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Recent 3-Loop Heavy Flavor Corrections to Deep-Inelastic Scattering

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We report on recent progress in calculating the three loop QCD corrections of the heavy flavor contributions in deep–inelastic scattering and the massive operator matrix elements of the variable flavor number scheme. Notably we deal with the operator matrix elements $A_{gg,Q}^{(3)}$ and $A_{Qg}^{(3)}$ and technical steps to their calculation. In particular, a new method to obtain the inverse Mellin transform without computing the corresponding N–space expressions is discussed.

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1. Introduction

Since the last Loops and Legs conference in April 2022, [1, 2], a series of new results on two– and three–loop heavy flavor contributions to deep–inelastic scattering has been obtained. They concern the completion of the two–loop corrections to the polarized structure function $g_1(x, Q^2)$ and the two–mass polarized variable flavor number scheme [3], the three–loop unpolarized and polarized single–mass operator matrix elements (OMEs) $(\Delta)A_{gg,Q}^{(3)}$ [4], and first analytic steps beyond first–order factorizable contributions to $A_{Qg}^{(3)}$ in [5], given in Ref. [6]. Herewith, only the constant part in the dimensional parameter $\varepsilon = D - 4$ to the unrenormalized OMEs $(\Delta)A_{Qg}^{(3)}$ have still to be completed as a function of general values of N, to obtain the full description of the heavy flavor contributions to deep–inelastic scattering at three–loop order at large enough virtualities. The known contributions have recently been discussed in Ref. [8]. In this report we will concentrate on the results obtained in Refs. [4, 6].

2. The massive OMEs $A_{gg,Q}^{(3)}$ and $\Delta A_{gg,Q}^{(3)}$

These three–loop OMEs form an essential asset to the description of the three–loop variable flavor number scheme, allowing heavy quarks to become light in the asymptotic region $Q^2 \gg m^2$. With this all OMEs, except $(\Delta)A_{Qg}^{(3)}$, are known completely. A preliminary closed form N-space result for all even moments has been derived by us in 2015 [7]. However, there has been a problem with the analytic continuation to x–space with one Feynman diagram. We finally could obtain this by applying the methods of Ref. [6] in Ref. [4]. The final expression could then be transformed to N–space again, which is of advantage for N–space evolution codes.

The calculation of the gluonic OMEs proceeded by using a series of computation techniques, described in Refs. [63–73] of [4]. The problem at hand is finally characterized by the alphabet

$$\mathfrak{A}_1 = \left\{ \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{\sqrt{1-x}}{x}, \sqrt{x(1-x)}, \frac{1}{\sqrt{1-x}} \right\}. \tag{1}$$

In N–space also different finite (inverse) binomial sums are contributing, an example of which is

$$\mathsf{BS}_9(N) = \sum_{\tau_1=1}^{N} \frac{4^{-\tau_1}(2\tau_1)! \sum_{\tau_2=1}^{\tau_1} \frac{4^{\tau_2}(\tau_2!)^2 \sum_{\tau_3=1}^{\tau_2} \frac{1}{\tau_3}}{(2\tau_2)!\tau_2^2}}{(\tau_1!)^2\tau_1}. \tag{2}$$

These binomial sums obey recursively first–order difference equations and their asymptotic expansions can be computed analytically in their analyticity region. Furthermore, one may calculate their Mellin inversion to x–space analytically and obtain iterated integrals over $\mathfrak{A}_1$ there. The small and large x expansions can be obtained analytically.

Numerically, 50-term expansions around $x = 0$ and $x = 1$ are sufficient to obtain the constant parts of the unrenormalized OMEs $(\Delta)A_{gg,Q}^{(3)}$ to high precision, cf. Figure 1. The so–called BFKL limit thoroughly deviates from the complete results, which is a well–known fact observed in

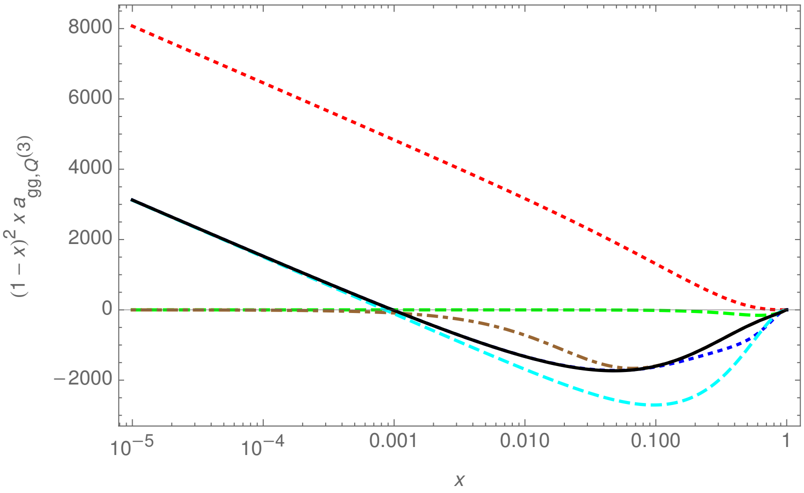

Figure 1: The non–N_F terms of $a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Solid line (black): complete result; upper dotted line (red): term $\propto \ln(x)/x$; lower dashed line (cyan): small x terms $\propto 1/x$; lower dotted line (blue): small x terms including all $\ln(x)$ terms up the constant term; upper dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): full large x contribution; from Ref. [4].

many phenomenological cases [9]. Several sub–leading terms are necessary to yield a quantitative description of the small x region. It is also visible that the expansions around $x = 0$ and $x = 1$ are sufficient to describe the full function as mentioned above.

3. Inverse Mellin transform from a resummed variable

Already in Ref. [4] it has been necessary to work in x–space in part. For this reason the resummation

$$\sum_{N=1}^{\infty} t^N (\Delta.p)^{N-1} = \frac{t}{1 - t\Delta.p} \tag{3}$$

has been performed [10]. This way, the original discrete Mellin variable N is transformed into the continuous variable t. The respective moments are obtained by a formal Taylor expansion. In Ref. [6] we have shown how to obtain from the resummed form the associated inverse Mellin transform in x–space. Since some of the contributing terms are distribution–valued, one should deal with them first. They can be structurally identified in t–space as those leading to the distributions $\propto \delta(1-x)$ and $[\ln^k(1-x)/(1-x)]_+$, $k \in \mathbb{N}, k \geq 0$. The remaining t–space expressions will then lead to regular contributions in x–space, if the complete amplitude is considered. The method of Ref. [6] works for contributions which obey both first–order factorizing and non first-order factorizing differential equations. In the former case the known classes are harmonic polylogarithms, generalized harmonic polylogarithms, cyclotomic harmonic polylogarithms, and iterated integrals also containing square–root valued letters. Non first–order factorizing cases are those starting with ${}_2F_1$–solutions, cf. e.g. [11], and generalizations thereof, obeying even higher order differential equations.

Let us consider an example in the first–order factorizing case, e.g. the regular function

$$\tilde{F}_1(t) = \mathrm{H}_{0,0,1}(t), \tag{4}$$

with $\mathrm{H}_{\vec{a}}(t)$ denoting a harmonic polylogarithm. One obtains

$$F_1(x) = \frac{1}{\pi}\mathsf{Im}\tilde{F}_1\left(\frac{1}{x}\right) = \frac{1}{2}\mathrm{H}_0^2(x). \tag{5}$$

Here one has to consider the monodromy around $t = 1$ only, while in general this is necessary for $t = \pm 1$. The Mellin transform of Eq. (5) is

$$\mathsf{M}[F_1(x)](N) = \int_0^1 dx x^{N-1} F_1(x) = \frac{1}{N^3}. \tag{6}$$

In accordance to this one obtains the formal Taylor series

$$\tilde{F}_1(t) = \sum_{k=1}^{\infty} \frac{t^k}{k^3}. \tag{7}$$

Let us now consider the case of non first–order factorizing differential equations. Here one has to deal with a higher linear system to be decoupled, where at least a 2×2 system remains. As has been shown in Ref. [6], it is important which of the solutions is dealt with first, since the ε–expansion of the solution in x–space may have a simpler structure than the case for other choices.

The alphabets over which the iterated integrals in t–space are obtained do now contain also factors of higher transcendental functions and derivatives thereof, unlike for the first–order factorizing cases. E.g. one has

$$\mathfrak{A}_2 = \left\{\frac{1}{t}, \frac{1}{1-t}, \frac{1}{8+t}, g_1(t), g_2(t), \frac{g_1(t)}{t}, \ldots\right\}, \tag{8}$$

where

$$g_1(t) = \frac{2}{(1-t)^{2/3}(8+t)^{1/3}} {}_2F_1\left[\begin{matrix} \frac{1}{3} & \frac{4}{3} \\ & 2 \end{matrix}; -\frac{27t}{(1-t)^2(8+t)}\right], \tag{9}$$

$$g_2(t) = \frac{2}{(1-t)^{2/3}(8+t)^{1/3}} {}_2F_1\left[\begin{matrix} \frac{1}{3} & \frac{4}{3} \\ & \frac{2}{3} \end{matrix}; 1 + \frac{27t}{(1-t)^2(8+t)}\right]. \tag{10}$$

The closed form solutions now allow to form the corresponding iterative integrals over $\mathfrak{A}_2$ and the analytic continuation of $t \rightarrow \pm 1/x$. In part, regularizations are necessary. Furthermore, iterative integrals at $x = 1$ will occur as constants in the description of the problem. Not all of them reduce to known special numbers and they have to be computed numerically to high precision in the end. The final expressions obtained can now be expanded around the points $x = 0, 1$ and e.g. also 1/2 to obtain fast and highly precise numerical representations in the region $x \in [0, 1]$, cf. Ref. [6], which is necessary for phenomenological applications.

4. Conclusions

The calculation of the unpolarized and polarized single– and two–mass three–loop contributions to deep–inelastic scattering started with a series of moments in 2009 [12] and then turned to the general N and x–space results in the limit $Q^2 \gg m_Q^2$ thereafter. In the region $Q^2/m_Q^2 \gtrsim 10$ this is sufficient for considering the structure function $F_2(x, Q^2)$, cf. [13]. A cut of this kind should be applied because of higher twist contributions in the data at lower scales.

Along with these calculations various mathematical and computer–algebraic technologies were developed, cf. Refs. [14, 15] for surveys. After a very successful treatment of the majority of problems in N space by using the method of arbitrary high Mellin moments [16], the method of guessing [17], and difference field theory as implemented in the package `Sigma` [18, 19], we had to refer to x–space in different cases to obtain the complete result. Here the method described in Ref. [6] has been used extensively. For a brief report on the physical status reached for the heavy flavor corrections see [8]. After having obtained the OMEs $(\Delta)A_{gg,Q}^{(3)}$, current work is dedicated to complete $(\Delta)A_{Qg}^{(3)}$, since now also the massless polarized three–loop Wilson coefficients are available [20] and the unpolarized ones [21] are confirmed. $(\Delta)A_{Qg}^{(3)}$ contains ${}_2F_1$–sectors requiring special attendance. In the internal representation also the different functional contributions to first–order factorizing terms may show exponential growth in the limit $N \rightarrow \infty$, which will cancel, however, between the different mathematical classes of contributions. One is advised to show this analytically.

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PoS(RADCOR2023)046

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