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Calculating the QED correction to the hadronic vacuum polarisation on the lattice

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Isospin-breaking corrections to the hadron vacuum polarization component of the anomalous magnetic moment of the muon are needed to ensure the theoretical precision of $g_\mu - 2$ is below the experimental precision. We describe the status of our work calculating, using lattice QCD, the QED correction to the light and strange connected hadronic vacuum polarization in a Dashen scheme. We report results using physical $N_f = 2 + 1 + 1$ HISQ ensembles at three lattice spacings and three heavier-than-light valence quark masses.

The 39th International Symposium on Lattice Field Theory (Lattice2022),
8-13 August, 2022
Bonn, Germany

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1. Introduction

The experimental results from the Fermilab Muon g-2 experiment [1] and E821 experiment [2], for the muon anomalous magnetic moment, motivates reducing the errors on lattice QCD calculations of the leading order hadronic vacuum polarization contribution to the muon anomalous magnetic moment ($a_\mu^{\mathrm{HVP,LO}}$). There is a comprehensive review [3] of the theoretical calculations of $a_\mu^{\mathrm{HVP,LO}}$.

This project is part of the Fermilab Lattice, HPQCD and MILC collaboration's [4–9] work on computing $a_\mu^{\mathrm{HVP,LO}}$. Reducing the theoretical uncertainty of $a_\mu^{\mathrm{HVP,LO}}$ below 1% requires the inclusion of isospin breaking effects. These arise from the up and down quarks unequal masses, $m_u \neq m_d$, and their unequal electric charges, $Q_u = 2e/3 \neq Q_d = -e/3$. This project aims to calculate the QED isospin breaking correction to the light and strange connected $a_\mu^{\mathrm{HVP,LO}}$.

We use the following definition for the QED correction to the $a_\mu^{\mathrm{HVP,LO}}$, δa_μ^f,

$$\delta a_\mu^{(f)} \equiv a_\mu^f(m_f, Q_f) - a_\mu^f(m_f, 0), \tag{1}$$

where f labels the quark flavour and the difference is evaluated at equal renormalised quark mass. The QED correction to the connected strange $a_\mu^{\mathrm{HVP,LO}}$is then

$$\delta a_\mu^{(s)} = a_\mu^s(m_s, -1/3e) - a_\mu^s(m_s, 0), \tag{2}$$

with corresponding formulas for the up and down quarks. The QED correction to the light connected $a_\mu^{\mathrm{HVP,LO}}$is the sum of the corrections for the up and down quarks,

$$\delta a_\mu^{(l)} = \delta a_\mu^{(u)} + \delta a_\mu^{(d)}. \tag{3}$$

We originally extract QED corrections to a_μ at fixed bare quark mass (Δa_μ) and then convert to δa_μ using

$$\delta a_\mu = \Delta a_\mu - \delta m_q \frac{\partial a_\mu}{\partial m_q} \tag{4}$$

A Dashen-like scheme is used to set the renormalized quark mass following [10, 11].

2. Simulation Details

We measured correlators on gauge field ensembles generated with the Highly Improved Staggered Quark (HISQ) action [12] with 2+1+1 flavours of dynamical sea quarks and physical pion masses. In order to take the continuum limit we took measurements on three ensembles with lattice spacings of approximately 0.15, 0.12, and 0.09 fm. The HISQ ensembles were generated by the MILC collaboration [13, 14]. The basic parameters of the ensembles are in Table 1. The lattice spacing is fixed using the Wilson flow parameter w_0=0.1715(9) fm [15].

name	L^3xT	w_0/a	$M_\pi L$	M_π (MeV)	N_{cfg}
very coarse	32^3x48	1.13215(35)	3.30	134.73(71)	1844
coarse	48^3x64	1.41490(60)	3.88	132.73(70)	967
fine	64^3x96	1.95180(70)	3.66	128.34(68)	596

Table 1: Properties of the gauge field ensembles used in the measurements. The lattice spacings, w_0/a, and pion masses are from [7].

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We use the electro-quenched approximation [10, 16, 17] to partially include the dynamics of QED. The quenched QED fields were fixed to the Feynman gauge with the QED$_L$ prescription [18]. We measure pseudoscalar and vector correlators with equal mass, oppositely charged quarks and antiquarks, so that all the mesons are neutral. The code first reads in a dynamical SU(3) gauge configuration and a quenched U(1) gauge configuration before multiplying the U(1) link fields into the SU(3) link fields, and gauge smearing as usual.

We use stochastic-wall sources projected onto the appropriate spin-taste quantum numbers. For the vector current we use the $\gamma_i \otimes \gamma_i$ operators, and for the pseudoscalar current we use the $\gamma_5 \otimes \gamma_5$ operator. The local vector current is not conserved, therefor it requires renormalising with a renormalisation factor Z_V. The required Z_V, including the QED corrections, have been calculated by HPQCD [17, 19]. To remove potential subjective bias, we do a “blinded analysis”. The blinding is done by multiplying the correlators on each ensemble by a random hidden number in [0.95,1.05].

Following BMW [20], in order to avoid the increased statistical noise incurred by simulating at the light quark mass, $m_l \equiv 1/2(m_u + m_d)$, we measure with valence quarks at multiples of m_l. We measure at $3m_l$, $5m_l$, and $7m_l$ and the strange quark mass, m_s, on each ensemble. We use a multi-shift solver so that for each charge all the masses can be solved in a single iterative process. As it is not prohibitively expensive we also measure at the physical m_u and m_d on the very coarse ensemble. The correlators measured, including the quark masses and electric charges, are summarised in Table 2.

name	Charges (e)	Quark masses (am_q)	sources
very coarse	$\pm 2/3, \pm 1/3, 0$	0.001524, 0.003328, 0.007278, 0.01213, 0.01698, 0.0677	16
coarse	$\pm 2/3, \pm 1/3, 0$	0.00552, 0.0092, 0.01288, 0.0527	16
fine	$\pm 2/3, \pm 1/3, 0$	0.0036, 0.006, 0.0084, 0.0364	16

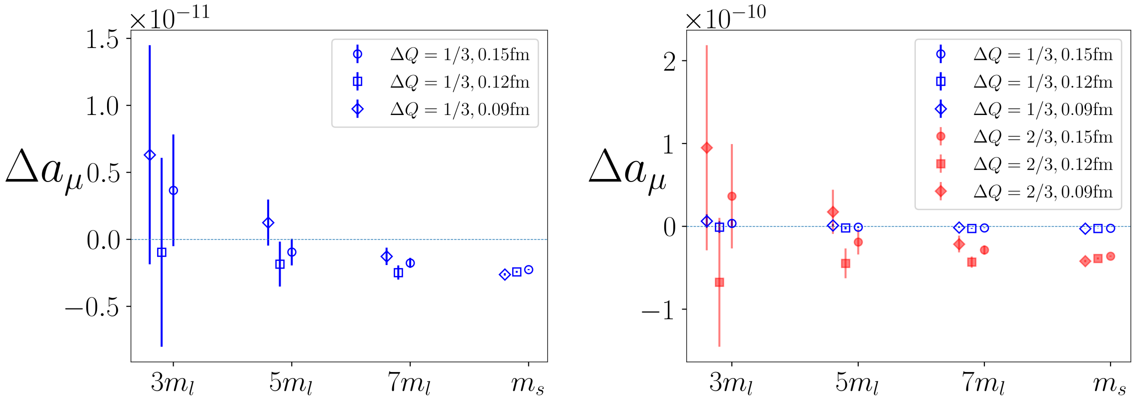
Table 2: The valence quark masses and charges used to compute the pseudoscalar and vector correlators.

We measure three sets of correlators, one uncharged where the SU(3) link fields are not multiplied by U(1) fields, and two charged with opposite electric charges. All the correlators are overall electrically neutral. We calculate two sets of correlators in this way as the charged correlators are noisier than the uncharged correlators. This is due to the presence of a QED noise term proportional to the electric charge, e, in the propagator. To suppress this noise term we average over the two correlators with opposite charges.

We use 16 time-sources on each field configuration to improve the statistics. All other things being equal this would increase the resources needed to run the simulations by a factor of 16. To mitigate this we use the truncated solver method (TSM) [21][13]. We use 16 sloppy solves with a residual of 10^{-3} and 1 precise solve with a residual of 10^{-6} before averaging over all the solves using the TSM method.

3. Results

The QED corrections at equal bare quark mass, Δa_μ are shown in Figure 1. The uncertainty of Δa_μ increases when the size of the charge is doubled and, as expected, grows rapidly with de-



(a) Δa_μ for down-like ($Q = -1/3e$) electric charges.

(b) Δa_μ for down-like ($Q = -1/3e$) and up-like ($Q = 2/3e$) electric charges.

Figure 1: The blinded QED corrections to $a_\mu^{\text{HVP,LO}}$ for all the ensembles and quark masses at fixed bare quark mass.

creasing quark mass. Figure 1 shows that the $\Delta Q = 1/3e$ and $\Delta Q = 2/3e$ QED corrections are highly correlated, not unexpected because the same stochastic-wall source is used for the neutral and charged correlators.

The strange quark contribution makes up around 7% of $a_\mu^{\text{HVP,LO}}$. An advantage of computing the QED corrections to the connected strange $a_\mu^{\text{HVP,LO}}$, is that the larger mass of the strange quark compared to the light quarks causes reduced errors for the strange $a_\mu^{\text{HVP,LO}}$, so it is potentially easier to determine the QED contribution. No chiral extrapolation is required at the mass of the strange quark.

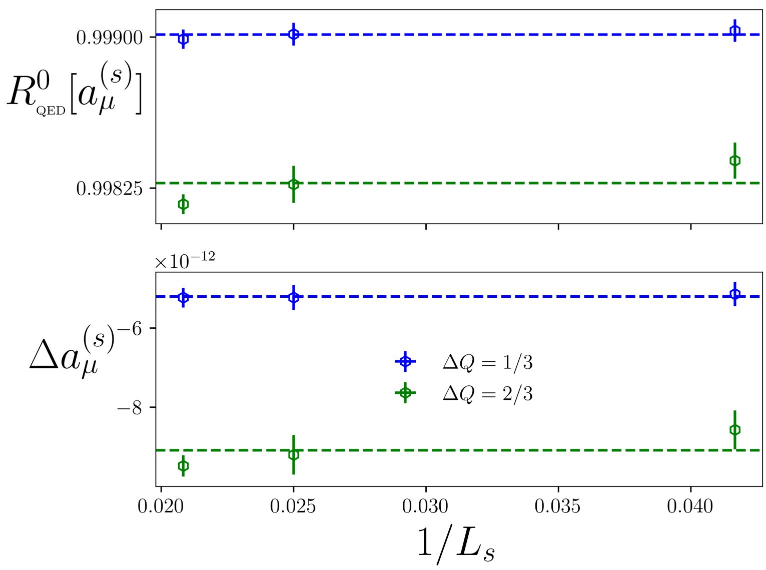


Figure 2: How $R^0_{\text{QED}}[a_\mu^{(s)}]$ and $\Delta a_\mu^{(s)}$ vary with inverse box size for $Q = -1/3e, 2/3e$. The dashed lines are the mean of the three values.

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In order to assess the magnitude of possible QED finite volume effects on $\Delta a_\mu^{(s)}$ we computed vector correlators on ensembles with varying spatial volumes. Hatton et al. [17] have preformed a similar finite volume study for the charmonium $\Delta a_\mu^{(c)}$. In this study we used the coarse physical ensemble with two ensembles with a similar lattice spacing (0.12 fm) and unphysically heavy pions with lattice volumes: $24^3 \times 64$ and $40^3 \times 64$. We can define the ratio of a quantity with and without QED,

$$R^0_{\text{QED}}[X] \equiv \frac{X[\text{QCD+qQED}]}{X[\text{QCD}]} \quad \text{at fixed } am_s, \tag{5}$$

and look at how this ratio varies with lattice volume. Figure 2 shows how $\Delta a_\mu^{(s)}$ and $R^0_{\text{QED}}[a_\mu^{(s)}]$ vary with the inverse lattice side ($1/L_s$). Figure 2 shows that the QED finite volume effects on $\Delta a_\mu^{(s)}$ are negligible for the statistics used. Even with the unphysical (for the strange quark) larger electric charge $Q_s = -2/3e$ the maximum deviation is only slightly above 1σ. The results of the finite volume study are similar to what was found for charm quarks [17] and is consistent with expectations from effective field theory [22].

The QED contribution, $\delta a_\mu^{(s)}$, to $a_\mu^{(s)}$ is obtained by taking the $Q = 0$ and $Q = -1/3e$ strange vector correlators and computing $a_\mu^{(s)}$ and Δa_μ, before converting to the Dashen scheme to obtain δa_μ. We note that the scheme adjustment on the fine ensemble is very imprecise, because the chiral extrapolation of the mass of the pseudoscalar meson to m_l needed to get the quark mass shift is less constrained on the fine ensemble. After the scheme adjustment the continuum limit needs to be taken. As the HISQ action has lattice artifacts of $O(a^2)$ the data is fit to a simple function, linear in a^2. The extrapolation function used is:

$$\delta a_\mu^{(s)}(a^2) = c_0 \left(1 + c_1 (a\Lambda)^2\right), \tag{6}$$

where the c_i are parameters to be fitted, a is the lattice spacing, and $\Lambda = 0.5$ GeV for the typical QCD scale.

The extrapolation in a^2 is plotted in Figure 3 and the fitted parameters are listed in Table 3. The fit is satisfactory with goodness of fit parameters $\chi^2/\text{dof} = 0.21$ for the $a_\mu^{(s)}$ extrapolation and $\chi^2/\text{dof} = 0.0014$ for the $\delta a_\mu^{(s)}$ extrapolation. Figure 3 shows that the slope in a^2 is mild and the posterior for c_1 is consistent with a horizontal band.

param	prior	posterior
	$\delta a_\mu^{(s)}$	
c_0	$0(1)\times10^{-10}$	$-0.0092(81)\times10^{-10}$
c_1	0(100)	0.3(6.4)
	$a_\mu^{(s)}$	
c_0	$0(1)\times10^{-8}$	$54.38(80)\times10^{-10}$
c_1	0(1)	−0.22(14)

Table 3: Priors and preliminary results for the parameters of the continuum extrapolations of $\delta a_\mu^{(s)}$ and $a_\mu^{(s)}$.

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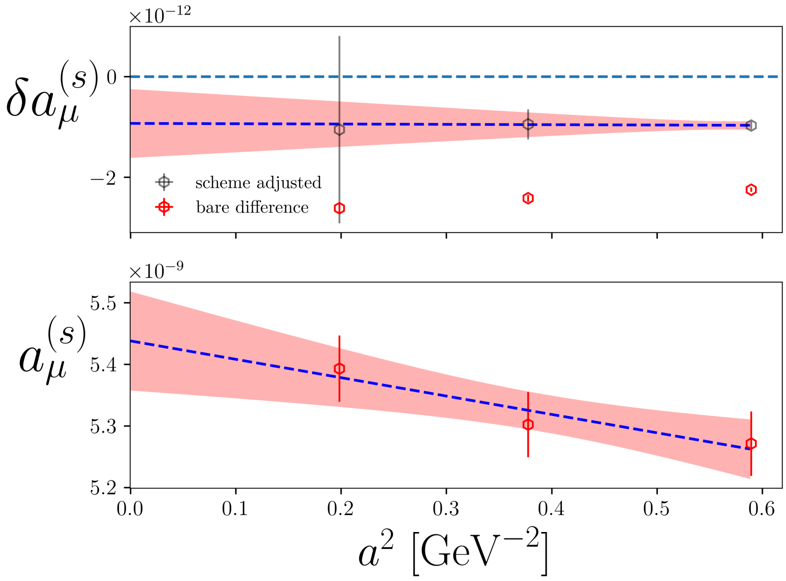

Figure 3: The blinded continuum extrapolations of $\delta a_\mu^{(s)}$ and $a_\mu^{(s)}$. See equation 6 and Table 3 for the fit function and fitted parameter values.

The extrapolated continuum values are,

$$\begin{aligned} \delta a_\mu^{(s)} &= -0.0092(81)\times 10^{-10} \\ a_\mu^{(s)} &= 54.38(80)\times 10^{-10}\,, \end{aligned} \tag{7}$$

where we remind the reader that these are blinded results. Our computed absolute uncertainty, 0.0081×10^{-10}, contributes a tiny amount to the overall uncertainty of a_μ.

The light quark contribution to a_μ comes from quark loops formed from up and down quarks $a_\mu^{(l)}$ makes up the lion's share, around 90%, of the total value of a_μ. We work in the limit where the up and down quarks have the same mass (in QCD), $m_l = \frac{1}{2}(m_u^{\text{phys}} + m_d^{\text{phys}})$. We measure vector correlators at $3m_l$, $5m_l$, $7m_l$ for all ensembles, because measurements at the physical pion mass are noisy.

The procedure to obtain $\delta a_\mu^{(l)}$ is essentially the same as that described above for $\delta a_\mu^{(s)}$. To aid the calculation we split up $\delta a_\mu^{(l)} = \delta a_\mu^{(u)} + \delta a_\mu^{(d)}$, where $\delta a_\mu^{(d)}$ is calculated using the $Q = 0, 1/3e$ correlators and $\delta a_\mu^{(u)}$ is calculated from the $Q = 0, 2/3e$ correlators. To find the physical value of $\delta a_\mu^{(l)}$ we do a combined chiral-continuum extrapolation for each piece before adding them together. We fit our data to the following functional form,

$$\begin{aligned} \delta a_\mu^{(d)}(a^2, m_q/m_l) &= c_0^{(d)}\left(1 + c_1^{(d)}(a\Lambda)^2 + c_2^{(d)} m_q/m_l\right) \\ \delta a_\mu^{(u)}(a^2, m_q/m_l) &= c_0^{(u)}\left(1 + c_1^{(u)}(a\Lambda)^2 + c_2^{(u)} m_q/m_l\right), \end{aligned} \tag{8}$$

which has a similar form to the extrapolation used for the strange quark contribution. The $c_2^{(u)} m_q/m_l$ and $c_2^{(d)} m_q/m_l$ terms control the extrapolation in quark mass. We again use $\Lambda = 0.5$ GeV. The fit has six parameters for 3 masses × 3 ensembles × 2 (u/d) = 18 pieces of data. We fit all six parameters simultaneously to account for correlations between measurements on the same ensemble. We plot both the continuum and chiral extrapolations of $\delta a_\mu^{(u)}, \delta a_\mu^{(d)}$ in Figure 4 and list the fitted values of the c_i in Table 4.

param	prior	posterior
	$\delta a_\mu^{(u)}$	
c_0	$0(1){\times}10^{-9}$	$-1.2(1.6){\times}10^{-11}$
c_1	$0(1)$	$-0.17(86)$
c_2	$0(1)$	$0.06(23)$
	$\delta a_\mu^{(d)}$	
c_0	$0(1){\times}10^{-10}$	$-0.5(1.0){\times}10^{-12}$
c_1	$0(1)$	$-0.06(94)$
c_2	$0(1)$	$0.14(51)$

Table 4: Priors and preliminary fitted results for the parameters of the chiral-continuum extrapolation of $\delta a_\mu^{(l)}$. See equations 8 for fit function.

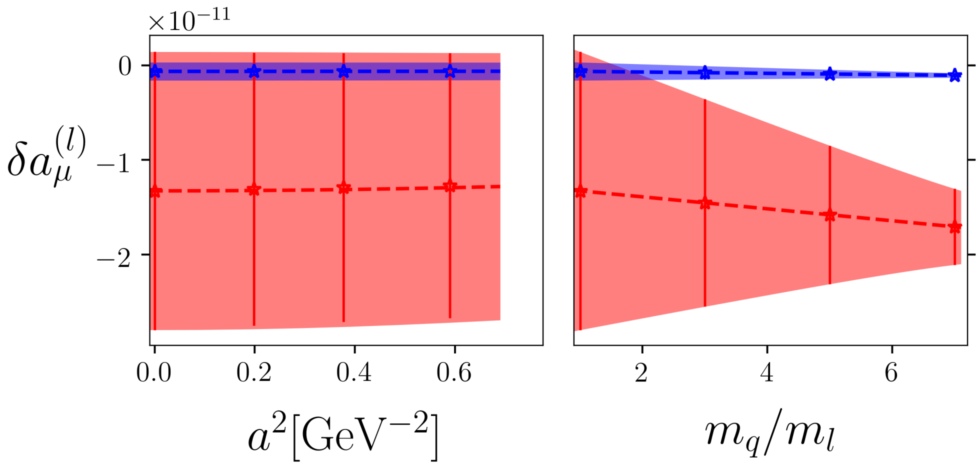

Figure 4: The blinded chiral-continuum extrapolations of $\delta a_\mu^{(u)}$ (red) and $\delta a_\mu^{(d)}$ (blue). On the left is the extrapolation in the lattice spacing at $m_q/m_l = 1$ and on the right is the extrapolation in quark mass at $a = 0$. See equations 8 and Table 4.

Our extrapolated value for the quenched QED correction to the light connected HVP is

$$\delta a_\mu^{(l)}(a^2 = 0, m_q/m_l = 1) = -1.3(1.5) \times 10^{-11} . \tag{9}$$

If the correlations between quark masses and charges are turned off we find an extrapolated $\delta a_\mu^{(l)} = 1.9(3.4) \times 10^{-11}$.

4. Conclusions

We have used staggered quarks, gluon fields generated with the HISQ action, and quenched U(1) fields gauge fixed with the $\mathrm{QED_L}$ prescription to measure vector correlators at a series of lattice spacings and light quark masses. From these correlators we have computed the QED corrections to the light and strange connected $a_\mu^{\mathrm{HVP,LO}}$.

To compare our results on the QED contributions to $a_\mu^{\mathrm{HVP,LO}}$ in the Dashen scheme with those from other collaborations requires work on converting the results to a consistent scheme. Our results, even though they are still blinded, imply that the QED correction to the connected $a_\mu^{\mathrm{HVP,LO}}$ are small, with an absolute uncertainty less than 1×10^{-10}. This is well below the threshold necessary for sub percent precision on the total a_μ ($\leq 5 \times 10^{-10}$). We are working on reducing the errors on the other contributions to $a_\mu^{\mathrm{HVP,LO}}$ to below sub percent precision, which when combined with the planned future experimental measurements of a_μ will maximize the test of the theoretical prediction of a_μ from the standard model [3].

We also plan to study the contribution of QED to the windows on the hadronic vacuum polarization [9], compute the QED contributions to the disconnected diagrams and the effect of QED in the sea.

Acknowledgements

This work was supported in part by grants and contracts from the U.S. Department of Energy, Office of High Energy Physics, from the U.S. National Science Foundation, and from the Simons Foundation.

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