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Feynman Integrals and Relative Cohomologies

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The mathematical field of intersection theory has provided an important new perspective on Feynman integrals and the relations between them. With this approach it is possible to derive such relations with a direct projection, providing a promising alternative to the IBP-based approach. These proceedings will discuss a new development based on *relative cohomology* which is a perfect fit for the mathematical structure of Feynman integrals and which can further accelerate the intersection-based approach to master integral decomposition.

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1. Introduction

Recently, a new and promising approach to the derivations of linear relations between Feynman integrals has been developed. This approach is based on the mathematical concept of the *intersection number* and it works as an alternative to, and with time perhaps a replacement of the traditional IBP-based method. These proceedings will explore a variant of the intersection approach, which utilizes *relative cohomology* that turns out to be a mathematical framework that is more suited to Feynman integrals than the original "non-relative" approach, and which allows for the decomposition of Feynman integrals of higher complexity than was possible before.

The use of intersection theory to describe Feynman integrals was pioneered in ref. [2] by Mastrolia and Mizera. That study was performed on the maximal cut, and so was a follow-up study [3] that performed the reduction of a large variety of Feynman integrals on their maximal cuts. The first applications of intersection theory to a complete reduction were done in refs. [4, 5]. These developments have sparked a lot of interest, and the introduction of relative cohomology is in particular inspired by refs. [6, 7]. The discussion in these proceedings follows an upcoming work, ref. [1] expected to be published later in 2023.

2. Background

In this section, we will give a brief overview of the use of the intersection framework to derive linear relations between Feynman integrals. In parametric representations, having the Baikov representation in mind particularly, a Feynman integral may be written as

$$I = \int_C u\phi \tag{1}$$

where u is a multivalued function of n integration variables, and ϕ a rational n-form. For the case of the (standard) Baikov representation u will be the Baikov polynomial $\mathcal{B}$ raised to a d-dependent power ensuring the multivaluedness, and ϕ will contain the propagators along with a potential numerator factor. In the language of twisted homology and cohomology theory, we may write such an integral as

$$I = \langle \phi | C] \tag{2}$$

where $\langle \phi|$ and $|C]$ are representatives of the twisted cohomology and homology groups respectively, known as cocycles and cycles. For different integrals in the same integral family, the parametrizations can be chosen such that u and C are the same for all members, and the difference is present in ϕ only, so in that language, a master integral decomposition may be written

$$I = \sum c_i I_i \qquad \Leftrightarrow \qquad \langle \phi| = \sum_i c_i \langle \phi_i| \tag{3}$$

The intersection number is a different kind of pairing, not between a cocycle and a cycle but rather between a cocycle and a *dual* cocycle, written as $\langle \phi | \xi \rangle$. For the exact mathematical definition of this object see the mathematical literature e.g. [8, 9] but here let us use an equivalent, derived

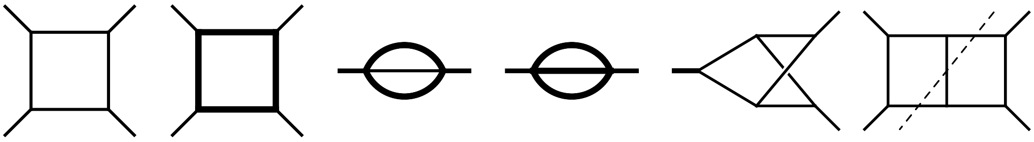

Figure 1: Feynman integrals for which the reduction with the intersection based approach was discussed in ref. [5].

expression valid in the univariate case:

$$\langle\phi|\xi\rangle = \sum_{p\in\mathcal{P}} \mathrm{Res}_{z=p}(\psi\xi) \qquad \text{where} \qquad (\mathrm{d} + \mathrm{d}\log(u))\psi = \phi \tag{4}$$

where $\mathcal{P}$ is defined as the set of poles of $\mathrm{d}\log(u)$. Similar (but more involved) expressions are valid for the multivariate case [4, 5].

The intersection number acts as an *inner product* of the cohomology group, and effectively between Feynman integrals. And with such an inner product, the coefficients of the master integral decomposition may be extracted directly using the *master decomposition formula* [2]

$$c_i = \sum_{j=1}^{\nu} \langle\phi|\xi_j\rangle \left(C^{-1}\right)_{ji} \qquad \text{with} \qquad C_{ij} = \langle\phi_i|\xi_j\rangle \tag{5}$$

With this method, the complete reductions have been performed for a number of Feynman integrals. Until now the state of the art with this approach may be the work of ref. [5] where the integrals reduced are depicted in fig. 1.

Judging from the results depicted in that figure, there is quite a gap from what at this stage can be done with intersection up to the state-of-the-art reductions that can be performed with IBP [10, 11] based tools such as FIRE [12] and Kira [13]. One reason for this discrepancy is a subtlety that I have not discussed: One assumption for the twisted cohomology framework to be valid and the integral of eq. (1) to be described by the language of intersection theory, is that all poles of ϕ are regulated by u. But for uncut Feynman integrals that assumption is not fulfilled as ϕ has poles when the propagators go on shell, in Baikov parametrization corresponding to $z_i = 0$, whereas these points in general will not be zeros of the Baikov polynomial. This can be fixed, as first done in ref. [3], by introducing *regulators* ρ by making the replacement

$$u \;\longrightarrow\; u_{\text{reg}} = u \prod_i z_i^{\rho_i} \tag{6}$$

and use that regulated u for the reductions, and then only at the end put the regulators to zero $\rho \to 0$. While this works, as shown in refs. [4, 5], it has the obvious downside that it introduces a new scale, ρ_i, for each uncut propagator in the problem. Looking for instance at the cut of the (fully massless) double box shown furthest right at fig. 1, what should be a two-scale problem (from kinematic variables s and t) becomes effectively a six-scale problem since a regulator is needed for each of the four uncut propagators.

This motivates the need for a new approach, in which these regulators are no longer needed.

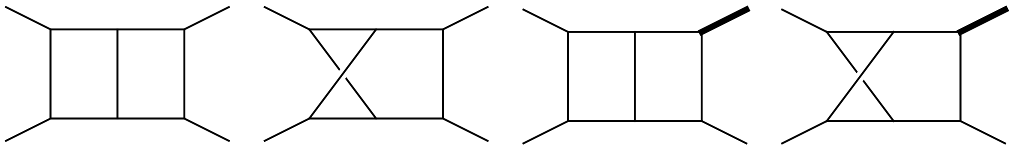

Figure 2: Four examples of Feynman integrals whose reductions, as performed using the relative cohomology framework, are discussed in ref. [1].

3. Relative cohomology

Relative twisted cohomology is such a new approach. In the mathematical literature it was introduced in ref. [14], and first used in the context of Feynman integrals in refs. [6, 7]. In relative cohomology, the space in which the integration variables are defined, is studied modulo a different space, in this context the *zero locus* of the propagators $\bigcup_i (z_i = 0)$. In that way the points which are unregulated by the original u will be excluded *a priori*, removing the need for the ρ-regulators.

What this does in practice is that it allows for the introduction of a new kind of dual form, which we call *delta-forms* $\delta_{z_a z_b \ldots}$. These forms have the property that when using them in an intersection number they act as residue operators. In the univariate case

$$\langle \phi | \delta_z \rangle = \mathrm{Res}_{z=0}(\phi) \tag{7}$$

and for the multivariate (where ξ in a function only of z_{m+1} to z_n)

$$\langle \phi | \delta_{z_1 \ldots z_m} \xi \rangle_n = \langle \mathrm{Res}_{z_1 \ldots z_m = 0}(\phi) | \xi \rangle_{n-m} \tag{8}$$

where the intersection number on the RHS involves m variables less. If ϕ has higher poles in the propagators, a slight refinement is needed that we will not discuss here.

Aside from removing the dependence on the ρ-regulators, this relative setup has a number of benefits. One is that many of the intersection numbers needed for a complete reduction, will, as by the above equation (8), have fewer variables to be treated non-trivially, which simplifies the traditional intersection computation. Likewise, many intersection numbers involving delta-forms are trivially zero (due to the residue). If the master forms ϕ_i are ordered sector by sector, this will make the C-matrix bloc triangular

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & 0 & 0 & \cdots & 0 & 0 \\ C_{21} & C_{22} & 0 & \ddots & 0 & 0 \\ C_{31} & C_{32} & C_{33} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ C_{h1} & C_{h2} & C_{h3} & \cdots & C_{hh} & 0 \\ C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nh} & C_{nn} \end{bmatrix} \tag{9}$$

(where $h = n-1$) which again makes its inversion needed for eq. (5) and elsewhere, much easier to perform.

With this approach based on relative cohomology, we may do the master integral decomposition of Feynman integrals of a much higher degree of complexity compared to the previous method. In ref. [1] we discuss a number of reductions performed with this approach, four of which are depicted in fig. 2. Also, further reductions have been performed, including some of the *spanning cuts* needed for the reduction of the pentabox.

4. Perspectives

There are also further advancements discussed in ref. [1]. One is a polynomial approach to series expanding of the solutions ψ of eq. (4), which avoid the explicit introduction of *algebraic extensions* otherwise needed when specifying $\mathcal{P}$, i.e. the locations where the residues of eq. (4) have to be taken. This approach follows the developments of ref. [15] which also discusses the application of rational reconstruction methods to the computation of the intersection number, something that has been of great use in the public IBP codes, and which will be of equal help in the context of intersection theory.

Another development discussed in ref. [1] is a way of connecting the delta-forms to a limit of the ρ-regulators, thus in a sense bridging the gap between the relative and non-relative approaches.

In order for intersection-based approaches to outcompete IBPs there is still some way left to go. One problem stems from the presence of *magic relations*, which relate integrals in different sectors, thus making master integrals no longer confined to a specific sector. This seems to make the correct delta-forms less well defined, and that contradiction will have to be resolved before the intersection approach becomes generally applicable.

There are also other places where refinements of the approach discussed here might be needed. This might involve the truly multivariate approach to the computation of the intersection number discussed in ref. [16] replacing the fibration-based approach of refs. [4, 5].

Yet the developments discussed here and in ref. [1] do provide a significant step forward, and the day when the intersection-based approach can outcompete IBPs is, in my opinion, within sight.

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