PROCEEDINGS OF SCIENCE

c_{SW} at One-Loop Order for Brillouin Fermions

Maximilian Ammer[a,*] **and Stephan Dürr**[a,b]

[a] *Physics Department, University of Wuppertal, D-42119 Wuppertal, Germany*
[b] *IAS/JSC, Forschungszentrum Jülich, D-52425 Jülich, Germany*

E-mail: ammer(AT)uni-wuppertal.de

Wilson-like Dirac operators can be written in the form $D = \gamma_\mu \nabla_\mu - \frac{ar}{2}\Delta$. For Wilson fermions the standard two-point derivative $\nabla_\mu^{(\mathrm{std})}$ and 9-point Laplacian $\Delta^{(\mathrm{std})}$ are used. For Brillouin fermions these are replaced by improved discretizations $\nabla_\mu^{(\mathrm{iso})}$ and $\Delta^{(\mathrm{bri})}$ which have 54- and 81-point stencils respectively. We derive the Feynman rules in lattice perturbation theory for the Brillouin action and apply them to the calculation of the improvement coefficient c_{SW}, which, similar to the Wilson case, has a perturbative expansion of the form $c_{\mathrm{SW}} = 1 + {c_{\mathrm{SW}}}^{(1)} g_0^2 + O(g_0^4)$. For $N_c = 3$ we find ${c_{\mathrm{SW}}}^{(1)}_{\mathrm{Brillouin}} = 0.12362580(1)$, compared to ${c_{\mathrm{SW}}}^{(1)}_{\mathrm{Wilson}} = 0.26858825(1)$, both for $r = 1$.

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1. Introduction

The massless Wilson Dirac Operator can be expressed in terms of the standard two-point derivative ∇_μ^{std} and the standard 9-point Laplacian Δ^{std}

$$D_W(x,y) = \sum_\mu \gamma_\mu \nabla_\mu^{\text{std}}(x,y) - \frac{ar}{2}\Delta^{\text{std}}(x,y). \tag{1}$$

The massless Brillouin Dirac operator is obtained by replacing the derivative and Laplacian by different, more complicated discretizations called ∇_μ^{iso} and Δ^{bri} [1][2]

$$D_B(x,y) = \sum_\mu \gamma_\mu \nabla_\mu^{\text{iso}}(x,y) - \frac{ar}{2}\Delta^{\text{bri}}(x,y). \tag{2}$$

This way the Brillouin Dirac operator has an 81-point stencil containing all *off-axis* points that are 1-,2-,3-, and 4-hops away from x. The contributing points are weighted by the coefficients

$$(\rho_1,\rho_2,\rho_3,\rho_4) = \frac{1}{432}(64,16,4,1) \tag{3}$$

$$(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \frac{r}{64}(-240,8,4,2,1), \tag{4}$$

such that

$$\begin{aligned}
D_B(x,y) = &-\frac{\lambda_0}{2}\delta(x,y) \\
&+ \sum_{\mu=\pm1}^{\pm4}\left(\rho_1\gamma_\mu - \frac{\lambda_1}{2}\right) W_\mu(x)\delta(x+\hat\mu,y) \\
&+ \sum_{\substack{\mu,\nu=\pm1\\|\mu|\neq|\nu|}}^{\pm4}\left(\rho_2\gamma_\mu - \frac{\lambda_2}{4}\right) W_{\mu\nu}(x)\delta(x+\hat\mu+\hat\nu,y) \\
&+ \sum_{\substack{\mu,\nu,\rho=\pm1\\|\mu|\neq|\nu|\neq|\rho|}}^{\pm4}\left(\frac{\rho_3}{2}\gamma_\mu - \frac{\lambda_3}{12}\right) W_{\mu\nu\rho}(x)\delta(x+\hat\mu+\hat\nu+\hat\rho,y) \\
&+ \sum_{\substack{\mu,\nu,\rho,\sigma=\pm1\\|\mu|\neq|\nu|\neq|\rho|\neq|\sigma|}}^{\pm4}\left(\frac{\rho_4}{6}\gamma_\mu - \frac{\lambda_4}{48}\right) W_{\mu\nu\rho\sigma}(x)\delta(x+\hat\mu+\hat\nu+\hat\rho+\hat\sigma,y),
\end{aligned} \tag{5}$$

where $|\mu| \neq |\nu| \neq \ldots$ is used in a transitive way i.e. the sums are over indices with pairwise different absolute values. The Ws are the average of the products of the link variables U along the paths:

$$W_\mu(x) = U_\mu(x) \tag{6}$$

$$W_{\mu\nu}(x) = \frac{1}{2}\left(U_\mu(x)U_\nu(x+\hat\mu) + \text{perm}\right) \tag{7}$$

$$W_{\mu\nu\rho}(x) = \frac{1}{6}\left(U_\mu(x)U_\nu(x+\hat\mu)U_\rho(x+\hat\mu+\hat\nu) + \text{perms}\right) \tag{8}$$

$$W_{\mu\nu\rho\sigma}(x) = \frac{1}{24}\left(U_\mu(x)U_\nu(x+\hat\mu)U_\rho(x+\hat\mu+\hat\nu)U_\sigma(x+\hat\mu+\hat\nu+\hat\rho) + \text{perms}\right). \tag{9}$$

2. Feynman Rules of the Brillouin Action

We have derived the Feynman rules of the Brillouin action using a computer algebra system (Mathematica), supplemented by some analytical calculations by hand.

2.1 The Fermion Propagator

The Fourier transforms of the "free" derivative ∇_μ^{iso} and Laplace operator Δ^{bri} are [1]:

$$\nabla_\mu^{\text{iso}}(k) = \frac{i}{27}\sin(k_\mu)\prod_{\nu\neq\mu}(\cos(k_\nu)+2) \tag{10}$$

$$\Delta^{\text{bri}}(k) = 4\left(\cos^2(\tfrac{1}{2}k_1)\cos^2(\tfrac{1}{2}k_2)\cos^2(\tfrac{1}{2}k_3)\cos^2(\tfrac{1}{2}k_4)-1\right), \tag{11}$$

where k_μ is the fermion momentum in lattice units. Then the Fermion propagator is

$$S_{\text{Brillouin}}(k) = a\left(\sum_\mu \gamma_\mu \nabla_\mu^{\text{iso}}(k) - \frac{r}{2}\Delta^{\text{bri}}(k)\right)^{-1} = a\frac{-\sum_\mu \gamma_\mu \nabla_\mu^{\text{iso}}(k) - \frac{r}{2}\Delta^{\text{bri}}(k)}{\frac{r^2}{4}\Delta^{\text{bri}}(k)^2 - \left(\sum_\mu \nabla_\mu^{\text{iso}}(k)^2\right)}. \tag{12}$$

2.2 The Vertices

For the calculation of c_{SW} to one-loop order below the three vertices with one, two and three gluons coupling to a quark and anti-quark are needed (see Figure 1). We define the following notations to express the vertex Feynman rules:

$$s(k_\mu) = \sin\left(\tfrac{1}{2}k_\mu\right) \qquad c(k_\mu) = \cos\left(\tfrac{1}{2}k_\mu\right) \tag{13}$$

$$\bar{s}(k_\mu) = \sin\left(k_\mu\right) \qquad \bar{c}(k_\mu) = \cos\left(k_\mu\right) \tag{14}$$

$$K_{\mu\nu}^{(fg)}(p,q) = f(p_\mu+q_\mu)\left[\bar{g}(p_\nu)+\bar{g}(q_\nu)\right] \tag{15}$$

$$K_{\mu\nu\rho}^{(fgh)}(p,q) = f(p_\mu+q_\mu)\left\{\bar{g}(p_\nu)\bar{h}(p_\rho)+\bar{g}(q_\nu)\bar{h}(q_\rho)+[\bar{g}(p_\nu)+\bar{g}(q_\nu)][\bar{h}(p_\rho)+\bar{h}(q_\rho)]\right\} \tag{16}$$

$$\begin{aligned} K_{\mu\nu\rho\sigma}^{(fghj)}(p,q) = f(p_\mu+q_\mu)&\left\{2\left[\bar{g}(p_\nu)\bar{h}(p_\rho)\bar{j}(p_\sigma)+\bar{g}(q_\nu)\bar{h}(q_\rho)\bar{j}(q_\sigma)\right]\right.\\ &\left.+[\bar{g}(p_\nu)+\bar{g}(q_\nu)][\bar{h}(p_\rho)+\bar{h}(q_\rho)][\bar{j}(p_\sigma)+\bar{j}(q_\sigma)]\right\} \end{aligned} \tag{17}$$

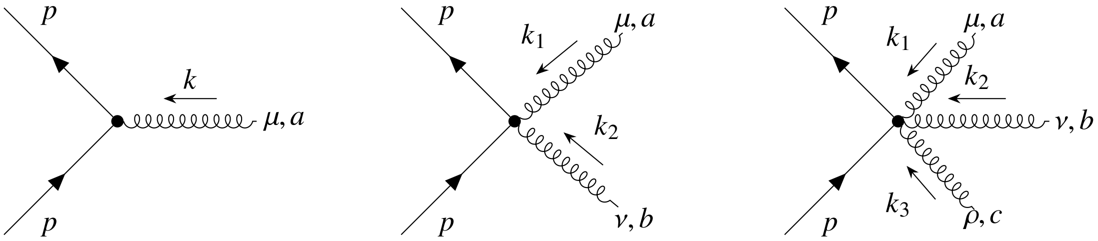

Figure 1: Momentum assignments for the vertices with one, two and three gluons.

$$
\begin{aligned}
L^{(fg)}_{\mu\nu}(p,q,k) &= f(p_\mu + q_\mu + k_\mu) g(2p_\nu + k_\nu) && (18)\\
L^{(fgh)}_{\mu\nu\rho}(p,q,k) &= f(p_\mu + q_\mu + k_\mu) g(2p_\nu + k_\nu) \left[\bar{h}(p_\rho) + \bar{h}(p_\rho + k_\rho) + \bar{h}(q_\rho)\right] && (19)\\
L^{(fghj)}_{\mu\nu\rho\sigma}(p,q,k) &= f(p_\mu + q_\mu + k_\mu) g(2p_\nu + k_\nu)\\
&\quad \times \left\{\bar{h}(p_\rho)\bar{j}(p_\sigma) + \bar{h}(p_\rho + k_\rho)\bar{j}(p_\sigma + k_\sigma) + \bar{h}(q_\rho)\bar{j}(q_\sigma)\right.\\
&\quad \left. + [\bar{h}(p_\rho) + \bar{h}(p_\rho + k_\rho) + \bar{h}(q_\rho)][\bar{j}(p_\sigma) + \bar{j}(p_\sigma + k_\sigma) + \bar{j}(q_\sigma)]\right\} && (20)
\end{aligned}
$$

$$
\begin{aligned}
M^{(fgh)}_{\mu\nu\rho}(p,q,k_1,k_2) &= f(p_\mu + q_\mu + k_{1\mu} + k_{2\mu}) g(2p_\nu + k_{1\nu} + 2k_{2\nu}) h(2p_\rho + k_{2\rho}) && (21)\\
M^{(fghj)}_{\mu\nu\rho\sigma}(p,q,k_1,k_2) &= f(p_\mu + q_\mu + k_{1\mu} + k_{2\mu}) g(2p_\nu + k_{1\nu} + 2k_{2\nu}) h(2p_\rho + k_{2\rho})\\
&\quad \times \left\{\bar{j}(p_\sigma) + \bar{j}(p_\sigma + k_{2\sigma}) + \bar{j}(p_\sigma + k_{1\sigma} + k_{2\sigma}) + \bar{j}(q_\sigma)\right\} && (22)
\end{aligned}
$$

with $f, g, h, j \in \{s, c\}$.

The $q\bar{q}g$-vertex:

$$
\begin{aligned}
V^a_{1\mu}(p,q) = -g_0 T^a \Bigg[& \lambda_1 s(p_\mu + q_\mu) + 2i\rho_1 c(p_\mu + q_\mu)\gamma_\mu\\
& + \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{4} \left\{\lambda_2 K^{(sc)}_{\mu\nu}(p,q) + 2i\rho_2 \left(K^{(cc)}_{\mu\nu}(p,q)\gamma_\mu - K^{(ss)}_{\mu\nu}(p,q)\gamma_\nu\right)\right\}\\
& + \frac{1}{3} \sum_{\substack{\nu,\rho=1\\ \neq(\nu,\rho;\mu)}}^{4} \left\{\lambda_3 K^{(scc)}_{\mu\nu\rho}(p,q) + 2i\rho_3 \left(K^{(ccc)}_{\mu\nu\rho}(p,q)\gamma_\mu - 2K^{(ssc)}_{\mu\nu\rho}(p,q)\gamma_\nu\right)\right\}\\
& + \frac{1}{9} \sum_{\substack{\nu,\rho,\sigma=1\\ \neq(\nu,\rho,\sigma;\mu)}}^{4} \left\{\lambda_4 K^{(sccc)}_{\mu\nu\rho\sigma}(p,q) + 2i\rho_4 \left(K^{(cccc)}_{\mu\nu\rho\sigma}(p,q)\gamma_\mu - 3K^{(sscc)}_{\mu\nu\rho\sigma}(p,q)\gamma_\nu\right)\right\} \Bigg] \quad (23)
\end{aligned}
$$

The $q\bar{q}gg$-vertex:

$$
\begin{aligned}
V^{ab}_{2\mu\nu}(p,q,k_1,k_2) = a g_0^2 T^a T^b \Bigg\{ & -\frac{1}{2}\lambda_1 \delta_{\mu\nu} c(p_\mu + q_\mu) + i\rho_1 \delta_{\mu\nu} s(p_\mu + q_\mu)\gamma_\mu\\
& + \lambda_2 \left((1-\delta_{\mu\nu}) L^{(ss)}_{\mu\nu}(p,q,k_2) - \frac{1}{2}\delta_{\mu\nu} \sum_{\substack{\alpha=1\\ \alpha\neq\mu}}^{4} K^{(cc)}_{\mu\alpha}(p,q) \right)\\
& + \lambda_3 \left(\frac{2}{3}(1-\delta_{\mu\nu}) \sum_{\substack{\rho=1\\ \neq(\rho,\mu,\nu)}}^{4} L^{(ssc)}_{\mu\nu\rho}(p,q,k_2) - \frac{1}{6}\delta_{\mu\nu} \sum_{\substack{\alpha,\rho=1\\ \neq(\alpha,\rho,\mu)}}^{4} K^{(ccc)}_{\mu\alpha\rho}(p,q) \right)\\
& + \lambda_4 \left(\frac{1}{6}(1-\delta_{\mu\nu}) \sum_{\substack{\rho,\sigma=1\\ \neq(\rho,\sigma,\mu,\nu)}}^{4} L^{(sscc)}_{\mu\nu\rho\sigma}(p,q,k_2) - \frac{1}{18}\delta_{\mu\nu} \sum_{\substack{\alpha,\rho,\sigma=1\\ \neq(\alpha,\rho,\sigma,\mu)}}^{4} K^{(cccc)}_{\mu\alpha\rho\sigma}(p,q) \right)
\end{aligned}
$$

$$
\begin{aligned}
&+i\rho_2\Bigg(2(1-\delta_{\mu\nu})\Big[L^{(cs)}_{\mu\nu}(p,q,k_2)\gamma_\mu + L^{(sc)}_{\mu\nu}(p,q,k_2)\gamma_\nu\Big]\\
&\qquad + \delta_{\mu\nu}\sum_{\substack{\alpha=1\\ \alpha\neq\mu}}^{4}\Big[K^{(sc)}_{\mu\alpha}(p,q)\gamma_\mu + K^{(cs)}_{\mu\alpha}(p,q)\gamma_\alpha\Big]\Bigg)\\
&+i\rho_3\Bigg(\frac{4}{3}(1-\delta_{\mu\nu})\sum_{\substack{\rho=1\\ \neq(\rho;\mu,\nu)}}^{4}\Big[L^{(csc)}_{\mu\nu\rho}(p,q,k_2)\gamma_\mu + L^{(scc)}_{\mu\nu\rho}(p,q,k_2)\gamma_\nu - L^{(sss)}_{\mu\nu\rho}(p,q,k_2)\gamma_\rho\Big]\\
&\qquad + \frac{1}{3}\delta_{\mu\nu}\sum_{\substack{\alpha,\rho=1\\ \neq(\alpha,\rho;\mu)}}^{4}\Big[K^{(scc)}_{\mu\alpha\rho}(p,q)\gamma_\mu + 2K^{(ccs)}_{\mu\alpha\rho}(p,q)\gamma_\rho\Big]\Bigg)\\
&+i\rho_4\Bigg(\frac{1}{3}(1-\delta_{\mu\nu})\sum_{\substack{\rho,\sigma=1\\ \neq(\rho,\sigma,\mu,\nu)}}^{4}\Big[L^{(cscc)}_{\mu\nu\rho\sigma}(p,q,k_2)\gamma_\mu + L^{(sccc)}_{\mu\nu\rho\sigma}(p,q,k_2)\gamma_\nu - 2L^{(sssc)}_{\mu\nu\rho\sigma}(p,q,k_2)\gamma_\rho\Big]\\
&\qquad + \frac{1}{9}\delta_{\mu\nu}\sum_{\substack{\alpha,\rho,\sigma=1\\ \neq(\alpha,\rho,\sigma,\mu)}}^{4}\Big[K^{(sccc)}_{\mu\alpha\rho\sigma}(p,q)\gamma_\mu + 3K^{(ccsc)}_{\mu\alpha\rho\sigma}(p,q)\gamma_\rho\Big]\Bigg)\Bigg\}
\end{aligned}
\tag{24}
$$

The $q\bar{q}ggg$-vertex:

$$
\begin{aligned}
&V^{abc}_{3\,\mu\nu\rho}(p,q,k_1,k_2,k_3) = a^2 g_0^3 T^a T^b T^c\Bigg\{\frac{1}{6}\delta_{\mu\nu}\delta_{\mu\rho}\big(\lambda_1 s(p_\mu+q_\mu) + 2i\rho_1 c(p_\mu+q_\mu)\gamma_\mu\big)\\
&+\lambda_2\Bigg(\frac{1}{6}\delta_{\mu\nu}\delta_{\mu\rho}\sum_{\substack{\alpha=1\\ \alpha\neq\mu}}^{4}K^{(sc)}_{\mu\alpha}(p,q)\\
&\quad + \frac{1}{2}\delta_{\mu\nu}(1-\delta_{\mu\rho})L^{(cs)}_{\mu\rho}(p,q,k_3) + \frac{1}{2}\delta_{\nu\rho}(1-\delta_{\mu\nu})L^{(sc)}_{\mu\nu}(p,q,k_2+k_3)\Bigg)\\
&+\lambda_3\Bigg(-\frac{2}{3}(1-\delta_{\mu\nu})(1-\delta_{\mu\rho})(1-\delta_{\nu\rho})M^{(sss)}_{\mu\nu\rho}(p,q,k_2,k_3) + \frac{1}{18}\delta_{\mu\nu}\delta_{\mu\rho}\sum_{\substack{\alpha,\beta=1\\ \neq(\alpha,\beta,\mu)}}^{4}K^{(scc)}_{\mu\alpha\beta}(p,q)\\
&\quad + \frac{1}{3}\delta_{\mu\nu}(1-\delta_{\mu\rho})\sum_{\substack{\alpha=1\\ \neq(\alpha,\mu,\rho)}}^{4}L^{(csc)}_{\mu\rho\alpha}(p,q,k_3) + \frac{1}{3}\delta_{\nu\rho}(1-\delta_{\mu\nu})\sum_{\substack{\alpha=1\\ \neq(\alpha,\mu,\nu)}}^{4}L^{(scc)}_{\mu\nu\alpha}(p,q,k_2+k_3)\Bigg)\\
&+\lambda_4\Bigg(-\frac{1}{3}(1-\delta_{\mu\nu})(1-\delta_{\mu\rho})(1-\delta_{\nu\rho})\sum_{\substack{\sigma=1\\ \neq(\sigma,\mu,\nu,\rho)}}^{4}M^{(sssc)}_{\mu\nu\rho\sigma}(p,q,k_2,k_3)\\
&\quad + \frac{1}{54}\delta_{\mu\nu}\delta_{\mu\rho}\sum_{\substack{\alpha,\beta,\sigma=1\\ \neq(\alpha,\beta,\sigma,\mu)}}^{4}K^{(sccc)}_{\mu\alpha\beta\sigma}(p,q)\\
&\quad + \frac{1}{12}\delta_{\mu\nu}(1-\delta_{\nu\rho})\sum_{\substack{\alpha,\sigma=1\\ \neq(\alpha,\sigma;\mu,\rho)}}^{4}L^{(cscc)}_{\mu\rho\alpha\sigma}(p,q,k_3) + \frac{1}{12}\delta_{\nu\rho}(1-\delta_{\mu\nu})\sum_{\substack{\alpha,\sigma=1\\ \neq(\alpha,\sigma;\mu,\nu)}}^{4}L^{(sccc)}_{\mu\nu\alpha\sigma}(p,q,k_2+k_3)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+i\rho_2\Bigg(\frac{1}{3}\delta_{\mu\nu}\delta_{\mu\rho}\sum_{\substack{\alpha=1\\ \alpha\neq\mu}}^{4}\Big[K^{(cc)}_{\mu\alpha}(p,q)\gamma_\mu-K^{(ss)}_{\mu\alpha}(p,q)\gamma_\alpha\Big]\\
&\qquad+\delta_{\mu\nu}(1-\delta_{\mu\rho})\Big[L^{(cc)}_{\mu\rho}(p,q,k_3)\gamma_\rho-L^{(ss)}_{\mu\rho}(p,q,k_3)\gamma_\mu\Big]\\
&\qquad+\delta_{\nu\rho}(1-\delta_{\mu\nu})\Big[L^{(cc)}_{\mu\rho}(p,q,k_2+k_3)\gamma_\mu-L^{(ss)}_{\mu\rho}(p,q,k_2+k_3)\gamma_\nu\Big]\Bigg)\\
&+i\rho_3\Bigg(-\frac{4}{3}(1-\delta_{\mu\nu})(1-\delta_{\mu\rho})(1-\delta_{\nu\rho})\\
&\qquad\times\Big[M^{(css)}_{\mu\nu\rho}(p,q,k_2,k_3)\gamma_\mu+M^{(scs)}_{\mu\nu\rho}(p,q,k_2,k_3)\gamma_\nu+M^{(ssc)}_{\mu\nu\rho}(p,q,k_2,k_3)\gamma_\rho\Big]\\
&\qquad+\frac{1}{9}\delta_{\mu\nu}\delta_{\mu\rho}\sum_{\substack{\alpha,\beta=1\\ \neq(\alpha,\beta,\mu)}}^{4}\Big[K^{(ccc)}_{\mu\alpha\beta}(p,q)\gamma_\mu-K^{(ssc)}_{\mu\alpha\beta}(p,q)\gamma_\alpha-K^{(scs)}_{\mu\alpha\beta}(p,q)\gamma_\beta\Big]\\
&\qquad-\frac{2}{3}\delta_{\mu\nu}(1-\delta_{\mu\rho})\sum_{\substack{\alpha=1\\ \neq(\alpha,\mu,\rho)}}^{4}\Big[L^{(ssc)}_{\mu\rho\alpha}(p,q,k_3)\gamma_\mu-L^{(ccc)}_{\mu\rho\alpha}(p,q,k_3)\gamma_\rho+L^{(css)}_{\mu\rho\alpha}(p,q,k_3)\gamma_\alpha\Big]\\
&\qquad-\frac{2}{3}\delta_{\nu\rho}(1-\delta_{\mu\nu})\sum_{\substack{\alpha=1\\ \neq(\alpha,\mu,\nu)}}^{4}\Big[-L^{(ccc)}_{\mu\nu\alpha}(p,q,k_2+k_3)\gamma_\mu+L^{(ssc)}_{\mu\nu\alpha}(p,q,k_2+k_3)\gamma_\nu\\
&\qquad\qquad+L^{(scs)}_{\mu\nu\alpha}(p,q,k_2+k_3)\gamma_\alpha\Big]\Bigg)\\
&+i\rho_4\Bigg(-\frac{2}{3}(1-\delta_{\mu\nu})(1-\delta_{\mu\rho})(1-\delta_{\nu\rho})\sum_{\substack{\sigma=1\\ \neq(\sigma,\mu,\nu,\rho)}}^{4}\Big[M^{(cssc)}_{\mu\nu\rho\sigma}(p,q,k_2,k_3)\gamma_\mu+M^{(scsc)}_{\mu\nu\rho\sigma}(p,q,k_2,k_3)\gamma_\nu\\
&\qquad\qquad+M^{(sscc)}_{\mu\nu\rho\sigma}(p,q,k_2,k_3)\gamma_\rho-M^{(ssss)}_{\mu\nu\rho\sigma}(p,q,k_2,k_3)\gamma_\sigma\Big]\\
&\qquad+\frac{1}{27}\delta_{\mu\nu}\delta_{\mu\rho}\sum_{\substack{\alpha,\beta,\sigma=1\\ \neq(\alpha,\beta,\sigma,\mu)}}^{4}\Big[K^{(cccc)}_{\mu\alpha\beta\sigma}(p,q)\gamma_\mu-K^{(sscc)}_{\mu\alpha\beta\sigma}(p,q)\gamma_\alpha\\
&\qquad\qquad-K^{(scsc)}_{\mu\alpha\beta\sigma}(p,q)\gamma_\beta-K^{(sccs)}_{\mu\alpha\beta\sigma}(p,q)\gamma_\sigma\Big]\\
&\qquad-\frac{1}{6}\delta_{\mu\nu}(1-\delta_{\mu\rho})\sum_{\substack{\alpha,\sigma=1\\ \neq(\alpha,\sigma,\mu,\rho)}}^{4}\Big[L^{(sscc)}_{\mu\rho\alpha\sigma}(p,q,k_3)\gamma_\mu-L^{(cccc)}_{\mu\rho\alpha\sigma}(p,q,k_3)\gamma_\rho\\
&\qquad\qquad+L^{(cssc)}_{\mu\rho\alpha\sigma}(p,q,k_3)\gamma_\alpha+L^{(cscs)}_{\mu\rho\alpha\sigma}(p,q,k_3)\gamma_\sigma\Big]\\
&\qquad+\frac{1}{6}\delta_{\nu\rho}(1-\delta_{\mu\nu})\sum_{\substack{\alpha,\sigma=1\\ \neq(\alpha,\sigma,\mu,\nu)}}^{4}\Big[L^{(cccc)}_{\mu\nu\alpha\sigma}(p,q,k_2+k_3)\gamma_\mu-L^{(sscc)}_{\mu\nu\alpha\sigma}(p,q,k_2+k_3)\gamma_\nu\\
&\qquad\qquad+L^{(scsc)}_{\mu\nu\alpha\sigma}(p,q,k_2+k_3)\gamma_\alpha+L^{(sccs)}_{\mu\nu\alpha\sigma}(p,q,k_2+k_3)\gamma_\sigma\Big]\Bigg)\Bigg\}
\end{aligned}
\tag{25}
$$

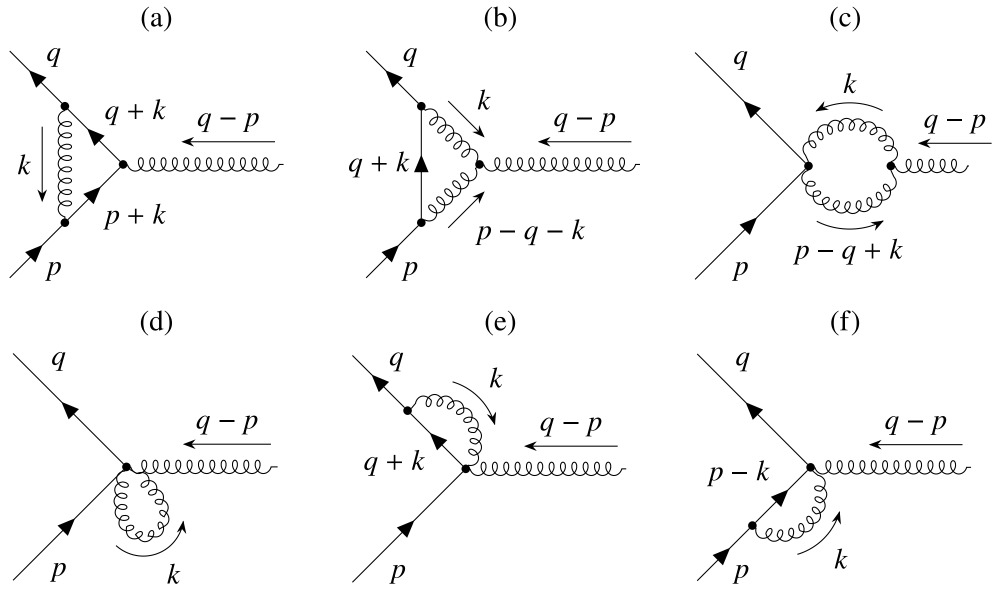

Figure 2: The six one-loop diagrams contributing to the vertex function in lattice perturbation theory.

Purely gluonic Feynman rules (gluon propagator and ggg-vertex) as well as contributions to the vertex Feynman rules coming from the clover term containing the improvement coefficient c_{SW} are unaffected by the fermion formulations and thus identical to the Wilson case (see Ref. [3]).

3. Perturbative Determination of c_{SW}

Adding the usual clover-term to the Brillouin action we obtain the $O(a)$-improved Brillouin action (with chromo-hermitean field strength $F_{\mu\nu}$ for fixed $\mu < \nu$):

$$\mathcal{S}^{\text{Clover}}_{\text{Brillouin}}[\overline{\psi},\psi] = \mathcal{S}_{\text{Brillouin}}[\overline{\psi},\psi] + c_{SW} \cdot \sum_x \sum_{\mu<\nu} \bar{\psi}(x)\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}(x)\psi(x). \tag{26}$$

The improvement coefficient has a perturbative expansion $c_{SW} = {c_{SW}}^{(0)} + g_0^2 {c_{SW}}^{(1)} + O(g_0^4)$ and at tree level in both the Wilson and the Brillouin case:

$${c_{SW}}^{(0)} = 1(= r) \tag{27}$$

At one-loop level ${c_{SW}}^{(1)}$ is calculated from the one-loop vertex function $\Lambda_\mu^{a(1)}$ comprised of six one-loop Feynman diagrams (see Figure 2). The sum of all diagrams results in a finite integral, while diagrams (a),(b),(c),(e), and (f) on their own lead to IR-divergent integrals. To see the cancellation of the divergencies explicitly, we used a small gluon mass μ as a regulator (following Ref. [3]) and split off an analytically solvable divergent integral from the diagrams. Table 1 shows the results for each diagram for $N_c = 3$ and $r = 1$, comparing the Wilson and the Brillouin actions.

Diagram	Divergent part	Constant part Wilson	Constant part Brillouin
(a)	$-L/3$	0.004569626(1)	0.0047576939(1)
(b)	$-9L/2$	0.083078349(1)	0.055554134(1)
(c)	$+9L/2$	−0.081307544(1)	−0.057741323(1)
(d)	0	0.297394534(1)	0.142461144(1)
(e)	$L/6$	−0.017573359(1)	−0.010702925(1)
(f)	$L/6$	−0.017573359(1)	−0.010702925(1)
Sum	0	0.26858825(1)	0.12362580(1)

Table 1: Divergent and constant contributions from each diagram for $N_c = 3$ and $r = 1$. The logarithmic divergence is encoded in $L := \frac{1}{16\pi^2} \ln\left(\frac{\pi^2}{\mu^2}\right)$.

The sum of all diagrams finally results in the following value

$$c_{\mathrm{SW}}{}^{(1)}_{\mathrm{Brillouin}} = 0.045785517(3)N_c - 0.041192255(3)\frac{1}{N_c} = 0.12362580(1), \tag{28}$$

where we have set $N_c = 3$ in the last step. Compare to

$$c_{\mathrm{SW}}{}^{(1)}_{\mathrm{Wilson}} = 0.0988424712(4)N_c - 0.083817496(3)\frac{1}{N_c} = 0.26858825(1) \tag{29}$$

from Ref. [3], one sees that $c_{\mathrm{SW}}{}^{(1)}_{\mathrm{Brillouin}}$ is about half of $c_{\mathrm{SW}}{}^{(1)}_{\mathrm{Wilson}}$ for any value of N_c.

References

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