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Real time evolution of scalar fields in semiclassical gravity

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We report on the development of a lattice formalism for studying the real time behavior of radially symmetric configurations of massless scalar fields in radially symmetric, curved space times in 3+1 dimensions. It is intended to numerically study back reaction effects due to semiclassical gravity in the time evolution of scalar field configurations, especially for those that will eventually evolve into black holes.

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1. Introduction

We study the semiclassical collapse of a spherical symmetric configuration of a free, scalar, massless, quantum field in general relativity. Classically, provided the initial configuration is appropriately chosen [1], such a collapse would lead to the formation of a black hole. In his seminal paper [2], Hawking predicted that such a black hole would radiate Hawking radiation. This lead to the information paradox. For recent reviews, see [3, 4].

Here we are interested in numerically studying the full time evolution of of such a system, starting in the collapse phase. We consider initial field configurations that feature an energy density that is localized on a spherical shell with finite width. The configurations are chosen such that the majority of the energy density is inmoving at the initial time. The radius of the spherical shell is chosen large enough such that, at the the same initial time, the system is very far from exhibiting a horizon. The goal is to be able to evolve the system in time while taking into account the unique features exhibited by a quantum field theory in a time dependent and curved space time. We also aim at studying the back reaction induced by quantum effects in the stress energy tensor within a semiclassical approximation where the metric is assumed to be classical. A similar approach is used in [5, 6].

At first, we discretize the system in radial and temporal direction and decompose the field in the angular direction into spherical harmonics. We then introduce the mode decomposition of the field and discuss a formalism which allows to calculate the real time evolution of the scalar field in a time independent background metric. Then, we discuss how the metric can be updated as a function of the Hamiltonian density of the field. We first solve this problem in the limit where only the $l = 0$ modes contribue. For related work see [7]. Then, we comment on ongoing research to include all the $l > 0$ modes of the scalar field and the additional difficulty created by divergences of the energy momentum tensor.

2. The formalism

The semicalassical time evolution is governed by the equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \langle \psi \mid T^{\text{ren}}_{\mu\nu} \mid \psi \rangle. \tag{1}$$

Here $R_{\mu\nu}$ and R are the Ricci tensor and scalar derived from a classical metric $g_{\mu\nu}$ and $T^{\text{ren}}_{\mu\nu}$ are the quantum operators corresponding to the renormalized energy momentum tensor. These operators are evaluated in a given state $\mid \psi \rangle$ for which the resulting expectation value is spherical symmetric.

We chose the metric to be of the form [1, 8–10]

$$\mathrm{d}s^2 = \alpha^2(t, r)\,\mathrm{d}t^2 - a^2(t, r)\,\mathrm{d}r^2 - \mathrm{d}\Omega^2. \tag{2}$$

Here $\alpha(t, r)$ and $a(t, r)$ are functions determined by the semiclassical Einstein equation (1) and $\mathrm{d}\Omega$ is the surface element of a two-sphere. This choice of coordinates allows to track the time evolution of a general spherical symmetric system in a region of space time. Note that this region of space time might be bounded by coordinate singularities. In the case of a Schwarzschild black hole, this choice of coordinates leads to the standard parametrization of the Schwarzschild metric where

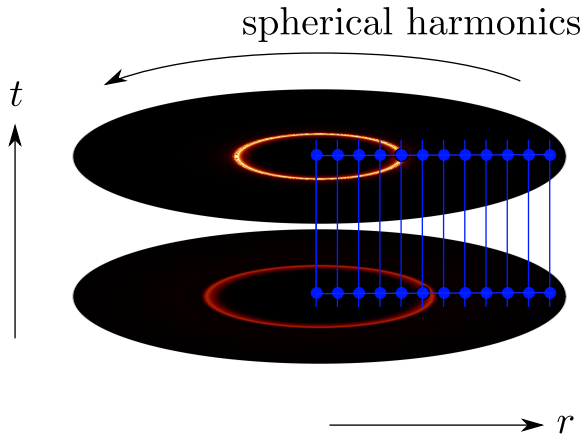

Figure 1: The lattice discretizations of the system. The radial and temporal coordinates are discretized. The angular modes are described by spherical harmonics. The tilted cirlces are two dimensional cross sections through the three dimensional space at two fixed times. The colors indicate the energy density of a purely classical collapsing scalar field.

$\alpha(t,r) = a^{-1}(t,r) = 1 - \frac{r_s}{r}$ with the Schwarzschild radius r_s. Therefore, we expect to be able to describe the outside region of a black hole but not the inside of a possibly forming event horizon. The scalar field behind such a horizon is expected to be frozen in time in this coordinate system. It would be interesting to supplement our calculation with one carried out in a metric where this problem is absent.

We sliced our space-time into space-like hypersurfaces of constant coordinate time t and use the Hamiltonian formalism to quantize the scalar field. More details can be found in [11]. We discretized the radial coordinate r by introducing a lattice with N_r equidistant points with distance Δr. To account for the angular directions, we decomposed the field into spherical harmonics. Each spherical harmonic is labeled by two integer numbers l and m where $l \geq 0$ and $|m| \leq l$. Note that spherical symmetry demands that all the $2l + 1$ modes for one given l are identical. Therefore, we droped the m dependence in all expressions. Our discretization scheme is depicted in figure 1.

The Hamiltonian of the field can than be written as a function of the angular modes of ϕ_l and their canonical conjugate momenta Π_l as

$$\mathcal{H} = \sum_{l=0}^{\infty}(2l+1)\left(\frac{\alpha a_0}{a\alpha_0}\Pi_l\Pi_l^\dagger + \phi_l^\dagger\sqrt{\frac{a\alpha_0}{\alpha a_0}}K\sqrt{\frac{a\alpha_0}{\alpha a_0}}\phi_l\right). \tag{3}$$

where we dropped the m index of the field and used a vector notation for the radial dependence. The kernel

$$K = q^T q + \frac{l(l+1)}{r^2}\alpha^2 \tag{4}$$

describes the radial derivatives and the effect of angular momentum. Here, the operator

$$q = \sqrt{\frac{\alpha}{a}}r\nabla_r\sqrt{\frac{\alpha}{a}}\frac{1}{r} \tag{5}$$

contains a suitable discretization ∇_r of the radial derivative. For simplicity, we used

$$\nabla_r\phi(r,t) = \frac{1}{\Delta r}(\phi(r+\Delta r) - \phi(r)). \tag{6}$$

It is useful to write the scalar field in terms of eigenfunctions of the kernel K. The eigendecomposition of K is given by

$$K = V\omega^2 V^T$$

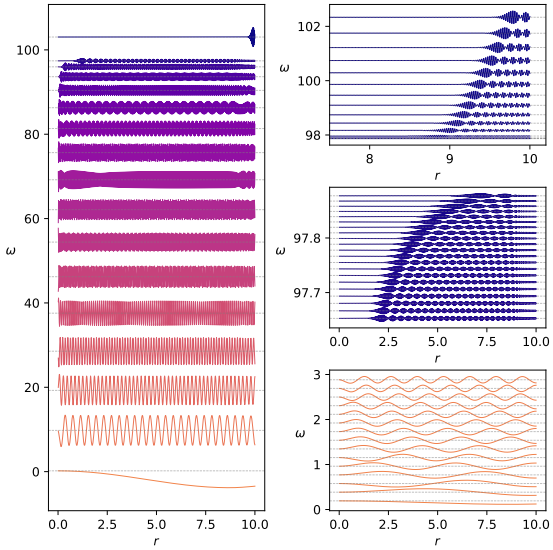

Figure 2: The eigenmodes of the operator K at the initial time in a typical simulation. Each solid line corresponds to one of the eigenmodes V_k. Each eigenmode is centered around a dotted line that indicates the corresponding eigenfreqency ω_k. On the left hand side, representative eigenmodes in the whole range of k are shown. On the right hand side, the top panel shows the eigenmodes with the largest ω_k and the lowest panel shows the eigenmodes with smallest ω_k. The middle panel shows intermediate eigenmodes.

where ω_k is the eigenfrequency of the k-th mode and the k-th column of V is the corresponding eigenmode. The eigenmodes are visualized in figure 2

The Hamiltonian from eq. (3) is quadratic in the fields and the conjugate momenta. Hence, in the eigenbasis defined by V, it becomes a collection of decoupled harmonic oscillators:

$$H = \sum_k (2l+1) \sum_l \sum_k \omega_{lk} (\hat{\phi}_{lk}^2 + \hat{\Pi}_{lk}^2) \tag{7}$$

where

$$\hat{\Pi}_{lk} = \sum_r \frac{1}{\sqrt{\omega_{lk}}} V_{rk}^{(l)} \sqrt{\frac{\alpha a_0}{a \alpha_0}} \Pi_{lr} \tag{8a}$$

$$\hat{\phi}_{lk} = \sum_r \sqrt{\omega_{lk}} V_{rk}^{(l)} \sqrt{\frac{\alpha_0 a}{a_0 \alpha}} \phi_{lr} \tag{8b}$$

The eigenbasis and the eigenfreqencies change in time when the functions $a(r,t)$ and $\alpha(r,t)$ are not constant in time. We quantized the harmonic oscillators appearing in eq. (7) in the standard fashion.

To stay as closely as possible to the classical situation, we chose, as an initial state of the simulation, a state in which all harmonic oscillators are in a coherent state, with no entanglement between the oscillators. Such a coherent state is uniquely characterized by the its classical expectation values for the field and the conjugate momenta. For all $l > 0$, we chose the classical expectation values of the field and momenta to be 0. For the $l = 0$ modes we chose them so that they produce the desired classical shape of the energy-momentum tensor in a way that is consistent with the initial metric. For more details see [11].

3. Time evolution of the scalar field for time independent metrics

Assuming that the function $a(r,t)$ and $\alpha(r,t)$ are independent of t, the time evolution of the scalar field can be determined exactly. We employed two ways to study this time evolution. The

first way is to employ the Heisenberg picture and evolve the creation and annihilation operators of the harmonic oscillators comprising the quantum field. The second way is to use the Schrödinger picture and evolve the quantum state of the system. Both ways of describing the time evolution are equivalent.

In the first case, it is useful to write the Hamiltonian in the form

$$\mathcal{H} = \sum_l b_+^{(l)} W^{(l)} b_+^{(l)\dagger} + b_-^{(l)\dagger} W^{(l)} b_-^{(l)} + b_+^{(l)} X^{(l)} b_-^{(l)} + b_-^{(l)\dagger} X^{(l)} b_+^{(l)\dagger}. \tag{9}$$

Here, $W^{(l)}$ and $X^{(l)}$ are $N_r \times N_r$ coefficient matrices in the momentum space and the $b_\pm^{\dagger(l)}$ and $b_\pm^{(l)}$ are vectors of creation and annihilation operators for the N_r eigenmodes. Note that $W^{(l)}$ and $X^{(l)}$ can be determined by comparison of coefficients with eq. (7). It is useful to consider the linear combinations $b_u^{(l)}(t)$, $b_v^{(l)}(t)$, $b_u^{(l)\dagger}(t)$, and $b_v^{(l)\dagger}(t)$ In terms of these linear combinations the time evolution can be written as

$$b_u(t) = \frac{1}{\sqrt{2}}(b_+ u(t) + b_i^\dagger u^*(t)) V \sqrt{\omega} \tag{10a}$$

$$b_u(t) = \frac{1}{\sqrt{2}}(b_+ v(t) - b_-^\dagger v^*(t)) V \sqrt{\omega^{-1}} \tag{10b}$$

Therefore, the problem reduces to finding the time evolution of the matrices $u(t)$ and $v(t)$. Once initialized by

$$u(0) = \frac{1}{\sqrt{\omega}} V^T \quad \text{and} \quad v(0) = \sqrt{\omega} V^T \tag{11}$$

their time evolution is given by

$$\begin{pmatrix} u(t+\Delta t) & v(t+\Delta t) \end{pmatrix} = \begin{pmatrix} u(t) & v(t) \end{pmatrix} \exp\left(-\mathrm{i} \begin{pmatrix} 0 & \sqrt{\frac{a\alpha_0}{\alpha a_0}} K \sqrt{\frac{a\alpha_0}{\alpha a_0}} \\ \frac{\alpha a_0}{a\alpha_0} & 0 \end{pmatrix} \Delta t \right) \tag{12}$$

Note that eq. (12) describes a Bogolyubov transformation.

The second approach is to use the properties of Gaussian states. Since the initial state is Gaussian and the Hamiltonian is quadratic, the Wigner functional has the form of a Gaussian probability density function in phase space. Therefore, it is convenient to study the time evolution in terms of this Gaussian. For a review on Gaussian states, see e.g. [12].

It is useful to introduce a vector in phase space

$$\vec{x}^{(l)} = \begin{pmatrix} \Pi_1^{(l)\dagger} & \Pi_2^{(l)\dagger} & \dots & \phi_1^{(l)} & \phi_2^{(l)} & \dots \end{pmatrix}^T \tag{13}$$

where the indices of the fields and the momenta corresponds to the radial coordinate. With this vector, the Hamiltonian can be written as

$$H = \vec{x}^{(l)\dagger} M^{(l)} \vec{x}^{(l)} \tag{14}$$

where

$$M^{(l)} = \begin{pmatrix} A & 0 \\ 0 & A^{-\frac{1}{2}} K A^{-\frac{1}{2}} \end{pmatrix} \quad \text{with} \quad A = \frac{\alpha a_0}{a \alpha_0}. \tag{15}$$

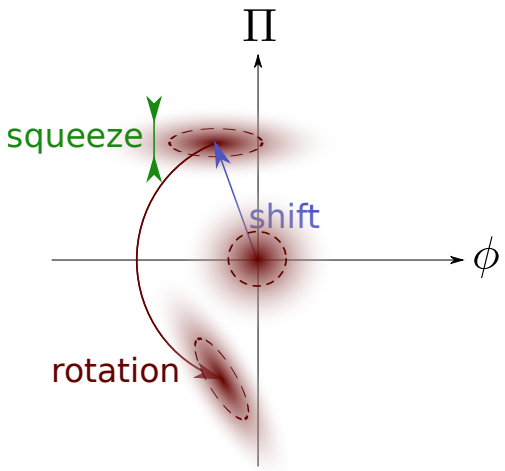

Figure 3: The construction of a Gaussian state from the ground state of a harmonic oscillator. Shown are sketches of the Wigner distributions of three Gaussian states. Each Gaussian state can be characterized uniquely by a mean value in the phase space and a covariance matrix. The state shown at the origin is the harmonic oscillator ground state. By a shift, a coherent state can be constructed. After squeezing, the resulting state has a minimal uncertainty non-equally distributed between coordinate and momentum. The time evolution of is achieved by a rigid rotation around the origin.

The time evolution can be realized by first performing a canonical transformation with the matrix

$$T = \begin{pmatrix} \frac{1}{\sqrt{\omega}} V^T \sqrt{A} & \\ & \sqrt{\omega} V^T \frac{1}{\sqrt{A}} \end{pmatrix} \tag{16}$$

which brings the Hamiltonian to a form where only decoupled oscillators appear. Then, the time evolution is described by a rigid rotation in phase space:

$$R = \begin{pmatrix} \cos(\omega_1 \Delta t) & & \sin(\omega_1 \Delta t) & & \\ & \cos(\omega_2 \Delta t) & & \sin(\omega_2 \Delta t) & \\ & & \ddots & & \ddots \\ -\sin(\omega_1 \Delta t) & & \cos(\omega_1 \Delta t) & & \\ & -\sin(\omega_2 \Delta t) & & \cos(\omega_2 \Delta t) & \\ & & \ddots & & \ddots \end{pmatrix} \tag{17}$$

To get back into momentum space, one has to invert the canonical transformation so that the full time evolution can be described by $S = T^{-1} R T$. In terms of the mean value and the covariance matrix, the time evolution is given by

$$\vec{\mu}^{(l)}(t + \Delta t) = S \vec{\mu}^{(l)}(t), \tag{18a}$$

$$C^{(l)}(t + \Delta t) = S C^{(l)}(t) S^T. \tag{18b}$$

In general, such a time evolution can be decomposed into shifts, squeezes and rotations. For the case of one oscillator, these operations are visualized in figure 3. Note that for more then one oscillators, the covariance matrix has more degrees of freedoms than visualized. These additional degrees of freedoms correspond to the entanglement of different oscillators. Also, note that in a time independent background metric, using either eq. (12) or eqns. (18a) and (18b) is exact.

4. Time evolution of the metric

Since the energy momentum tensor changes with time when performing the time evolution discussed in the previous section, the metric must be updated accordingly at each time step. For the time evolution of the metric, it is useful to define the functions

$$\hat{\alpha}(r,t) = \alpha(r,t)a(r,t),$$
$$d(r,t) = \frac{r}{a^2(r,t)}.$$

In terms of these functions, the Einstein equations can be written as

$$\ln'(\hat{\alpha}) = \langle h_r^0 \rangle_\psi, \tag{20a}$$
$$d' + d\langle h_r^0 \rangle_\psi = 1 - r\langle m_r \rangle_\psi, \tag{20b}$$

where a prime indicates radial derivatives. The quantities h_r^0 and m_r are linear combinations of the energy momentum tensors. In terms of the Bogolyubov coefficients, they are

$$\langle h_r^0 \rangle_\psi = \frac{1}{d^0 \hat{a}^0} \left(|l_{ur}|^2 + |l_{vr}|^2 + \sum_{l=0}^{\infty} (2l+1) \left((v_l^\dagger v_l)_r r + (q^0 u_l^\dagger u_l q^{0T})_r r \right) \right), \tag{21}$$

$$\langle m_r \rangle_\psi = \frac{d^0 \hat{\alpha}^0}{r^2} \sum_{l=0^\infty} (2l+1) \frac{l(l+1)}{r^2} (u_l^\dagger u_l)_{rr}. \tag{22}$$

Here, l_{ur} and l_{vr} encode the classical expectation values of the coherent state and are present only in the $l = 0$ case. Equivalent formulations in terms of the $\vec{\mu}$ and C can also be derived. Eqs. (20a) and (20b) can be numerically integrated.

To combine this metric update with the field evolution, we developed the following procedure. First, we take the scalar field at time t and do one field update with time step Δt. Then, we use eqns. (20a) and (20b) to construct a new metric from the evolved fields. Then, we go back to the time t and repeat the update but instead of the original metric, we use the newly determined one. After this evolution, we construct again new metric. We repeat this process until the metric converges. When convergence is reached, we update the field at time t with a time step $2\Delta t$ using the metric we converged to. We use the result of this last time evolution as the final result for the field at time $t + 2\Delta t$.

To understand the effect of the changing metric to the scalar field, it is most useful to use the Wigner functional formulation for Gaussian states. A change in the metric changes the quadratic form that defines the Hamiltonian. This in turn means that both the the eigenmodes- and frequencies of K change. Therefore, what was a vacuum in one oscillator basis, is not a vacuum in another basis. The situation is visualized for one oscillator in figure 4. In addition to the squeezing depicted in this figure, in the higher dimensional case there will also be mixing between the different oscillators. In figure 5 the covariance matrix of a sample system after some simulation time can be seen. Note that the covariance matrix contains the full information on equal time correlation functions.

5. Renormalization

The components of the energy momentum tensor that appear in eqs. (20a) and (20b) are divergent. They require renormalization to produce a finite result.

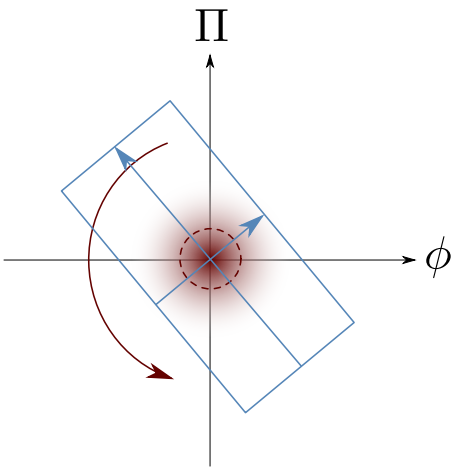

Figure 4: A vacuum state for one harmonic oscillator corresponds to a squeezed state after the harmonic oscillator frequency has changed. The black coordinate system corresponds to the phase space space of the initial harmonic oscillator with canonically conjugate variables (ϕ, Π). The blue coordinate system corresponds to a harmonic oscillator with changed frequency.

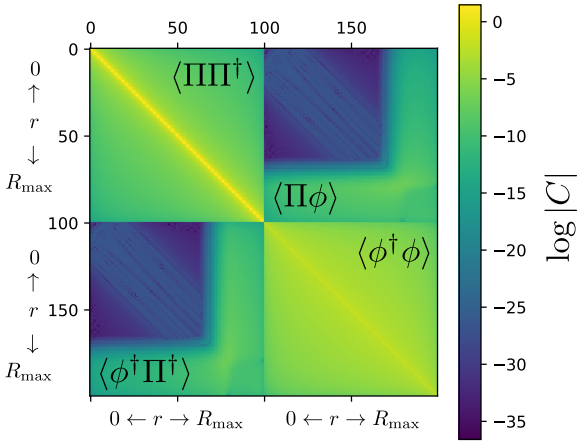

Figure 5: Covariance matrix characterizing the $l = 0$ part of a Gaussian state after some time. The diagonal blocks contain the connected field and canonical conjugate momenta equal-time correlation functions. The of diagonal blocks correspond to mixed correlation function.

In the approximation where only the $l = 0$ mode is taken into account, normal ordering at the initial time is sufficient to make these components finite. For more details, see [11]. In the general case, there are additional divergences that must be subtracted. These additional divergences are related to the inifite number of $l > 0$ modes. Each of these modes is finite on its own. But because the modes with a given l come in multiples of $2l + 1$, the contribution per l increases as l increases. And because infinitely many l modes have to be summed, the whole energy momentum tensor diverges. The contribution of the first l modes to h^0 can be seen in figure 6 in the case of a typical simulation.

In [13] is was shown that for a certain set of states, which are called Hadamard states, the two point function of the field has the form

$$\lim_{x' \to x} \langle \psi \mid G(x, x') \mid \psi \rangle = \frac{u(x, x')}{\sigma(x, x')} + v(x, x') \ln \sigma(x, x') + w(x, x') \tag{23}$$

Here, $\sigma(x, x')$ is the geodesic distance between x and x' and $u(x, x')$, $v(x, x')$ are state-independent function that depend only on the metric and $w(x, x')$ is regular. Since the energy momentum tensor

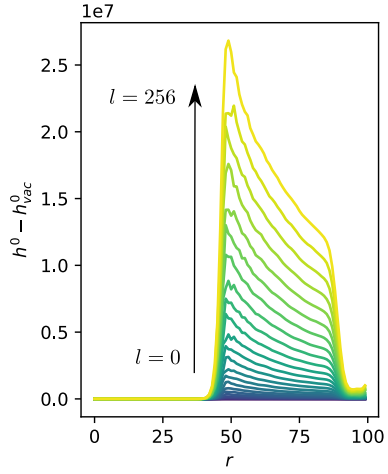

Figure 6: Divergent contribution of higher l modes. While every single mode is finite, the sum over all modes divergence as the function of a l-cutoff. Shown is the h^0 part of the normal ordered energy density. Normal ordering is carried out with respect to the creation/annihilation operators at the initial time.

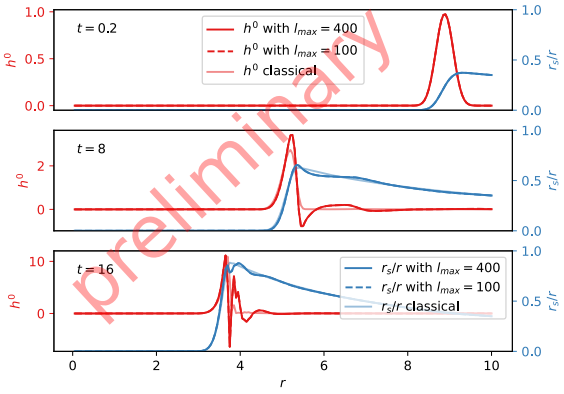

Figure 7: A preliminary simulation of the semiclassical collapse of a massless scalar field with Pauli-Villars regularization for three times. In red the combination h^0 is shown. The classical case is potted transparently while the semiclassical case is shown with 100 and 400 l-mode in dashed and solid lines. In blue, the the ratio of the fictitious Schwarzschild radius of all matter inside a sphere with radius r over this radius is shown. A value of 1 indicates the formation of a horizon.

can be defined in terms of the coincidence limit of $G(x, x')$, this allows in principle to subtract the divergent terms in $G(x, x')$ before constructing the quantities of interest.

The translation of this structure to the decomposition into spherical harmonics is complicated. See [14] for work in that direction. In [5, 6] it was suggested to introduce a set of Pauli-Villars fields to the system. These Paul-Villars fields are massive copies of the original scalar fields. The contributions of these additional fields to the energy momentum tensors can weighted with either a positive or a negative sign. If one employs a total of of 5 Pauli-Villars fields with alternating signs and with the masses fulfilling the conditions [5, 6]

$$m_2^2 + m_4^2 = m_1^2 + m_3^2 + m_5^2, \tag{24a}$$
$$m_2^4 + m_4^4 = m_1^4 + m_3^4 + m_5^4, \tag{24b}$$

the energy momentum tensor becomes finite.

We carried out a simulation with a set of Pauli-Villars masses chosen according to this condition. We set the first mass $m_1 = 1$ in Planck units. Preliminary results can be found in figure 7. The possible dependence of the result on the regulator masses is currently under investigation.

6. Conclusions and outlook

We presented a formalism to numerically study the real time evolution of a free, scalar field in a curved, non stationary space times. As a source term in the semiclassical Einstein equations, we used the energy momentum tensor of the quantum field. In the approximation where only the $l = 0$ mode contributes, renormalization can be done via normal ordering. In the full system, additional divergences appear that must be canceled. One scheme for canceling these divergences is the use of Pauli-Villars fields. Other regularization schemes are currently under investigation.

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