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Chiral Symmetry Breaking in QED induced by an External Magnetic Field

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We simulate Lattice QED in a constant and homogeneous external magnetic field using the Rational Hybrid Monte-Carlo (RHMC) algorithm developed for Lattice QCD. Our current simulations are directed towards observing chiral symmetry breaking in the limit of zero electron bare mass as predicted by approximate (Schwinger-Dyson) methods. Our earlier simulations were performed on a 36^4 lattice at the fine structure constant $\alpha = 1/137$, close to its physical value, with 'safe' electron masses $m = 0.1$ and $m = 0.2$. At this α, the dynamical electron mass produced by the external magnetic field, which is an order parameter for this chiral symmetry breaking, is predicted to be far too small to be measurable. Hence we are now simulating at the larger $\alpha = 1/5$, where the predicted dynamical electron mass at strong external magnetic fields accessable on the lattice is large enough to be measurable. However this requires electron masses down to $m = 0.001$. Such a small m requires lattices larger than 36^4, but at magnetic fields large enough to produce measurable dynamical electron masses, 36 is an adequate spatial extent for the lattice in the plane orthogonal to the magnetic field because the electrons preferentially occupy the lowest Landau level. We are therefore performing finite size analyses using $36^2 \times N_{\parallel}^2$ lattices with $N_{\parallel} \geq 36$. We measure the chiral condensate $\langle\bar{\psi}\psi\rangle$ as our order parameter for chiral symmetry breaking, since it should remain finite as $m \to 0$ if chiral symmetry is broken by the magnetic field, but vanish otherwise. Our preliminary results strongly suggest that chiral symmetry *is* broken by the external magnetic field. In all our simulations, as well as measuring other observables during these simulations, we are storing configurations at regular intervals for further analysis. One such measurement planned for these stored configurations is the determination of the effects that an external magnetic field has on the coulomb field of a charged particle placed in this magnetic field.

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1. Introduction

We study Lattice QED in external electromagnetic fields using methods developed for Lattice QCD. Since QED in background electric fields has a complex action, because the vacuum is unstable against decays into electron-positron pairs – the Sauter-Schwinger effect, [1, 2] – standard simulation methods which rely on importance sampling cannot be used. Because of this we start by considering Lattice QED in background magnetic fields where the action is real and bounded below. This enables us to perform simulations using standard methods. We use the Rational Hybrid Monte-Carlo (RHMC) method of Clark and Kennedy [3] whose implementation we describe in the appendix.

Relativistic quantum mechanical studies of electrons in external electromagnetic fields and the modified actions this produces for those fields, by Sauter, Euler and Heisenberg [1, 4] and formalized by Schwinger [2], are some of the earliest QED calculations. See Dunne [5] for some of the cases where exact solutions are known.

Phenomena such as the Sauter-Schwinger effect only become significant when the electric field $E \sim E_{cr} = m^2/e$ or larger and/or the magnetic field $B \sim B_{cr} = m^2/e$ or larger. Interest in extending such studies to full QED including non-perturbative effects have been revived by planned experiments colliding electron beams with intense beams of light from petawatt lasers [6, 7], where the electromagnetic fields are of this magnitude, or larger at LBNL, SLAC and possibly ELI. In addition it has been realized that compact astronomical X- and γ-ray sources are probably neutron-stars with magnetic fields of order B_{cr} or larger (magnetars). See for example the review [8]. Finally it has been noted that beam-beam interactions in the next generation of electron/positron colliders could produce electromagnetic fields orders of magnitude larger than their critical values [9]. Here, multi-electron-loop contributions become as, or more important than, single loop contributions and all conventional QED calculations break down.

For our simulations of QED in an external magnetic field, we choose a magnetic field which is constant over all space and time. For definiteness, we choose a magnetic field oriented in the z (3) direction. Classically the orbit of a charged particle (electron) in such a magnetic field is a helix around a fixed magnetic field line, whose projection on the (x, y) ((1, 2)) plane is a circle, while the motion in the z direction is free. Quantum mechanically the motion in the (x, y) plane is quantized into a set of levels whose transverse energies squared are evenly spaced with spacing $|2eB|$ – the Landau levels[10]. The lowest level has a single helicity, while the higher levels have both helicities. The the motion along the z direction is free field so that the z momenta have a continuous spectrum. Including QED means adding a dynamical photon field. The electron field feels both the external and the dynamical photon fields, while only the dynamical photon field has a kinetic term. For details of the lattice transcription of QED in this external field and its simulation, see the appendix. The most important feature is that at large $|eB|$ all electrons preferentially occupy the lowest Landau level whose orbit has a finite extent (proportional to $1/\sqrt{|eB|}$, leading to an effective dimensional reduction from 3 + 1 dimensions to 1 + 1 dimensions.

One of the most theoretically interesting non-perturbative effects predicted by truncated Schwinger-Dyson analyses of QED in such constant magnetic fields is that, in the limit $m \to 0$, chiral symmetry is broken by the magnetic field leading to a dynamical electron mass $\propto \sqrt{|eB|}$ [11–18] and a chiral condensate $\langle\bar{\psi}\psi\rangle \propto |eB|^{3/2}$ [19, 20]. This non-perturbative effect is often referred to as 'magnetic catalysis'. For a good review article with a more complete set of references see Miransky and Shovkovy [21]. We note that in $3+1$ dimensions, for massless electrons in an external magnetic field without QED, i.e. without internal photons, chiral symmetry is unbroken for all eB. In fact, as $m \to 0$ the chiral condensate vanishes $\propto m\log(m)$, so QED is essential for the breaking of chiral symmetry in a magnetic field. This contrasts with the situation in $2+1$ dimensions where chiral symmetry is broken with a finite chiral condensate $\propto eB$ for massless electrons in an external magnetic field, even without QED.

In our RHMC lattice QED simulations, we measure this chiral condensate, since as a local operator, it is easier to measure than the dynamical mass, which would require measuring the electron propagator itself. At physical $\alpha = e^2/(4\pi) \approx 1/137$, the predicted dynamical mass is more than 30 orders of magnitude less than any value we could possibly measure. We therefore perform simulations with a stronger $\alpha = 1/5$, which appears to be in the perturbative regime for $eB = 0$, and for which the predicted dynamical electron mass and chiral condensate although small, should be measurable for the eB value we choose. Following our earlier simulations at $\alpha = 1/137$ we simulate on a 36^4 lattice at $\alpha = 1/5$ and with masses in the range $0.001 \le m \le 0.2$. We choose $|eB| = 2\pi \times 100/36^2 = 0.4848...$, which is large but safely in the range $|eB| < 0.65$ required to keep discretization errors under control. For these parameters a lattice of size 36 in both the x and y directions is considerably larger than the projection of the lowest Landau level on the (x, y) plane and therefore adequate. However, 36 is too small a value in the z and t directions to accommodate the smallest masses, so a finite size scaling analysis is needed, however it is only necessary to increase the lattice sizes in the z and t directions. Preliminary results of such a finite size scaling analysis are presented in the next section, and strongly suggest that there is chiral symmetry breaking in the $m \to 0$ limit.

2. Simulations and Results

Here we discuss only the simulations and results from our simulations with $\alpha = 1/5$. For simulations and results with $\alpha = 1/137$ and for 'free' electrons in an external magnetic field see the proceedings from our talk at Lattice 2021 [22].

We perform RHMC simulations with $\alpha = 1/5$ aimed primarily at searching for evidence for chiral symmetry breaking in the presence of a constant and uniform magnetic field in the limit $m \to 0$. For this we measure the chiral condensate $\langle\bar{\psi}\psi\rangle$, which should remain finite and non zero as $m \to 0$ if chiral symmetry is broken in this limit.

If chiral symmetry is unbroken in the $m = 0$ limit, the chiral condensate is dominated by the short distance (ultraviolet) regime and should vanish proportional to m possibly times some power of $\log(m^2)$ as $m \to 0$. In this case it should be insensitive to the size of the lattice. Therefore we should be able to run with arbitrarily small masses without observing finite lattice size effects. To test this we perform $\alpha = 1/5$ simulations with $0.001 < m < 0.2$ on a 36^4 lattice at $eB = 0$, where chiral symmetry is believed to be unbroken at $m = 0$. Note that the normal requirement that $mN_\mu >> 1$ ($\mu = 1, 2, 3, 4$) to avoid finite size effects is not true for the lower part of this range. In figure 1 we plot the condensate as a function of mass and see that it does appear to be approaching zero for small m. We repeat these simulations on a 48^4 lattice at the lowest mass $m = 0.001$. The condensate on the 36^4 lattice is $4.3259(7) \times 10^{-4}$, while that on the 48^4 lattice is $4.3294(7) \times 10^{-4}$,

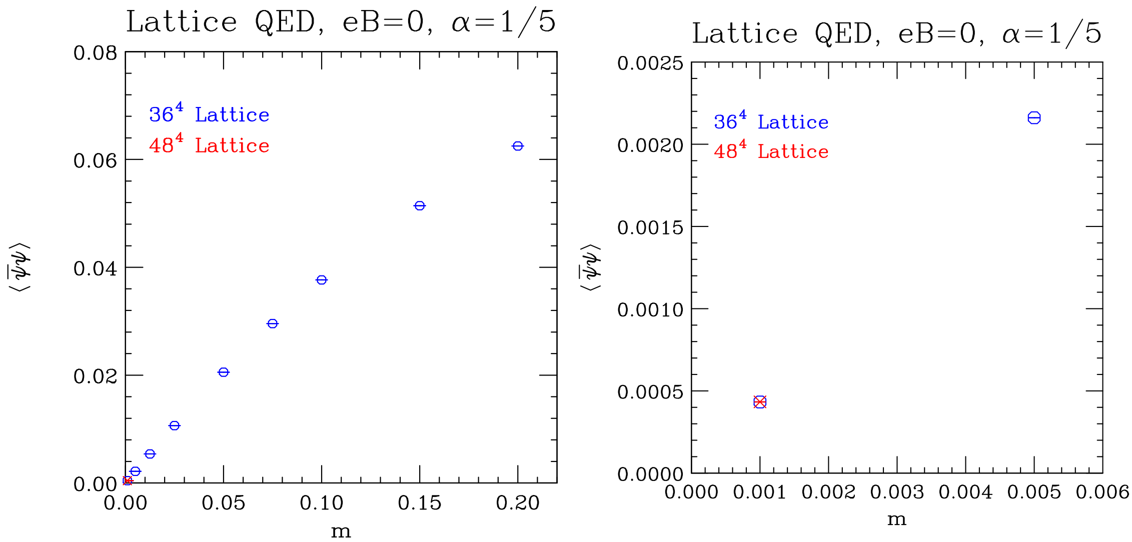

Figure 1: $\langle\bar{\psi}\psi\rangle$ as a function of mass at $eB = 0$, showing lattice size dependence.

a mere less than 0.1% difference. Such insensitivity to this finite size scaling analysis we take as evidence that the condensate is zero and chiral symmetry remains unbroken in the chiral ($m = 0$) limit, at least to within the precision of our simulations. In fact, a linear extrapolation to $m = 0$ from the points at $m = 0.005$ and $m = 0.001$ which, based on the curvature of this graph, should yield an upper estimate of the value of the condensate at $m = 0$, gives $\langle\bar{\psi}\psi\rangle \approx 10^{-6}$ which is only about 3 standard deviations from zero, giving further evidence that chiral symmetry remains unbroken as $m \to 0$.

We now turn to the case of large eB and choose $|eB| = 2\pi \times 100/36^2 = 0.4848...$, near the upper end of the range of $|eB|$ values where discretization errors are small. Again we run our RHMC simulations with $\alpha = 1/5$ and $0.001 < m < 0.2$ on a 36^4 lattice. As indicated above, the extent of the lattice in the x and y directions is large enough to contain the lowest Landau levels. However the extent of the lattice in the z and t directions is insufficient to prevent finite size effects. If chiral symmetry is broken in the limit $m \to 0$ then this indicates that there are modes with momenta of order $\sqrt{|eB|}$ or less which contribute to the condensate at small m. These modes will make a contribution of order $|eB|^{3/2}$ to the chiral condensate, keeping it non-zero as $m \to 0$ and making it sensitive to increases in the lattice extent in the z and t directions [19, 20].

Since the condensate on the 36^4 lattice appears to be headed towards zero as $m \to 0$, a sign that chiral symmetry is broken in this limit is that the chiral condensate at small enough m should increase when the lattice sizes in the z and t directions are increased. We have therefore performed simulations with the same parameters on $36^2 \times 64^2$ lattices. For $m = 0.025$ the change in the chiral condensate in going from the 36^4 lattice to the $36^2 \times 64^2$ lattice is very small, indicating that we need only simulate on the larger lattice at masses less than 0.025. At $m = 0.0125$ there is a small but significant increase in the condensate in going to the larger lattice, while at $m = 0.005$ the increase in the condensate in going to the larger lattice is relatively large. At m=0.001 the condensate on the larger lattice is approximately twice that on the smaller lattice. At $m = 0.005$, we have increased our lattice size even further to $36^2 \times 96^2$. While this leads to a further significant increase in the

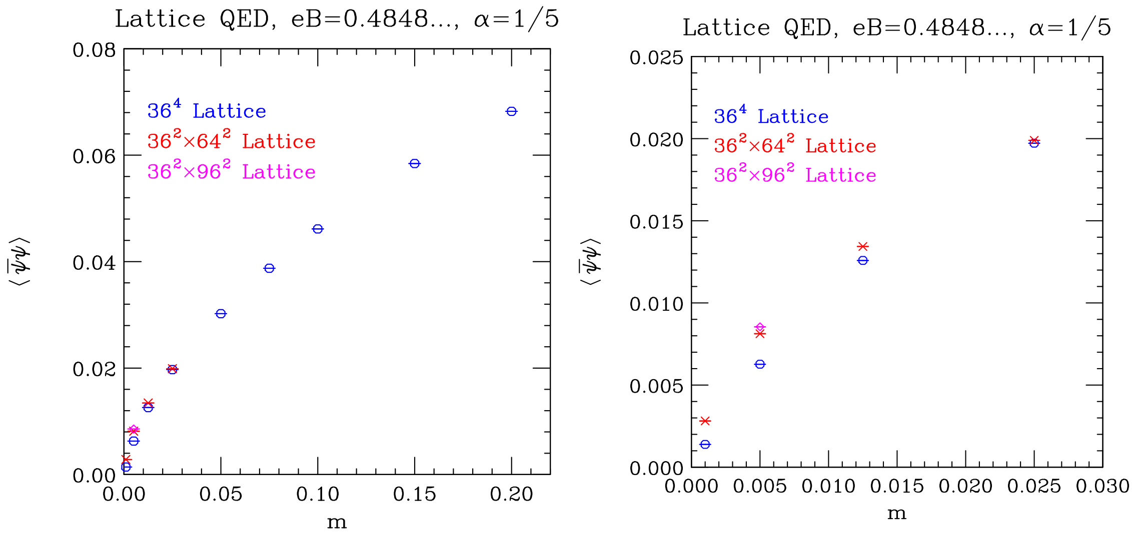

Figure 2: $\langle\bar{\psi}\psi\rangle$ as a function of mass at $eB = 2\pi\times100/36^2 = 0.4848...$, showing dependence on lattice size in the z and t directions.

condensate, it is only by $\approx 5\%$, and so going to an even larger lattice is unnecessary. Our next task is to increase the lattice size at $m = 0.001$ to $36^2\times96^2$ and possibly go to a lattice with z and t extents of 128, which should make the case for chiral symmetry breaking even more compelling and allow us to estimate the value of the chiral condensate at $m = 0$. Figure 2 shows the mass dependence of the chiral condensate as a function of mass m from these simulations at $|eB| = 2\pi\times100/36^2$ on our chosen lattice sizes. Even with simulations on only 36^4 and $36^2\times64^2$ lattices at $m = 0.001$, this graph does suggest that the condensate remains finite and non-zero as $m\to0$.

3. Summary, Discussion and Conclusions

We simulate lattice QED on lattice sizes 36^4, $36^2\times64^2$, and $36^2\times96^2$ with $\alpha = 1/5$, in an external magnetic field $eB = 2\pi\times100/36^2$ and masses in the range $0.001 \le m \le 0.2$. The simulations on the $36^2\times96^2$ lattice at $m = 0.001$ have yet to be performed. Preliminary results of this finite size scaling analysis show that for $m \le 0.0125$, the chiral condensates increase as the lattice size is increased, which is evidence that the condensate remains finite in the chiral ($m\to0$) limit. The forthcoming larger lattice simulations at $m = 0.001$ should confirm this and predict the value of this condensate at $m = 0$. We also perform simulations with the same α and masses, but with $eB = 0$ on 36^4 lattices and for $m = 0.001$ on a 48^4 lattice. Here the condensate shows no significant finite size effects, and the data strongly indicate that this condensate vanishes for $m\to0$.

The size of the chiral condensate at $m = 0$ appears to be larger than that predicted by Schwinger-Dyson methods. This might mean that $\alpha = 1/5$ is too large for the approximations made in the Schwinger-Dyson calculations or the derivation of the condensate from the dynamical electron mass predicted by these methods. It could also be due to the limitations of the lattice methods at these parameters.

Because the size of the (x, y) projection of lowest Landau levels for the chosen eB are appreciably smaller than 36^2, we are simulating with the same parameters on lattices with dimensions

18^2 in the (x, y) plane for comparison, in particular comparing the chiral condensates for small m.

We plan to perform further analyses on stored configurations. Of particular interest is the measurement of the distortion and screening of the coulomb fields of charged particles in the presence of external magnetic fields [23–26]

It is of interest to repeat our simulations in different external magnetic fields to test if the $m = 0$ chiral condensate scales as $|eB|^{3/2}$. We should also study the chiral behavior of QED extended to include multiple electron 'flavours'. If chiral symmetry is broken in the $m \to 0$ limit, this includes the spontaneous breaking of chiral flavour symmetry, complete with flavoured massless Goldstone bosons. These are allowed because, being uncharged, they do not feel the magnetic field and are thus not subject to the dimensional reduction from $3+1$ to $1+1$ dimensions, and hence are massless $3+1$ dimensional excitations.

We will explore the possibility of designing an effective action which incorporates the assumption that only the lowest Landau level contributes, but is otherwise fully $3+1$ dimensional.

We will explore the inclusion of external electric fields. Because these make the action complex, we will need to resort to simulation methods such as the complex Langevin equation (CLE). Because the non-compact gauge action describes a free field which is a collection of harmonic oscillators then, in the absence of fermions, the real Lagrangian is an attractive fixed point of the CLE, rather than a repulsive fixed point as is the case for QCD and probably lattice QED with a compact action. Therefore one might hope that this might remain true when fermions are included, possibly with a modified fermion action, at least for weak coupling. This would allow the study of the Sauter-Schwinger effect in full QED in an external electric field using Lattice QED simulations.

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A. Lattice QED in an external magnetic field

We simulate using the non-compact gauge action

$$S(A) = \frac{\beta}{2} \sum_{n,\mu<\nu} [A_\nu(n+\hat{\mu}) - A_\nu(n) - A_\mu(n+\hat{\nu}) + A_\mu(n)]^2$$

where n is summed over the lattice sites, and μ and ν run from 1 to 4 subject to the restriction $\mu < \nu$. $\beta = 1/e^2$. The expectation value of an observable $O(A)$ is then

$$\langle O \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} \Pi_{n,\mu} dA_\mu(n) e^{-S(A)} [\det \mathcal{M}(A + A_{ext})]^{1/8} O(A)$$

where $\mathcal{M} = M^\dagger M$, A is the dynamic photon field and A_{ext} is the external photon field while

$$M(A + A_{ext}) = \sum_\mu D_\mu(A + A_{ext}) + m$$

where the operator D_μ is defined by

$$\begin{aligned}[D_\mu(A + A_{ext})\psi](n) &= \frac{1}{2}\eta_\mu(n)\{e^{i(A_\mu(n)+A_{ext,\mu}(n))}\psi(n + \hat{\mu}) \\ &- e^{-i(A_\mu(n-\hat{\mu})+A_{ext,\mu}(n-\hat{\mu}))}\psi(n - \hat{\mu})\}\end{aligned}$$

and η_μ are the staggered phases.

We use the RHMC simulation method of Clark and Kennedy, using rational approximations to $\mathcal{M}^{-1/8}$ and $\mathcal{M}^{\pm 1/16}$. To account for the range of normal modes of the non-compact gauge action, we randomly vary the trajectory lengths over the range of periods of these modes [27]. A_{ext} are chosen in the symmetric gauge in the x-y plane so that the magnetic fields from each plaquette are in the +z-direction and have the value eB modulo 2π. This requires $eB = 2\pi n/(n_1 n_2)$, where n_1 and n_2 are the lattice dimensions in the x and y directions, and n is an integer in the range $[0, n_1 n_2/2]$ [28].

One of the observables we calculate is the electron contribution to the effective gauge action per site $\frac{-1}{8V}\mathrm{trace}[\ln(\mathcal{M})]$. For this we use a rational approximation to ln following Kelisky and Rivlin [29], and a stochastic approximation to the trace.

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