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# Topology and the Dirac Spectrum in Hot QCD

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It is known that contrary to expectations, the order parameter of chiral symmetry breaking, the Dirac spectral density at zero virtuality does not vanish above the critical temperature of QCD. Instead, the spectral density develops a pronounced peak at zero. We show that the spectral density in the peak has large violations of the expected volume scaling. This anomalous scaling and the statistics of these eigenmodes is consistent with them being produced by mixing instanton and antiinstanton zero modes. Consequently, we show that a nonvanishing topological susceptibility implies a finite density of eigenvalues around zero, which can have implications on the restoration of chiral symmetry above the critical temperature.

*The 39th International Symposium on Lattice Field Theory (Lattice2022),*
*8-13 August, 2022*
*Bonn, Germany*

*Speaker





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## 1. Introduction

The spectrum of the quark Dirac operator has long been known to provide information about the physics of QCD. In particular, the long-distance physics is encoded in the low-lying, infrared part of the Dirac spectrum. The spectral quantity that has probably the simplest direct connection to the physics is the spectral density at zero virtuality, $\rho(0)$, which – by the Banks-Casher relation – is the order parameter of chiral symmetry breaking [1]. Indeed, according to the standard picture of the finite temperature transition in QCD, in the low temperature, hadronic phase $\rho(0)$ is nonzero, signaling the spontaneous breaking of chiral symmetry. As the system is heated and enters the high temperature, quark-gluon plasma phase and chiral symmetry gets restored, $\rho(0)$ becomes zero.

Strictly speaking, the above picture applies only to an imaginary world, where quarks are massless and the chiral symmetry of the Lagrangian is exact. In the real world the quark masses are very small compared to the QCD scale, and chiral symmetry is only approximate. As a result, the finite temperature transition is only a crossover, and instead of jumping to zero at a well defined critical temperature, $\rho(0)$ only rapidly decreases through the crossover region, as the system enters the quark-gluon plasma state.

This simple picture, however, was later challenged. The first sign of a more complicated behavior started to emerge, when in the very early days of the overlap Dirac operator, it was noticed that in the high temperature phase of the quenched theory the spectral density of the overlap Dirac operator develops an unexpected sharp peak at zero virtuality [2]. Even though, this was in the quenched theory, where chiral symmetry is not straightforward to interpret, this behavior of the spectral density was rather counterintuitive. These simulations were performed on rather coarse lattices ($N_t = 4$, just above $T_c$), nevertheless the tentative interpretation of the unexpected spectral peak was that it was due to topology-related would-be zero modes of the Dirac operator, and resolving them was made possible only by the overlap operator, thanks to its exact chiral symmetry. We emphasize that these are not the chiral zero modes, corresponding to the net topological charge – those can be easily distinguished in the overlap spectrum. Rather, the peak in the spectral density was suggested to be due to mixing zero modes of instantons and antiinstantons[1], eigenmodes that come in complex conjugate pairs and can be close to zero if their splitting due to mixing is small [3]. One could also argue that these unusual small modes could be just cutoff effects caused by the coarse lattice, or a quenched artifact, since fluctuations of the topological charge are expected to be suppressed by dynamical quarks. This is, however, not the case, as was shown in several further works using much finer lattices, as well as light dynamical quarks [4–6]. It was even suggested that the approximate $1/\lambda$ shape of the spectral peak signals a previously unnoticed “phase” of QCD, intermediate between the hadronic and the quark-gluon plasma phase. In the present work we will show further evidence that the initial interpretation of the spectral peak as mixing topological near zero modes was correct.

[1]In the present work, for simplicity we will refer to lumps of unit topological charge as (anti)instantons, even though close to $T_c$ in the high temperature phase their charge profile might not even be close to that of calorons. For our discussion, the only important property of these objects is that they are isolated lumps of topological charge of unit magnitude.



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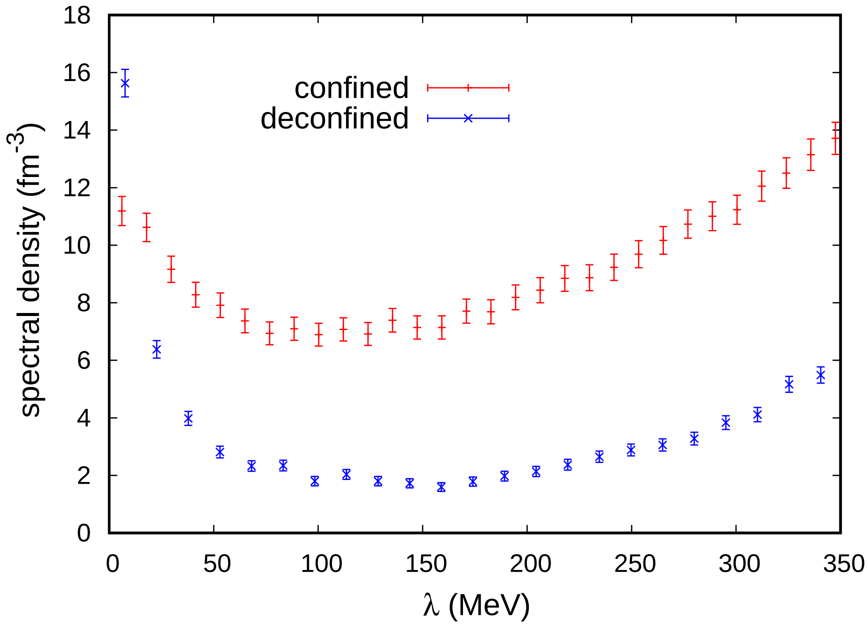


**Figure 1:** The spectral density of the overlap Dirac operator on an ensemble of $N_t = 8$ quenched lattices at the critical point. Based on the magnitude of the Polyakov loop, the confined and deconfined phases were separated, and their spectral densities are plotted separately.

## 2. The spectral peak

We work in the quenched approximation, but eventually will argue that our results also apply to the case with dynamical fermions, at least in a qualitative fashion. Let us first look at how the spectral density of the overlap Dirac operator changes across the finite temperature phase transition. In the quenched case with $SU(3)$ gauge group, this is a genuine first order phase transition, and tuning the coupling to the critical point allows us to sample both phases at the same value of the coupling. In Fig. 1 we show the spectral density for an ensemble of $N_t = 8$ lattices at Wilson gauge coupling $\beta = 6.063$, which is the critical value for the given temporal lattice extension. Based on the magnitude of the Polyakov loop, we separated the configurations into two sets, belonging to the two phases, and computed the spectral density separately in the two phases[2]. The difference is quite clear. For most of the spectrum the spectral density is much lower in the deconfined phase, a behavior, consistent with the naive expectation of the spectral density jumping to a lower value when the system enters the quark-gluon plasma phase. However, very close to zero virtuality, the deconfined phase shows a marked peak that goes much higher than the spectral density in the confined phase.

Let us further examine the peak by looking at its volume dependence. To be able to do so in a cleaner situation where we do not have to separate the two phases, we chose to do it at a slightly higher temperature, $T = 1.045T_c$. In Fig. 2 we show the spectral density at $N_t = 8$, Wilson

[2]In the deconfined phase, only configurations in the "physical"(real) Polyakov loop sector were used. In the complex sectors the spectral density and the properties of the corresponding eigenmodes are very different [7].

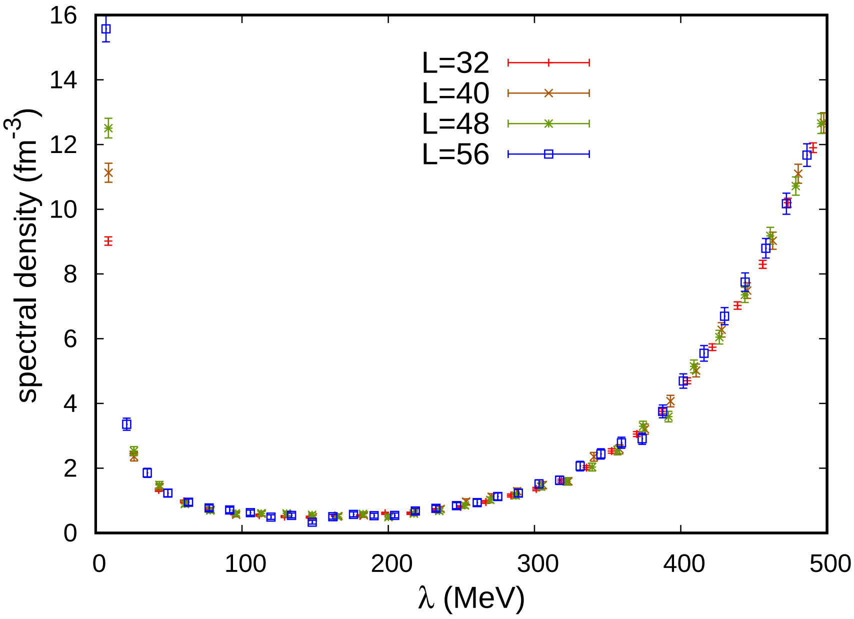


**Figure 2:** The spectral density of the overlap Dirac operator on four ensembles of $N_t = 8$ lattices at $T = 1.045T_c$. The spatial linear lattice sizes range from $L = 32$ to 56.

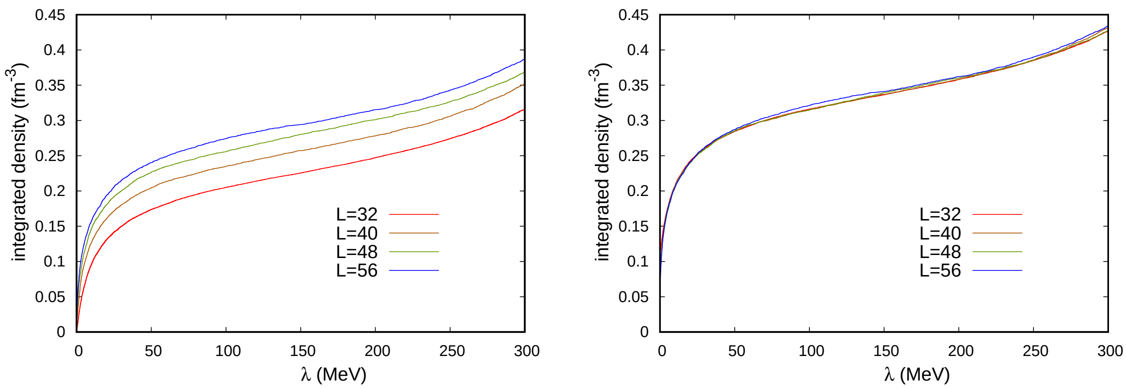


**Figure 3:** The integrated spectral density of the four ensembles of Fig. 2. In the left panel the exact zero modes were not included in the integral, while in the right panel they were.

$\beta = 6.09$ for ensembles of four different linear sizes, $L = 32, 40, 48, 56$. As expected, the (volume normalized) spectral density is volume independent, except at the spectral peak where it shows a rather strong volume dependence.

The lack of volume scaling at the very low-end of the spectral density is also manifest in the left panel of Fig. 3, where we show the integrated spectral density. It is important to keep in mind that both in Fig. 2, showing the spectral density and in the left panel of Fig. 3, showing its integral, we did not include the exact zero modes. In the density they would show up as a Dirac delta at the origin, which would shift the integrated density curves upward. To check how this affects the integrated spectral densities, in the right panel of Fig. 3 we also included these shifts due to the zero mode Dirac deltas, and – as can be seen in the figure – the resulting curves for the different volumes are thus shifted on top of one another. In other words, this means that the total number of low modes scales properly with the volume, and the "deficit" we observe in the near zero modes in the smaller volumes is made up by the relatively larger density of exact zero modes there.

This indicates that the strong volume dependence of the spectral density is connected to the zero modes and topological charge. However, these large finite-size corrections to the density still seem quite counter-intuitive, and it is not even clear from Fig. 2 whether the spectral density at zero converges to a finite value. Assuming that the modes in the spectral peak are mixing topological zero modes, we can have a more quantitative assessment of the expected finite-size effects in the spectral density. Indeed, the density of topological objects is proportional to the volume, and – on average – half of them are instantons, half antiinstantons. If in a configuration there are $n_+$ instantons and $n_-$ antiinstantons, their would be zero modes combine into $|n_+ - n_-|$ exact chiral zero modes, and the remaining would be zero modes will mix into non-chiral eigenmodes close to zero, making up the spectral peak. $\langle n_+ + n_- \rangle$ scales with the volume, but

$$\langle |n_+ - n_-| \rangle = \langle |Q| \rangle \propto V^{1/2}, \tag{1}$$

where $Q$ is the topological charge. So the volume scaling of the number of non-chiral close to zero modes is expected to have a large finite-size correction, vanishing in the thermodynamic limit only as $V^{-1/2}$. This is because some of the zero modes carried by the (anti)instantons combine into exact zero modes and do not show up in the spectral density arbitrarily close to zero. This deficit of close to zero modes is seen in the spectral density and causes the large finite-size correction. The missing modes, however, are not lost, they make up the exact zero modes, and – as we already saw – by their inclusion in the integrated spectral density, we can remove the large finite-size correction.

## 3. Ideal instanton gas

Above the critical temperature, the topological susceptibility rapidly decreases [8–10], the instanton gas becomes dilute and to a good approximation the (anti)instantons form an ideal (non-interacting) gas [11, 12]. In such a dilute gas, topological objects are typically far away from one another and the splitting from the origin of the non-chiral near zero modes of topological origin is expected to be small. Therefore, it is a reasonable assumption that these eigenvalues of topological origin are the ones closest to zero. They occupy a spectral region $[-\lambda_{zmz}, \lambda_{zmz}]$ that we call the zero mode zone.

Using the ideal instanton gas picture, we can estimate $\lambda_{zmz}$ by noting that in an ideal gas the number of instantons and antiinstantons follow independent and identical Poisson distributions. The parameter of this distribution, giving the density of topological objects is $\chi V$, where $\chi$ is the topological susceptibility and $V$ is the space-time volume of the system. It follows from the properties of Poisson distributions that the density of topological objects is

$$\langle n_+ + n_- \rangle = \chi V. \tag{2}$$

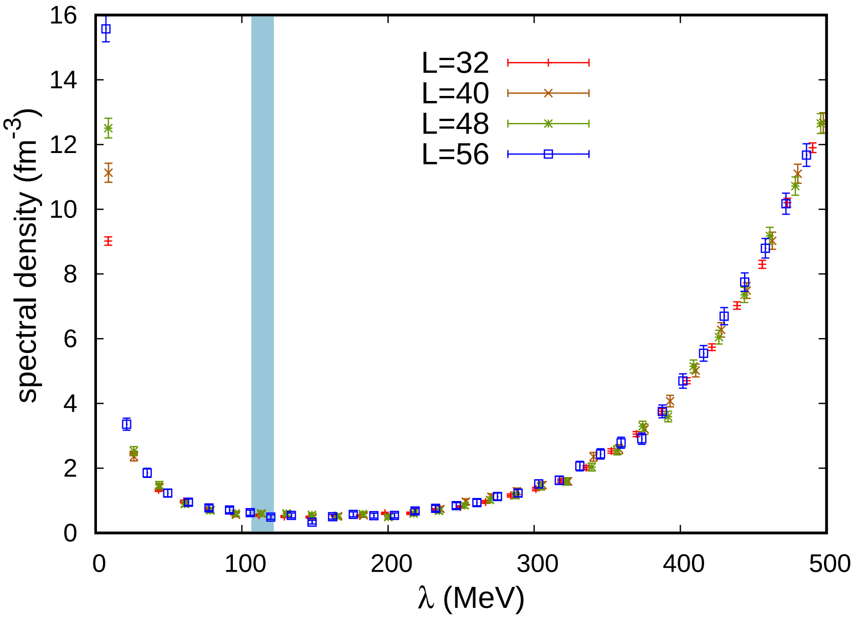


**Figure 4:** The spectral density of the overlap Dirac operator on four ensembles of $N_t = 8$ lattices at $T = 1.045T_c$. The blue vertical bar indicates $\lambda_{zmz}$, the boundary of the zero mode zone, its width is the uncertainty.

From the number of exact zero modes of the overlap Dirac operator, the topological susceptibility can be easily computed, and it is also related to the average number of topological objects as

$$\chi V = \langle Q^2 \rangle = \langle (n_+ - n_-)^2 \rangle. \tag{3}$$

Using the susceptibility, obtained from the exact zero modes, we can estimate $\lambda_{zmz}$ by assuming that to each instanton and antiinstanton, there is a corresponding eigenvalue in the zero mode zone. This is done by setting $\lambda_{zmz}$ such that on average there be $\chi V$ eigenvalues in the $[-\lambda_{zmz}, \lambda_{zmz}]$ interval, including the exact zero modes. In Fig. 4 we show $\lambda_{zmz}$, the boundary of the zero mode zone, computed in the above described way. As can be seen in the figure, the zero mode zone includes the whole of the spectral peak and the lower part of the wide valley in the spectral density, located between the peak at zero and the bulk of the spectrum, where the density starts to rise rapidly.

If all these eigenmodes in the zero mode zone are associated to topological lumps of unit charge that form an ideal gas, then the distribution of the number of these eigenmodes should follow a Poisson distribution with mean $\chi V$. We already fixed the mean to $\chi V$, but it is a nontrivial test whether the distribution is actually Poissonian. This can be easily checked in the lattice overlap spectra by counting the number of eigenvalues in the zero mode zone, configuration by configuration. We show the result in Fig. 5. Also shown in the figure is the expected Poisson distribution, that is the distribution of the number of topological lumps in an ideal instanton gas with the given susceptibility. Notice that there is no further fitting involved here, as the only parameter of the distribution, $\chi V$ has already been determined from the exact zero modes of the

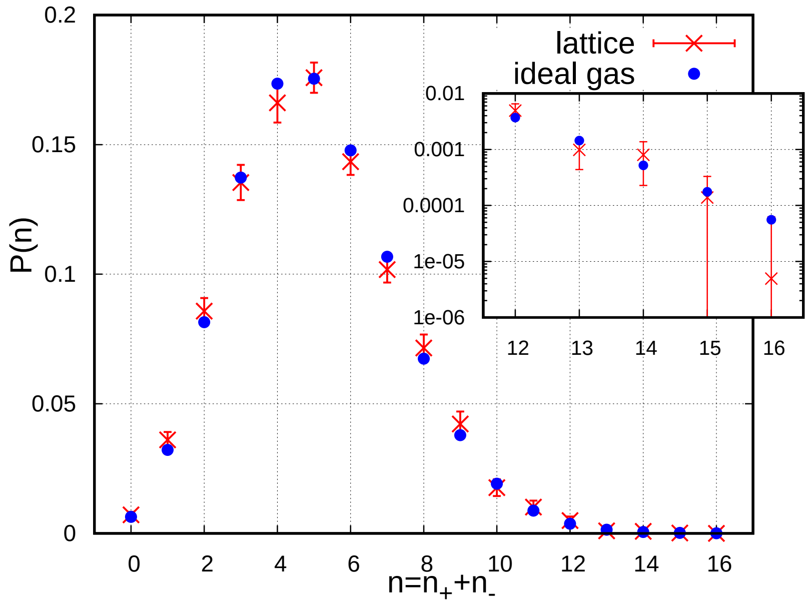


**Figure 5:** The distribution of the number of eigenmodes in the zero mode zone. The red crosses are the lattice data on an ensemble of $N_t = 8, V = 32^3$ lattices at a temperature of $T = 1.045T_c$. The blue circles represent the expected ideal gas distribution of the number of topological objects with their density equal to the density of eigenvalues in the zero mode zone.

overlap. The agreement with the lattice data is fairly good, indicating that the topological objects form a non-interacting gas.

At this point one could also ask the question whether $\lambda_{zmz}$ is really a special point of the spectrum in the sense that the number of eigenvalues in the $[-\lambda_{zmz}, \lambda_{zmz}]$ interval is Poisson distributed, but for other intervals it cannot be described with a Poisson distribution. To see this, we tried to fit the distribution of the number of eigenvalues in intervals $[-\lambda_{cut}, \lambda_{cut}]$ for different values of $\lambda_{cut}$. The chi squared per degree of freedom as a function of $\lambda_{cut}$ is shown in Fig. 6. Indeed, $\lambda_{zmz}$ is in the middle of the spectral region where the chi squared of the fit is acceptable, and the distribution is close to a Poisson distribution. Notice that this region is rather wide because of the simple reason that the spectral density is very small there, and by increasing $\lambda_{cut}$, so few eigenvalues are added to the data set that they do not influence the quality of the fit significantly.

## 4. Conclusions

In the present paper we studied the peak at zero virtuality in the spectral density of the QCD Dirac operator. We saw that the spectral density at the peak has unexpectedly large finite size corrections. However, this volume dependence of the spectral density can be understood if we assume that the eigenmodes in the spectral peak are mixing instanton-antiinstanton zero modes, originating from a non-interacting gas of topological objects. The distribution of the number of eigenmodes in the peak also supports this picture.

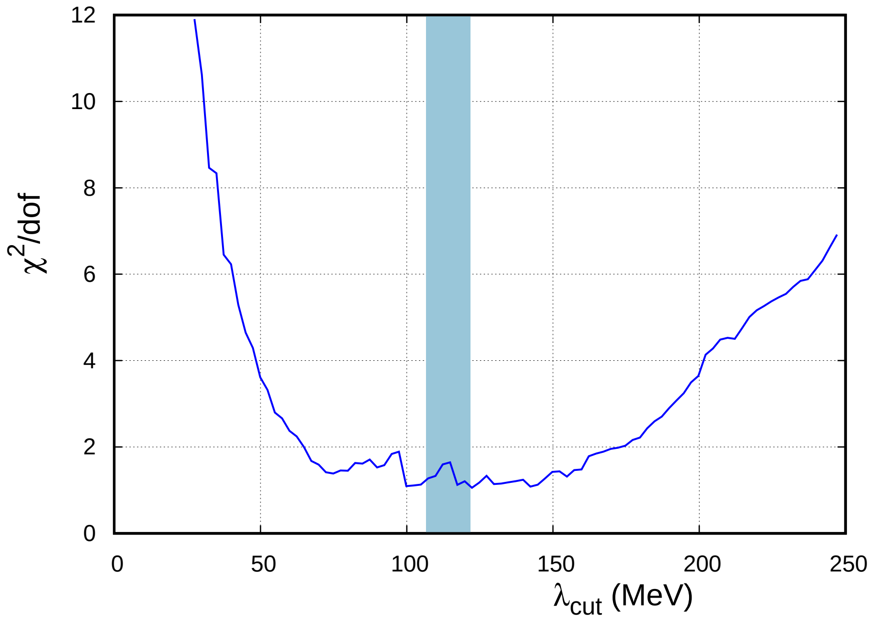


**Figure 6:** The chi squared per degree of freedom of a fit to a Poisson distribution of the number of eigenvalues in an interval $[-\lambda_{cut}, \lambda_{cut}]$ as a function of $\lambda_{cut}$. The blue bar represents the previously determined value of $\lambda_{zmz}$ with its width being the uncertainty.

The most important consequence of our findings is that there is a strong connection between the topological susceptibility and the total number of eigenvalues in the zero mode zone, most of which are in the spectral peak. In particular, a nonzero topological susceptibility implies the presence of a corresponding spectral peak, which means that chiral symmetry is not immediately restored in the high temperature phase.

We have to add some qualifications to the above statement. The numerical results we presented, were obtained in the quenched approximation. Dynamical fermions are expected to suppress fluctuations of the topological charge, and will definitely affect both the topological susceptibility and the spectral peak. However, as long as the fermions do not induce strong interactions among topological objects, the connection between the susceptibility and the spectral peak remains valid. Moreover, even if light dynamical quarks induce strong interactions among instantons, we expect this effect to suppress the net topological charge, i.e. the susceptibility, more than the near zero modes in the peak. This means that even in the presence of fermion-induced interactions, we expect a nonzero topological susceptibility to imply the presence of a spectral peak. It is a dynamical question how at a given value of the quark mass the breaking of chiral symmetry due to the spectral peak compares to the explicit breaking by the quark mass. Presently there is ongoing work to explore how dynamical quarks influence the spectral peak.

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