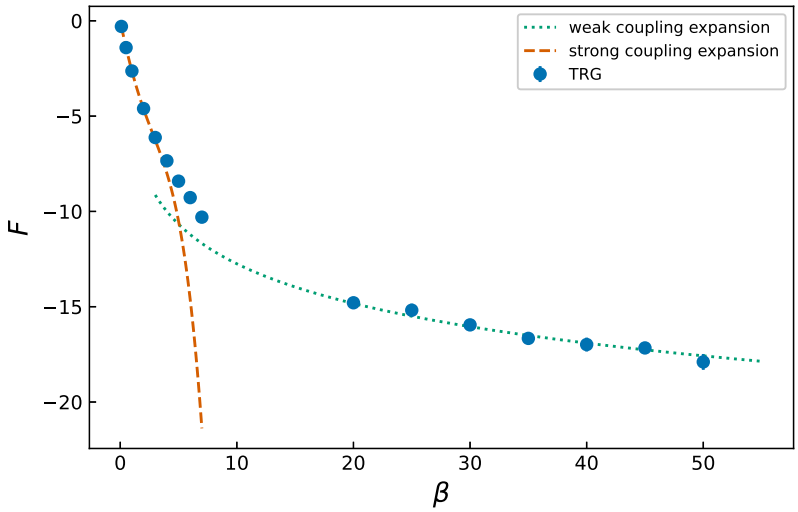

Figure 2: The free energy is plotted against β. The statistical errors are smaller than the symbol size. The strong coupling expansion is represented by the dashed line, while the weak coupling expansion by the dotted line.

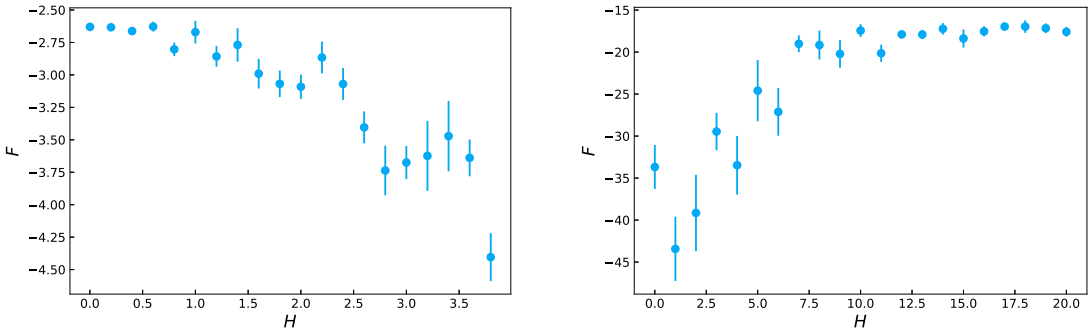

Figure 3: The H dependence of the free energy with $K = 16$ and $D = 12$ for $\beta = 1$(left) and $\beta = 50$(right).

Nagoya University. A.T. was supported in part by JSPS KAKENHI Grant Numbers JP18K03614 and JP21K03532 and by MEXT KAKENHI Grant in Aid for Transformative Research Areas A "Extreme Universe" No. 22H05253.

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