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Intermediate window observable for the muon g-2 from overlap valence quarks on staggered ensembles

A.Yu. Kotov[a,*] **for the Budapest-Marseille-Wuppertal Collaboration**

[a] *Jülich Supercomputing Centre, Forschungszentrum Jülich, 52428 Jülich, Germany*

E-mail: a.kotov@fz-juelich.de

The Budapest-Marseille-Wuppertal collaboration computed the leading hadronic vacuum polarization contribution to the anomalous muon magnetic moment with unprecedented accuracy on the lattice. The result was obtained using staggered fermions. Here we present an improved cross-check of the staggered result for the intermediate window observable using a mixed action setup: overlap valence quarks on staggered sea ensembles. We focus on the light connected contribution. Details of the overlap fermion formulation and of the methods used for the measurements of the hadronic vacuum polarization are described. We present first results for two different setups on lattices with a spatial extent of 3 fm.

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*Speaker

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a [fm]	L	T	# conf
0.1315	24	48	716
0.1116	28	56	887
0.0952	32	64	1110
0.0787	40	80	923
0.0640	48	96	577

Table 1: Summary of ensembles used in the current study.

1. Introduction

Possible discrepancy between theoretical and experimental determination of the muon anomalous magnetic moment would indicate the existence of new physics and is one of the most actively discussed topics nowadays [1]. Theoretical estimations based on the White Paper [2] are in 4.2 σ difference with the combined experimental results obtained recently at Fermilab [3] and previously at BNL[4]. On the other side, lattice calculation of the Budapest-Marseille-Wuppertal collaboration [5] for the leading order Hadron Vacuum Polarization (HVP) contribution to the anomalous muon magnetic moment significantly reduces tension between theory and experiment, making the current status of the muon $g-2$ problem more puzzling. In this proceeding we discuss the crosscheck of the results of [5], which used staggered fermion discretization, with the help of the overlap fermions, which possess exact chiral symmetry on the lattice.

2. Details of numerical calculations

Accurate determination of the HVP contribution to the anomalous magnetic moment requires careful consideration of a plenty of systematic uncertainties [6]. Analogous calculation with the help of the overlap fermions requires huge computer resources and is beyond currently available supercomputer resources. Instead, for the cross-check we concentrate only on one contribution to the anomalous magnetic moment a_μ: light connected intermediate window observable $a_\mu^{l,\mathrm{win}}$. Comparison is performed in the isospin symmetric point. We also restrict our study to the lattices with spatial volume $L \approx 3$ fm and we compare the continuum extrapolation obtained with staggered and overlap fermions. A short summary of the ensembles, used in the current analysis is presented in Tab. 1.

Ensembles were generated with $N_f = 2+1+1$ staggered 4-stout action, which is used in the analysis of [5]. For the crosscheck of staggered discretization we used overlap fermions [7] with 2 steps of HEX smearing [8] as valence quarks[1]:

$$\begin{aligned} D_{\mathrm{ov}} &= \frac{1}{2}(1 + \gamma_5 \operatorname{sgn}(\gamma_5 D_{\mathrm{w}}(-m_{\mathrm{w}}))) \\ D_{\mathrm{ov}}(m_{\mathrm{ov}}) &= \left(1 - \frac{m_{\mathrm{ov}}}{2m_{\mathrm{w}}}\right) D_{\mathrm{ov}} + \frac{m_{\mathrm{ov}}}{2m_{\mathrm{w}}} \end{aligned} \tag{1}$$

[1]For simplicity we omit here and later lattice spacing a.

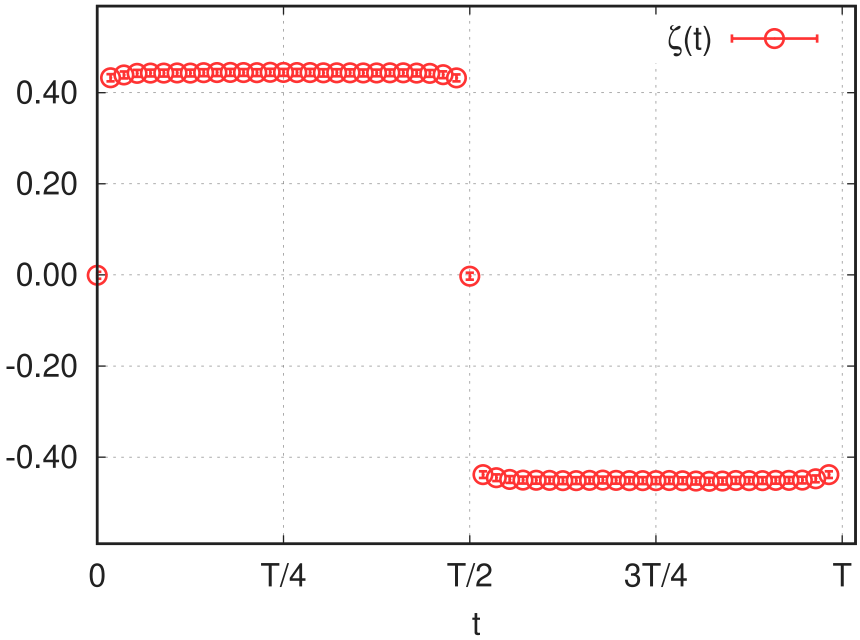

Figure 1: Ratio of three-point to two-point correlation functions $\zeta(t)$ from Eq. (3) as a function of time t. Lattice spacing $a = 0.1116$ fm, temporal lattice extent $T = 56$.

where $D_{\rm w}(-m_{\rm w})$ is the Wilson Dirac operator with large negative mass $-m_{\rm w}$. In this study we used $m_{\rm w} = 1.3$. For the overlap Dirac propagator $D_{\rm ov}$ we applied $O(a)$ improvement:

$$D_{\rm ov}^{-1}(m_{\rm ov}) \to (1 - D_{\rm ov})\, D_{\rm ov}^{-1}(m_{\rm ov}). \tag{2}$$

Mixed action setup requires tuning of the parameters of the valence quarks to match the sea quarks. In order to tune the quark mass of the overlap fermions we used pion mass, determined in both discretizations. We used two different prescriptions, matching overlap pion mass either to the Goldstone boson pion mass (GB), or to the root mean squared pion mass (RMS) determined with staggered fermions. We initially performed simulations with 4 values of overlap quark mass: $m_{\rm ov} = 0.002, 0.005, 0.010, 0.020$ and interpolated the overlap pion mass using the formula: $m_{\pi,\rm ov} = A m_{\rm ov}^B + C m_{\rm ov}^2$ with free parameters A, B and C. This expression is able to capture partial quenching effects [9].

For the measurements of the hadron vacuum polarization tensor with overlap fermions we used the local current definition. As a consequence, the current has to be renormalized. Renormalization factor Z_V is determined by measuring the pion electric charge. To this end we computed the following ratio of three- and two- point correlation functions:

$$\zeta(t) = \frac{\langle P(T/2) V_4(t) \bar{P}(0)\rangle}{\langle P(T/2)\bar{P}(0)\rangle}, \tag{3}$$

where $P(t) = \sum_{\vec{x}} (\bar{\psi}_2 \gamma_5 \psi_1)(\vec{x}, t)$, $\bar{P}(t) = \sum_{\vec{x}} (\bar{\psi}_1 \gamma_5 \psi_2)(\vec{x}, t)$ is the pseudoscalar density, $V_\mu(t) = \sum_{\vec{x}} (\bar{\psi}_1 \gamma_\mu \psi_1)(\vec{x}, t)$ is the local vector current expressed in terms of valence overlap quark fields ψ_1 and ψ_2. In the case of the conserved current $\zeta(t)$ equals to $1/2$ for $0 < t < T/2$ and is $-1/2$ for

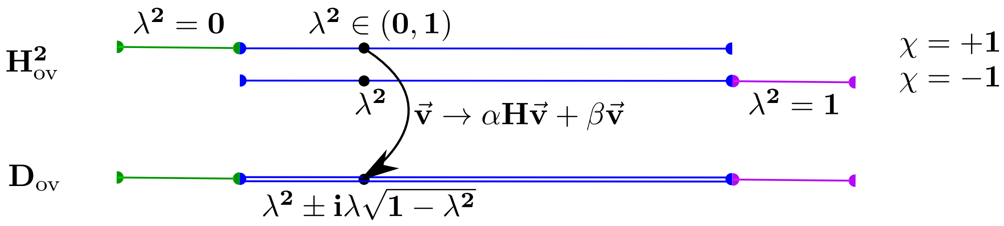

Figure 2: Schematic visualization of eigenpairs of the overlap operator D_{ov} and $H^2_{\text{ov}} = D^\dagger_{\text{ov}} D_{\text{ov}}$ ($H = \gamma_5 D_{\text{ov}}$).

$T/2 < t < T$. For other current discretizations we can determine the renormalization factor Z_V from matching $\zeta(t)$ to $\pm 1/2$ at some physical distance. In Fig. 1 we present the typical dependence of $\zeta(t)$ on the position t of the overlap local vector current insertion, determined on the ensemble with lattice spacing $a = 0.1116$ fm and lattice size $28^3 \times 56$. In practice we used the following symmetrized combination to determine Z_V: $Z_V = [\zeta(T/4) - \zeta(3T/4)]^{-1}$.

3. Low-mode averaging and overlap fermions

One way to improve the signal-to-noise ratio in lattice calculations of fermion observables is low-mode averaging (LMA) [5, 10]. Below we summarize the details of our implelentation of the LMA for the overlap Dirac operator[2]. The standard way to utilize LMA is based on the separation of the propagator D^{-1} into two parts:

$$D^{-1} = D_e^{-1} + D_r^{-1}, \tag{4}$$

where D_e^{-1} corresponds to the inversion of the Dirac operator on the low modes $|\lambda_i\rangle$ of the Dirac operator, $D|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$, as $D_e^{-1} = \frac{1}{\lambda_i}|\lambda_i\rangle\langle\lambda_i|$, and D_r^{-1} correspond to the rest part, $D_r^{-1} = D^{-1}(1 - |\lambda_i\rangle\langle\lambda_i|)$. In this case, the two point correlation function C can be separated into three terms:

$$C = C_{\text{ee}} + C_{\text{re}} + C_{\text{rr}}, \tag{5}$$

and eigen-eigen part C_{ee} can be calculated exactly, leading to significant noise reduction.

To calculate the low modes $|\lambda_i\rangle$ of the Dirac operator D, we first calculate the low modes of the Hermitian operator $H^2 = D^\dagger D$, where $H = \gamma_5 D$. Note that, for overlap Dirac fermions, the operator H^2 commutes with γ_5. Thus, except for zero-modes (and large eigenvalues $\lambda = 1$ which are irrelevant for LMA), eigenvectors come in degenerate pairs of opposite chirality [11]. Zero modes can be only of one chirality, which corresponds to the topological charge determined from the index of the overlap operator. If λ_i^2, $|\lambda_i\rangle$ is the eigenpair of $D^\dagger D$ of chirality $\chi = \pm 1$, then it corresponds to the eigenpairs $\bar\lambda_i$, $|\bar\lambda_i\rangle$ of the Dirac operator D itself [11]:

$$\begin{aligned} \bar\lambda_i = 0, |\bar\lambda_i\rangle = |\lambda_i\rangle, \text{ if } \lambda_i = 0, \\ \bar\lambda_i = \lambda_i^2 \pm i\lambda_i\sqrt{1-\lambda_i^2}, \quad |\bar\lambda_i\rangle \sim H|\lambda_i\rangle - \chi\bar\lambda_i|\lambda_i\rangle, \text{ if } \lambda_i \neq 0, \end{aligned} \tag{6}$$

[2]Note that for simplicity in this section we write all expressions for the massless overlap Dirac operator $m_{\text{ov}} = 0$. Extension to the massive case $m_{\text{ov}} \neq 0$ is straightforward.

	$t_0/2$	t_0	$2t_0$
Wilson	0.75	0.66	0.62
Symanzik	0.71	0.62	0.62
Iwasaki	0.80	0.71	0.76
DBW2	0.60	0.60	0.60

Table 2: Typical correlation between the sign of the topological charge determined by the index of the overlap Dirac operator and with the help of the Gradient Flow with different gauge actions (Wilson, Symanzik, Iwasaki and DBW2) and at various GF times, given in units of t_0 scale. Lattice spacing is $a = 0.0787$ fm, lattise size is $40^3 \times 80$. The correlation was determined on a small subset of 48 configurations.

Note that each zero mode of $D^\dagger D$ corresponds to one zero mode of D, which is of the same chirality, while every non-zero (and $\lambda_i \neq 1$) mode of $D^\dagger D$ give two modes of D. In Fig. 2 we present the schematic visualization of the eigenpairs of D and H^2. In order to determine low modes of the operator D one needs to know the low modes of $D^\dagger D$ of the chirality, that has zero modes, if there are any. Note also that for overlap Dirac operator, if we work in one chirality sector $\chi = \pm 1$ the application of the $D^\dagger D$ on a given vector can be rewritten as $D^\dagger D|v\rangle = \frac{1}{4}\left(2 + (\gamma_5 + \chi)\,\mathrm{sgn}(\gamma_5 D_\mathrm{w})\right)|v\rangle$, which contains only one application of the Wilson Dirac sign function, thus making it just as fast as a single application of D.

If one knew from the beginning which chirality sector contains zero modes of the overlap Dirac operator D, then one could calculate eigenpairs only in this sector. Otherwise one has to choose one sector and check afterwards for zero modes. In the case that it does not contain zero modes, one should recalculate the eigenpairs in the opposite sector. To decrease the required computer time one needs to make a good guess of the sign of the topological charge, which is based on the index of the overlap operator. A good proxy for this quantity could be another definition of the topological charge. We have found that the Gradient-Flow based definition of the topological charge provides a good approximation to this quantity, being also a very fast procedure, in comparison to the overlap eigenvalues determination. We have checked several variants of the Gradient Flow(GF), based on different gauge actions, used for the smoothing procedure. In Tab. 2 we present typical correlation between sign of the topological charge determined from the GF with different gauge actions and topological charge definition based on the index of the overlap Dirac operator. We have found, that the largest correlation is given by the GF with Iwasaki gauge action and at a GF time $t = t_0/2$, where t_0 is the standard GF scale [12]. Note that a somewhat similar conclusion about the Iwasaki gauge action was found in [13]. We would like to stress here that the exact matching of this procedure with the overlap topological charge or its sign is not needed. After the initial guess and the calculation of the overlap modes, we check that the selected chirality sector contains zero modes and if it does not, we recalculate the low modes in the other chirality sector.

4. Results

In our calculations, we typically compute 512 low modes of the H^2 operator, which correspond to $1024 - n_\mathrm{zero}$ modes of the overlap Dirac operator D, where n_zero is the number of zero modes. For the calculation of the rest-rest and the rest-eigen part we used 64 random sources.

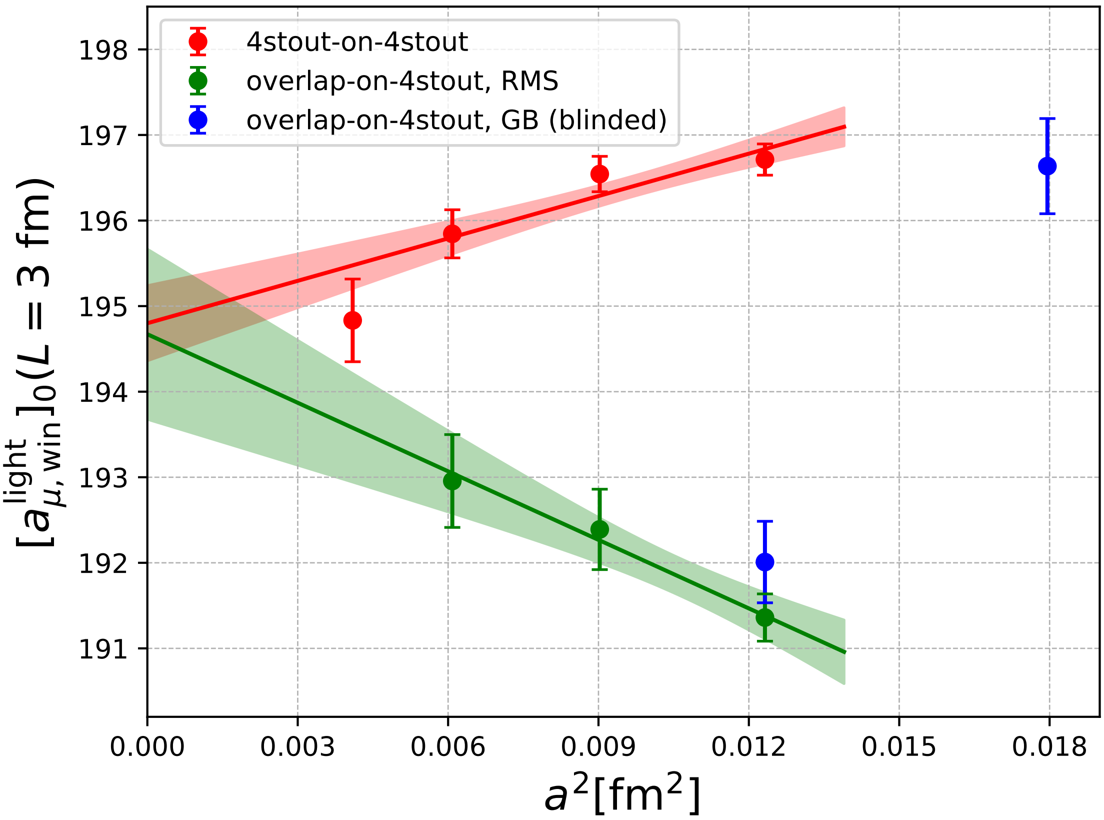

Figure 3: Continuum extrapolation of the light intermediate window observables determined on $L \approx 3$ fm ensembles with staggered fermions(red) and in a mixed overlap-staggered action setup with RMS pion mass matching(green) and GB pion mass matching(blue). The data for the GB pion mass matching are blinded by a factor α close to 1.

In Fig.3 we present the continuum extrapolation of the light connected intermediate window observable $a_\mu^{l,\mathrm{win}}$ determined on our $L \approx 3$ fm ensembles with the staggered fermions and overlap fermions. For the RMS pion mass matching the results correspond to the previous results [5] (with some increase in statistics). We also present first preliminary results obtained with GB pion mass matching for two coarse lattice spacings. These new data are blinded by a random factor α, which is close to 1. For the RMS pion matching the continuum extrapolation of the window observable with a simple $\sim a^2$ cutoff dependence describe the lattice data well and results for the staggered and overlap fermions are in agreement. The data for the overlap fermions with the GB pion matching are in the same ballpark as staggered and overlap results with RMS matching. However, the precise continuum extrapolation cannot be performed yet and is planned in the future, with the overlap results on finer lattices.

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