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Position-Space Renormalisation of the Energy-Momentum Tensor

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There is increasing interest in the study of nonperturbative aspects of three-dimensional quantum field theories (QFT). They appear as holographic dual to theories of (strongly coupled) gravity. For instance, in Holographic Cosmology, the two-point function of the Energy-Momentum Tensor (EMT) of a particular class of three-dimensional QFTs can be mapped into the power spectrum of the Cosmic Microwave Background in the gravitational theory. However, the presence of divergent contact terms poses challenges in extracting a renormalised EMT two-point function on the lattice. Using a ϕ^4 theory of adjoint scalars valued in the $\mathfrak{su}(N)$ Lie Algebra as a proof-of-concept motivated by Holographic Cosmology, we apply a novel method for filtering out such contact terms by making use of infinitely differentiable "bump" functions which enforce a smooth window that excludes contributions at zero spatial separation. The process effectively removes the local contact terms and allows us to extract the continuum limit behaviour of the renormalised EMT two-point function.

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1. Introduction

In holographic models of cosmology, the scalar power spectrum of the Cosmic Microwave Background (CMB) is computed from the two-point function of the Energy-Momentum Tensor (EMT) [14]. We can decompose the EMT two-point function as [8] in this case as

$$\langle T_{ij}(\bar{q}) T_{kl}(-\bar{q}) \rangle = A(\bar{q}) \Pi_{ijkl} + B(\bar{q}) \pi_{ij} \pi_{kl}, \tag{1}$$

where

$$\pi_{\mu\nu} = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \tag{2}$$

is the transverse projector and

$$\Pi_{\mu\nu\rho\sigma} = \frac{1}{2}(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho} - \pi_{\mu\nu}\pi_{\rho\sigma}) \tag{3}$$

is the transverse-traceless projector. The $B(q)$ form factor in eq. (1) maps into the CMB scalar power spectrum via the relation [11, 13]:

$$\Delta_R^2(q) = \frac{-q^3}{16\pi^2 \mathrm{Im} B(-iq)} = \frac{\Delta_0^2}{1 + \frac{gq*}{q} \log|\frac{q}{\beta g q*}|} \tag{4}$$

Perturbatively, it has been shown in [2, 3] that for high multipole momenta ($l \gtrsim 30$) the fit that the model gives to cosmological data is competitive with that of ΛCDM. The low multipole momentum region, however, maps into the non-perturbative regime of the QFT, and therefore a lattice treatment of it is rendered necessary.

2. Algorithm

The lattice simulations discussed in the following sections were written using the Grid library [5], whereupon a Heatbath-Overrelaxation algorithm was used to update the scalar fields. The algorithm consists of a heatbath update followed by n overrelaxation, or reflection, updates. These also contain Metropolis accept/reject steps where appopriate to account for non-gaussianities. These algorithms are discussed in more detail in [1], and the update prescription we used follows closely the one laid out in [6], with slight alterations for numerical stability and without the gauge updates.

3. The Discretised Energy-Momentum Tensor and Ward Identities

Motivated by Holographic Cosmology, we focus on the theory defined by the following lattice action:

$$S = \frac{a^3 N}{g} \sum_{x \in \Lambda^3} \mathrm{Tr} \left\{ \sum_\mu [\Delta_\mu \phi(x)]^2 + (m^2 - m_c^2)\phi^2(x) + \phi^4(x) \right\}. \tag{5}$$

Here, our scalar fields $\phi(x)$ are traceless hermitian $N \times N$ matrices valued in the $\mathfrak{su}(N)$ algebra and Δ_μ is the forward discrete derivative. Furthermore, the theory here is presented with large-N scaling. This theory is superrenormalisable and contains a continuum second-order phase transition between a symmetric and a broken phase at $m^2 = 0$. Furthermore, this theory is perturbatively IR-divergent, but it has been shown by the LatCos collaboration that it is nonperturbatively finite [9], in addition to the expectation that it should be IR-finite due to its superrenormalisability [4, 12]. A tentative form of the bare lattice Energy-Momentum Tensor $T^0_{\mu\nu}$ may be obtained by replacing the derivatives of the continuum $T_{\mu\nu}$ with discrete central derivatives, here denoted by $\bar{\Delta}_\mu$:

$$T^0_{\mu\nu} = \frac{N}{g}\mathrm{Tr}\Bigg\{2(\bar{\Delta}_\mu\phi)(\bar{\Delta}_\nu\phi) - \delta_{\mu\nu}\left[\sum_\rho(\bar{\Delta}_\rho\phi)^2 + (m^2 - m_c^2)\phi^2 + \phi^4\right] + \xi\left[\delta_{\mu\nu}\sum_\rho(\bar{\Delta}_\rho\phi)^2 - (\bar{\Delta}_\mu\phi)(\bar{\Delta}_\nu\phi)\right]\Bigg\}. \tag{6}$$

The last term in brackets, which multiplies the constant ξ, is the improvement term which accounts for non-minimal coupling of the QFT to gravity. ξ is a parameter to be fixed by comparing with CMB data. The bare lattice $T^0_{\mu\nu}$ as defined in eq. (6), however, does not satisfy the continuum Ward Identity (WI) due to the breaking of continuum translational symmetry. Explicitly,

$$\langle\bar{\Delta}_\mu T^0_{\mu\nu}(x)P(y)\rangle = -\left\langle\frac{\delta P(y)}{\delta\phi(x)}\bar{\Delta}_\nu\phi(x)\right\rangle + \langle X_\nu(x)P(y)\rangle, \tag{7}$$

where $P(x)$ and $X_\mu(x)$ are lattice operators. In order to restore the WI as the lattice regulator is removed (i.e. taking $a \to 0$), we require the second term in eq. (7) to vanish in this limit. However, due to radiative corrections inducing mixings with lower-dimensional operators than $T^0_{\mu\nu}$, the second term in fact diverges with a^{-1}. Therefore, it requires renormalisation. As a renormalisation prescription, we will require that the WI be restored when we remove the regulator by subtracting the divergent contribution from the bare lattice Energy-Momentum Tensor.

It has been shown [7] that in the 4D theory $T^0_{\mu\nu}$ mixes with 5 different lower-dimensional operators. Power-counting tells us that in the 3D theory there is only one operator with which it mixes. Namely, $\tilde{O} = \delta_{\mu\nu}\mathrm{Tr}\phi^2$. Therefore, we subtract from the bare EMT a divergent term that is proportional to this operator:

$$T^R_{\mu\nu} = T^0_{\mu\nu} - \frac{Nc_3}{a}\delta_{\mu\nu}\mathrm{Tr}\phi^2, \tag{8}$$

where c_3 is a constant to be determined. Perturbatively, we may obtain it, for instance to one loop:

$$c_3^{\text{1-loop}} = \left(2 - \frac{3}{N^2}\right)\left(\frac{6Z_0 - 1}{12}\right). \tag{9}$$

Nevertheless, since we wish to consider the theory at its critical point where the correlation length diverges, we cannot trust perturbative results to be accurate. Therefore, we need to turn to non-perturbative methods. Consider the following insertion of $T^0_{\mu\nu}$:

$$C^0_{\mu\nu}(q) = \frac{N}{g}a^3\sum_x e^{-iq\cdot x}\langle T^0_{\mu\nu}(x)\mathrm{Tr}\phi^2(0)\rangle = C_{\mu\nu}(q) + \frac{g}{a}c_3\delta_{\mu\nu}C_2(q) + \frac{\kappa}{a}\delta_{\mu\nu}, \tag{10}$$

where the κ/a factor is a contact term, and

$$C_2(q) = \left(\frac{N}{g}\right)^2 a^3\sum_x e^{-iq\cdot x}\langle\mathrm{Tr}\phi^2(x)\mathrm{Tr}\phi^2(0)\rangle. \tag{11}$$

Furthermore, $C_{\mu\nu}$ is the corresponding finite, continuum correlator and the expression in the last equality eq. (10) has been obtained by inserting eq. (8) into the definition of $C^0_{\mu\nu}$. Both the contact term and the WI-breaking term diverge with a^{-1}, so if we wish to nonperturbatively calculate the value of c_3, we need to untangle these two contributions. This can be done by filtering out the contact term in position-space with the use of a window function.

4. The Window Function

In order to remove the contributions from contact terms, we introduce the *position-space window function* $\Gamma_{r_0,\epsilon}(x)$, defined as follows:

$$\Gamma_{r_0,\epsilon}(x) = \begin{cases} 0, & 0 < x < r_0 \\ \bar{\Gamma}_{r_0,\epsilon}(x), & r_0 \leq x \leq r_0 + \epsilon \\ 1, & r_0 + \epsilon \leq x < \infty \end{cases} \tag{12}$$

where, between r_0 and ϵ, the function is defined as

$$\bar{\Gamma}_{r_0,\epsilon}(x) = 1 - \frac{\int_x^{r_0+\epsilon} \beta(u, r_0, \epsilon)\, du}{\int_{r_0}^{r_0+\epsilon} \beta(u, r_0, \epsilon)\, du} \tag{13}$$

where here we have

$$\beta(x, r_0, \epsilon) = f(x - r_0) f(r_0 + \epsilon - x), \tag{14}$$

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \exp\left(-\frac{1}{x}\right) & \text{for } x > 0 \end{cases}. \tag{15}$$

The relevant properties of this construction of $\Gamma_{r_0,\epsilon}(x)$ are as follows:

1. It is zero for any value of x less than a minimum radius r_0, and therefore will completely exclude any contribution coming from this window.

2. It is one for any value of x greater than $r_0 + \epsilon$, and therefore will not affect contributions to the function beyond this radius.

3. It smoothly interpolates between 0 and 1 in the window $r_0 \leq x \leq r_0 + \epsilon$, that is, it is C^∞ in this window, and therefore does not generate discontinuity artifacts.

4. Due to the Paley-Wiener theorem [15], the Fourier Transform $\mathcal{F}[\Gamma_{r_0,\epsilon}(p)]$ decays faster than any power of $\frac{1}{|p|}$, and goes asymptotically as $|p|^{-n} e^{-|p|^m}$ for some n, m for large $|p|$. The parameter ϵ determines the rate of decay of $\mathcal{F}[\Gamma_{r_0,\epsilon}(p)]$, i.e. the larger ϵ is, or equivalently, the smoother the window function, the more rapid its momentum-space representation decays.

In fig. 1 and fig. 2 it is possible to see the behaviour of the window and its Fourier Transform for different choices of ϵ.

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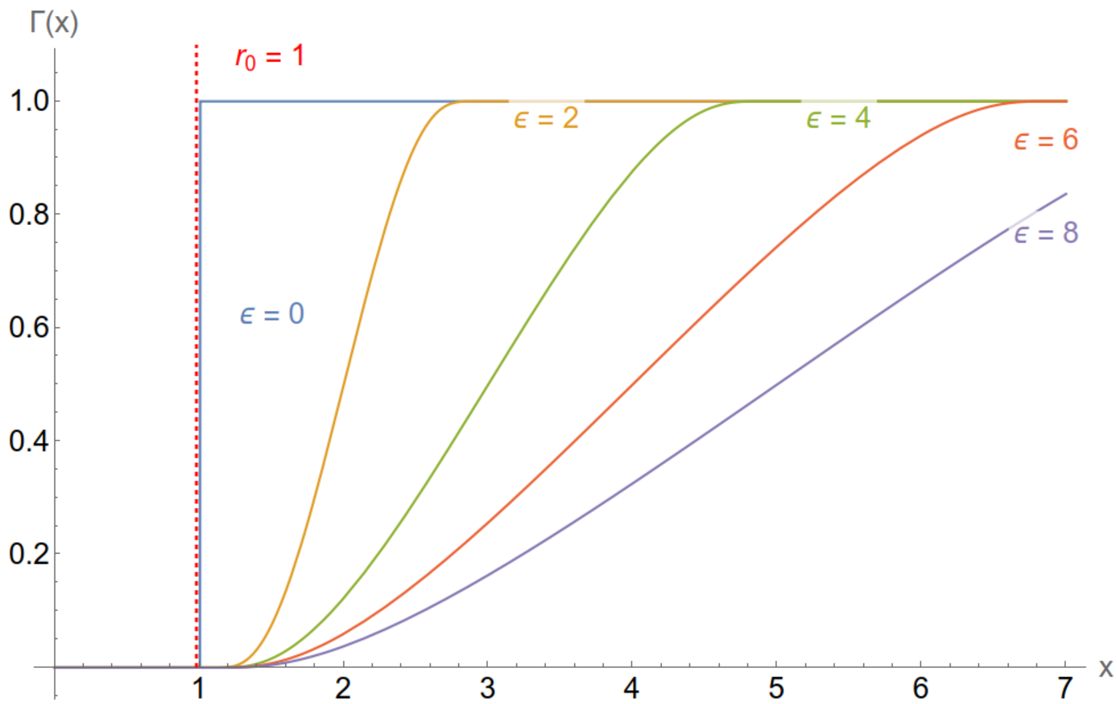


Figure 1: Plot of the position-space window function for different choices of ϵ, with a fixed $r_0 = 1$.

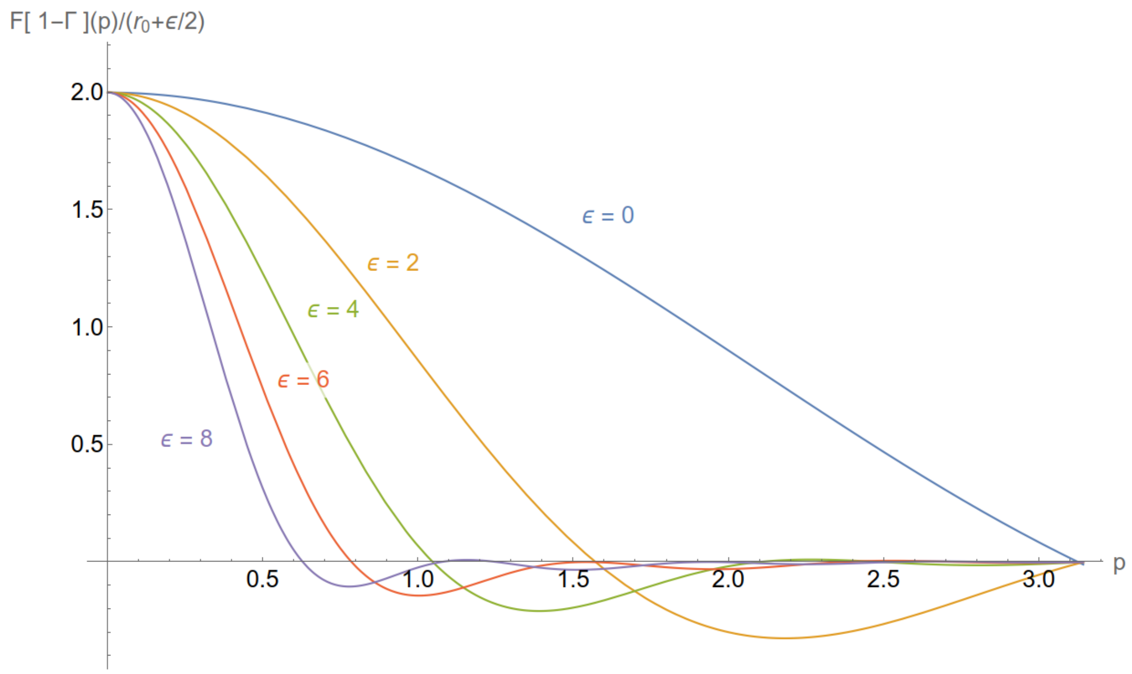


Figure 2: Plot of the Fourier Transform of $1 - \Gamma_{r_0,\epsilon}(x)$ for different choices of ϵ, with a fixed $r_0 = 1$. The subtraction is to restrict the Fourier Transform to a compact domain, and furthermore the curves are divided by a factor of $r_0 + \epsilon/2$ for better visualisation.

5. EMT Position-Space Renormalisation

In order to remove the contact term contributions from a given lattice operator $O(q)$, we define *windowing* as the following operation:

$$W_{r_0,\epsilon}[O](q) = \left(\frac{a}{L}\right)^3 \sum_x e^{-iq\cdot x}\, \Gamma_{r_0,\epsilon}(|x|) \sum_{q'} e^{iq'\cdot x}\, O(q'). \tag{16}$$

This windowing operation is linear, and therefore applying it to eq. (10) gives

$$W_{r_0,\epsilon}[C^0_{\mu\nu}](q) = W_{r_0,\epsilon}[C_{\mu\nu}](q) + \frac{g}{a} c_3 \delta_{\mu\nu} W_{r_0,\epsilon}[C_2](q) + W_{r_0,\epsilon}\left[\frac{\kappa}{a}\delta_{\mu\nu}\right]. \tag{17}$$

The last term in eq. (17) is a contact term and therefore yields zero when windowed. Rearranging this expression to isolate c_3,

$$c_3 = \frac{a}{g}\left(\frac{W_{r_0,\epsilon}[C^0_{\mu\nu}](q) - W_{r_0,\epsilon}[C_{\mu\nu}](q)}{W_{r_0,\epsilon}[C_2](q)}\right). \tag{18}$$

Restricting ourselves to the zero mode ($q = 0$) and taking the limit $\epsilon \to \infty$, it is possible to show that this expression behaves as

$$c_3 \sim \frac{a}{g}\left(\frac{C^0_{\mu\nu}(0)}{C_2(0)} - \frac{b_2}{\epsilon}\right) \tag{19}$$

where b_2 is some constant. This suggests that we can vary ϵ while measuring the ratio between the bare correlators and fit these results to the form

$$\frac{a W_{r_0,\epsilon}[C^0_{22}](q_l = 0)}{W_{r_0,\epsilon}[C_2](q_l = 0)} = \bar{c}_3 + \frac{b}{\epsilon}, \tag{20}$$

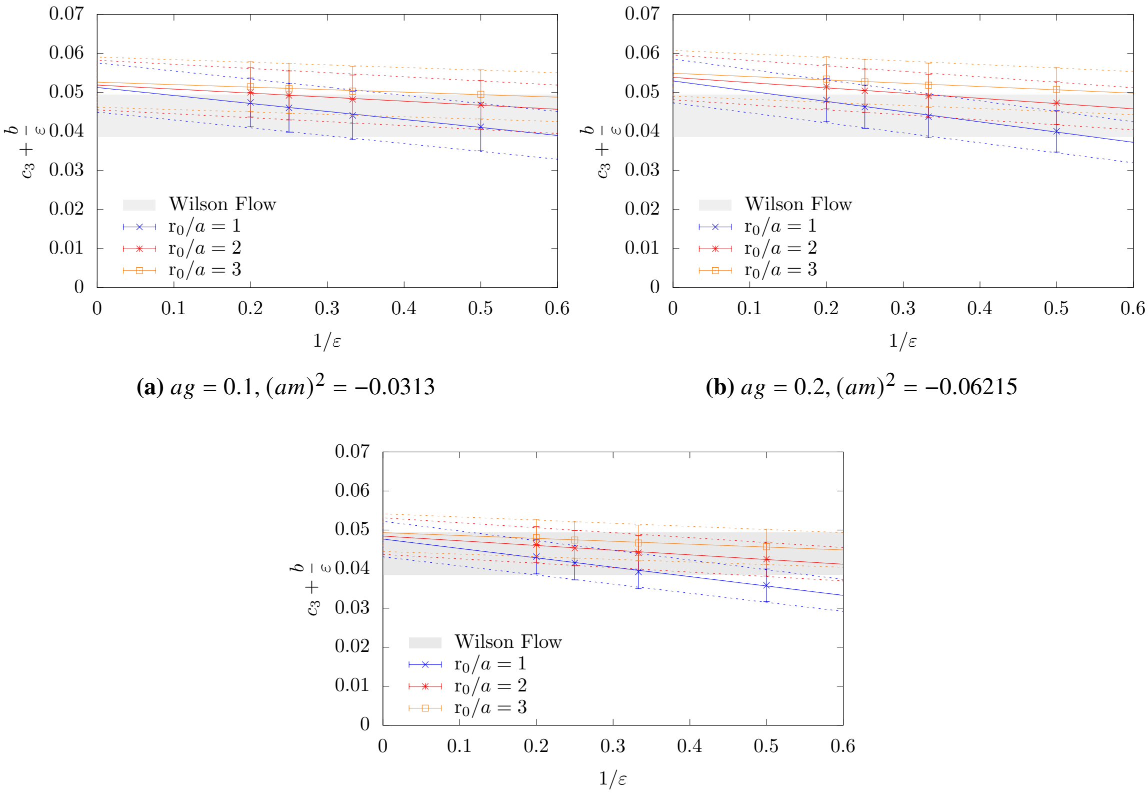
where b and $\bar{c}_3$ are fit parameters, whence we can extrapolate to find c_3 in the $1/\epsilon \to 0$ limit, as the left-hand side of eq. (20) contains only lattice observables.

6. EMT Renormalisation Results

On fig. 3 it is possible to see the results of the fit given by the ansatz in eq. (20) and their extrapolation to $\epsilon \to \infty$ for three different choices of ag. The simulated masses were in the vicinity of the critical point. It can be seen that the extrapolated value of c_3, given by the y-intercept, gives overlap between the error bands between the position-space method and the Wilson Flow method, as obtained in [10].

7. Two-Point Function Renormalisation on Synthetic Data

In the continuum, it is expected that the two-point function of the EMT will contain terms proportional to $(q/g)^3$ and $(q/g)^2 \log(q/g)$, in addition to contact terms. In principle, the windowing procedure can remove such contact terms from lattice data such that we are left only with the signal whence we may extract the form factors $A(q)$ and $B(q)$ from eq. (1). As a proof-of-concept, we generate synthetic data according to the distribution



(a) $ag = 0.1$, $(am)^2 = -0.0313$

(b) $ag = 0.2$, $(am)^2 = -0.06215$

(c) $ag = 0.3$, $(am)^2 = -0.09275$

Figure 3: Fit results for the renormalisation constant $\bar{c}_3$ using the ansatz eq. (20), which is the value of the y intercept. This is performed for three ensembles ($N_L = 256$) and different choices of window radius $r_0/a \in \{1, 2, 3\}$, as represented by the three colours. The grey bands correspond to the result for c_3 obtained using the Wilson flow.

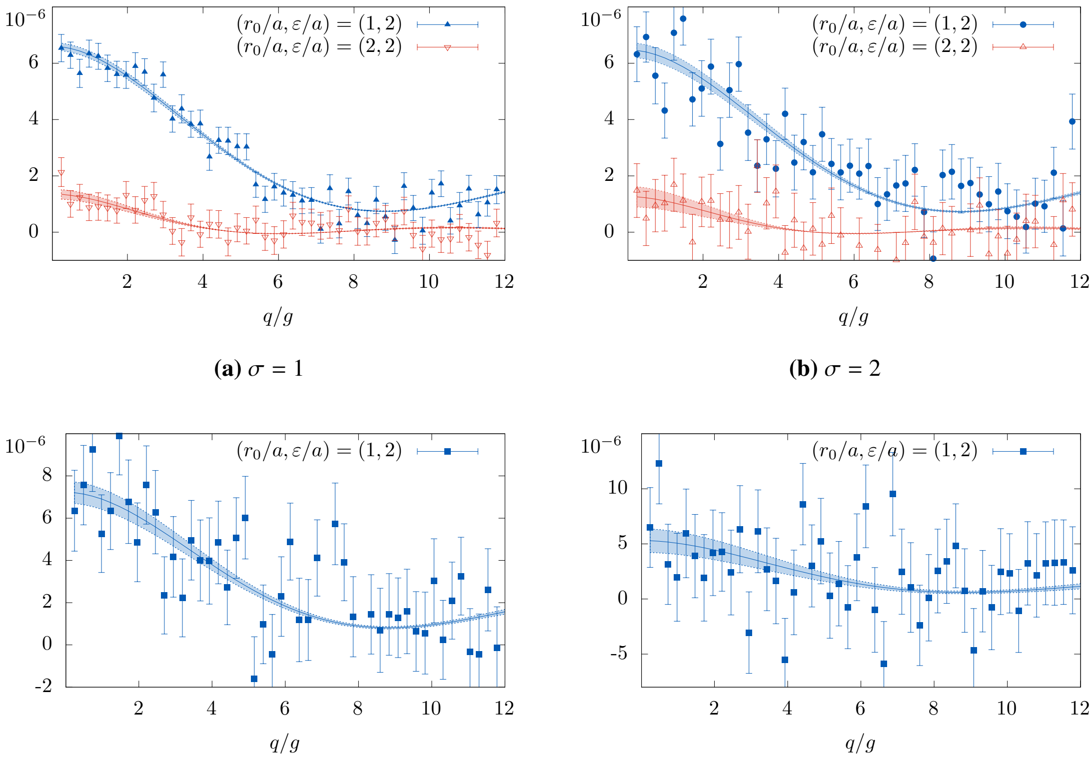
$$\frac{C(q)}{g^3} = \alpha_0 \left(\frac{\hat{q}}{g}\right)^3 + \frac{\beta_0}{ag}\left(\frac{\hat{q}}{g}\right)^2 + \frac{\gamma_0}{(ag)^3}, \tag{21}$$

where α_0, β_0, and γ_0 are parameters to be chosen. Only the first term on the right-hand side contains the relevant signal, as the other two are contact terms. To this generated momentum distribution, Gaussian noise with standard deviation σ is added. The resulting function is then windowed, and subsequently fitted against a windowed "pure" $\hat{q}^3$ distribution, with the intent of recovering the value of α_0. Some of those fits are shown in fig. 4.

The results of some such fits are given on table 1.

8. Conclusion

With the position-space method, we have managed to renormalise the EMT and obtain results that are compatible with those yielded by the Wilson Flow method. Furthermore, the method can also in principle get rid of contact term contributions in the two-point function to recover the continuum correlator parameters. We intend to show in future work that such a method can also



(a) $\sigma = 1$ **(b)** $\sigma = 2$

(c) $\sigma = 4$ **(d)** $\sigma = 8$

Figure 4: Fits against synthetic data with $\alpha_0 = \beta_0 = \gamma_0 = 0.01$ for increasing levels of added noise σ. The figures with $\sigma = 1$ and $\sigma = 2$ also show the fits for $(r_0/a, \epsilon/a) = (2, 2)$ since this window choice still contains some nonzero signal.

work on real lattice data and to apply it to other more complex theories, like one containing gauge fields alongside scalars. Thus, this method may pave the way for allowing tests of the predictions of Holographic Cosmology nonperturbatively.

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σ	r_0/a	ϵ/a		
		1	2	3
1	1	0.0100(1)	0.0097(2)	0.0103(4)
	2	0.0106(9)	0.0117(15)	0.0063(24)
	3	0.0095(33)	0.0126(49)	0.0048(75)
	4	-0.0001(87)	0.0143(121)	0.0208(158)
2	1	0.0103(2)	0.0096(4)	0.0105(8)
	2	0.0120(20)	0.0109(31)	0.0104(48)
	3	0.0141(65)	0.0020(106)	0.0195(143)
	4	0.0010(166)	-0.0057(243)	-0.0154(319)
4	1	0.0098(3)	0.0107(8)	0.0094(16)
	2	0.0142(38)	0.0072(62)	0.0142(97)
	3	0.0095(129)	-0.0176(196)	0.0448(286)
	4	-0.0133(345)	0.0078(458)	-0.1224(647)

Table 1: Some results of the best fit for α_0 for various windows and noise magnitudes. For sufficiently small windows, the fit accurately recovers the original value of $\alpha_0 = 0.01$.

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