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Infrared singularities of QCD amplitudes with a massive parton

Ze Long Liu[a,*] **and Nicolas Schalch**[b]

[a] *Theoretical Physics Department, CERN, 1211 Geneva 23, Switzerland*

[b] *Institut für Theoretische Physik & AEC, Universität Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland*

E-mail: zelong.liu@cern.ch, schalchn@itp.unibe.ch

We study the anomalous dimensions governing infrared divergences of multi-leg QCD amplitudes with one massive parton up to three loops. The behaviours of the kinematic functions for tripole and quadrupole correlations with a massive parton are studied in details. We analytically calculate the tripole correlation at three loops, and the result is expressed by Harmonic polylogarithms with transcendental weight four and weight five.

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1. Introduction

The infrared (IR) singularities of QCD amplitudes can be formulated to a general form, which can be applied to predict the IR poles and large logarithms for higher-order multi-leg QCD amplitudes. It plays a key role to precise determinations of the parameters in the Standard Model (SM). In addition, analytical results of IR singularities help to deepen our understanding on the structure of perturbative quantum field theory. IR singularities are governed by the soft anomalous dimension matrix, which has been well-known at three loops for multi-leg scattering of massless partons [1], but only at two loops for massive cases [2]. In this work [3], we focus on the three-loop anomalous dimension for QCD amplitudes with one massive parton, especially on the multi-leg correlations, and present the analytical result for the correlation between one massive and two massless partons. It can be used to improve theoretical predictions for top quark productions at hadron colliders.

2. Structure of anomalous dimensions

To investigate IR singularities of a QCD scattering amplitude $|\mathcal{M}(\{\underline{s}\},\epsilon)\rangle$, we start with a slightly off-shell n-parton amputated Green's function $G_n(\{\underline{p}\})$. In the framework of soft-collinear effective theory (SCET) [4–6], it can be factorized as

$$G_n(\{\underline{p}\}) \sim \boldsymbol{S}(\{\underline{\beta}\},\epsilon)\prod_i J(L_i^2,\epsilon)|\mathcal{M}(\{\underline{s}\},\epsilon)\rangle\,, \tag{1}$$

where $\{\underline{p}\}$ denotes the momenta of external legs, and $|\mathcal{M}(\{\underline{s}\},\epsilon)\rangle$ is the corresponding on-shell n-parton amplitude. $\boldsymbol{S}(\{\underline{\beta}\},\epsilon)$ describes the soft correlation between external legs, and $J(L_i^2,\epsilon)$ encodes collinear radiations along the ith massless external leg. $|\mathcal{M}(\{\underline{s}\},\epsilon)\rangle$ and $\boldsymbol{S}(\{\underline{\beta}\},\epsilon)$ are matrices in color space. To indicate the kinematic dependence in (1), we employ cusp angles between different pairs of massless or massive partons

$$\beta_{ij} = L_i + L_j - \ln\frac{\mu^2}{-s_{ij}}\,,\quad \beta_{Ij} = L_j - \ln\frac{m_I\mu}{-s_{Ij}}\,,\quad \beta_{IJ} = \cosh^{-1}\left(\frac{-s_{IJ}}{2m_I m_J}\right)\,, \tag{2}$$

where $L_i = \ln[\mu^2/(-p_i^2 - i0)]$ and $s_{ij} = 2\sigma_{ij}\, p_i\cdot p_j + i0$. The sign factor $\sigma_{ij} = +1$ if the momenta p_i and p_j are both incoming and outgoing, and $\sigma_{ij} = -1$ otherwise. Here and below, indices $I, J\cdots$ indicate massive partons, and lower-cases indices $i, j\cdots$ indicate massless ones. Because the soft and collinear divergences in the Green's function $G_n(\{\underline{p}\})$ are regulated by the off-shellness, the IR poles of $|\mathcal{M}(\{\underline{s}\},\epsilon)\rangle$ cancel with the ultraviolet (UV) poles of $J(L_i^2,\epsilon)$ and $\boldsymbol{S}(\{\underline{\beta}\},\epsilon)$. $J(L_i^2,\epsilon)$ is a color-singlet, and its UV poles can be simply derived from the collinear anomalous dimension $\Gamma_c^i = -\Gamma_{\rm cusp}^i L_i + \gamma_c^i$ [7], which linearly depends on the collinear logarithm. The color structure and kinematic dependence of UV poles of $\boldsymbol{S}(\{\underline{\beta}\},\epsilon)$ are strongly constrained by non-abelian exponentiation theorem and the rescaling invariance of soft Wilson lines. So it is more convenient to study the UV structure of soft function $\boldsymbol{S}(\{\underline{\beta}\},\epsilon)$ instead of studying IR singularities of hard scattering amplitudes directly.

For convenience, we use $\overline{\rm MS}$ renormalization scheme,

$$|\mathcal{M}(\{\underline{s}\},\mu)\rangle = \lim_{\epsilon\to 0}\boldsymbol{Z}^{-1}(\epsilon,\{\underline{s}\},\mu)|\mathcal{M}(\epsilon,\{\underline{s}\})\rangle\,, \tag{3}$$

where renormalization factors are determined by the anomalous dimensions as follow [8, 9]

$$\boldsymbol{Z}(\epsilon, \{\underline{p}\}, \{\underline{m}\}, \mu) = \mathbf{P} \exp\left[\int_{\mu}^{\infty} \frac{d\mu'}{\mu'} \boldsymbol{\Gamma}(\{\underline{p}\}, \{\underline{m}\}, \mu')\right], \tag{4}$$

Similarly, the soft and collinear matrix elements are renormalized by the factors $\boldsymbol{Z}_s$ and Z_J, respectively. The cancellation of poles implies that

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_s + \sum_i \Gamma_c^i \, \mathbf{1}, \tag{5}$$

where $\boldsymbol{\Gamma}_s$ and Γ_c^i denote the soft and collinear anomalous dimensions, respectively.

First, we discuss kinematic dependences of anomalous dimensions. Because the hard, soft and collinear interactions are decoupled in soft-collinear factorization, the anomalous dimension of hard scattering amplitudes does not depend on the collinear scale p_i^2. The eq. (5) implies that the soft matrix elements also linearly depends on L_i, i.e.

$$\frac{\partial \boldsymbol{\Gamma}_s}{\partial L_i} = -\frac{\partial \Gamma_c^i}{\partial L_i} \mathbf{1}. \tag{6}$$

Therefore soft matrix elements can only linearly depend on cusp angles $\{\underline{\beta}\}$, or depend on conformal cross ratios, e.g. $\beta_{ijkl} = \beta_{ij} + \beta_{kl} - \beta_{ik} - \beta_{jl}$ [9, 10], where all the collinear scales p_i^2 cancel out. For the tripole correlation between one massive and two massless partons, the only conformal cross ratio is give by

$$r_{ijI} \equiv \frac{v_I^2 \, (n_i \cdot n_j)}{2\,(v_I \cdot n_i)(v_I \cdot n_j)} \quad \text{with} \quad i \neq j\,, \tag{7}$$

where $v_I = p_I / m_I$ is the four-velocity of massive parton I, and $n_{i(j)}$ is the light-like unit vector along the momentum of massless parton $i(j)$. For the quadrupole correlation involving partons with indices $\{i, j, k, I\}$, the kinematic functions are determined by three independent cross ratios r_{ijI}, r_{ikI} and r_{jkI}.

Second, soft anomalous dimensions only receive contribution from maximally non-abelian parts of the conventional color factors, based on non-abelian exponentiation theorem [11, 12], which has already been generalized to multi-leg scattering [14, 19]. So we only need to consider fully connected color factors to build $\boldsymbol{\Gamma}_s$, and their attachments to Wilson lines can be further symmetrized by applying $[\boldsymbol{T}_i^a, \boldsymbol{T}_i^b] = i f^{abc} \boldsymbol{T}_i^c$ repeatedly. Up to three-loop order, we need to consider the following color structures

$$\mathcal{D}_{ij} = \boldsymbol{T}_i^a \boldsymbol{T}_j^a \equiv \boldsymbol{T}_i \cdot \boldsymbol{T}_j\,, \quad \mathcal{T}_{ijk} = i f^{abc} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c\right)_+, \quad \mathcal{T}_{ijkl} = f^{ade} f^{bce} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d\right)_+, \tag{8}$$

where $\left(\boldsymbol{T}_{i_1}^{a_1} \ldots \boldsymbol{T}_{i_n}^{a_n}\right)_+ \equiv 1/n! \sum_\sigma \boldsymbol{T}_{i_{\sigma(1)}}^{a_{\sigma(1)}} \ldots \boldsymbol{T}_{i_{\sigma(n)}}^{a_{\sigma(n)}}$, and σ goes through all the permutations of n objects. To indicate massive partons, we use $I, J, \ldots$ in the subscripts. Actually, $\mathcal{T}_{ijI}$ does not contribute to $\boldsymbol{\Gamma}_s$, because it is anti-symmetric under interchange of the two massless legs, while the corresponding kinematic function is symmetric. In addition, color conservation $\sum_i \boldsymbol{T}_i + \sum_I \boldsymbol{T}_I = 0$ in the color-space formalism [15] implies

$$\mathcal{T}_{ijII} = \frac{1}{2}\left(\mathcal{T}_{jjiI} + \mathcal{T}_{iijI}\right) - \frac{1}{2} \sum_{k \neq i,j} \left(\mathcal{T}_{ijkI} + \mathcal{T}_{jikI}\right) - \frac{1}{2} \sum_{J \neq I} \left(\mathcal{T}_{ijIJ} + \mathcal{T}_{jiIJ}\right), \tag{9}$$

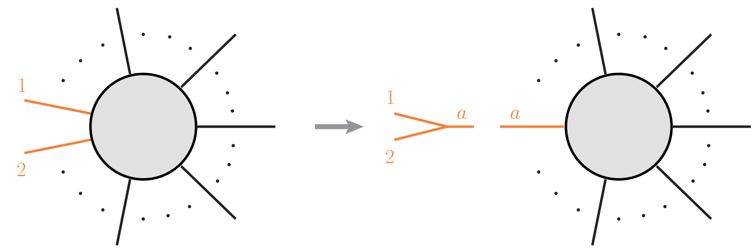

Figure 1: Two-particle collinear limit.

The last term vanishes if only one massive parton is involved in the scattering. Eventually, based on soft-collinear factorization and non-abelian exponentiation theorem, three-loop anomalous dimensions for QCD amplitudes are constrained to be

$$\begin{aligned}\mathbf{\Gamma}\left(\{\underline{p}\},\{\underline{m}\},\mu\right) &= \sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\,\gamma_{\text{cusp}}(\alpha_s)\ln\frac{\mu^2}{-s_{ij}} + \sum_{I,j}\boldsymbol{T}_I\cdot\boldsymbol{T}_j\,\gamma_{\text{cusp}}(\alpha_s)\ln\frac{m_I\mu}{-s_{Ij}} \\ &\quad - \sum_{(I,J)}\frac{\boldsymbol{T}_I\cdot\boldsymbol{T}_J}{2}\,\gamma_{\text{cusp}}(\beta_{IJ},\alpha_s) + \sum_i \gamma^i(\alpha_s)\,\mathbf{1} + \sum_I \gamma^I(\alpha_s)\,\mathbf{1} \\ &\quad + f(\alpha_s)\sum_{(i,j,k)}\mathcal{T}_{iijk} + \sum_{(i,j,k,l)}\mathcal{T}_{ijkl}\,F_4(\beta_{ijkl},\beta_{ijkl}-2\beta_{ilkj},\alpha_s) \\ &\quad + \sum_I\sum_{(i,j)}\mathcal{T}_{ijII}\,F_{\text{h2}}(r_{ijI},\alpha_s) + \sum_I\sum_{(i,j,k)}\mathcal{T}_{ijkI}\,F_{\text{h3}}(r_{ijI},r_{ikI},r_{jkI},\alpha_s) \\ &\quad + [\text{non-dipole contributions involving two or more massive partons}] + \mathcal{O}(\alpha_s^4)\,. \end{aligned} \tag{10}$$

Here $\gamma_{\text{cusp}}(\alpha_s)$ and $\gamma_{\text{cusp}}(\beta_{IJ},\alpha_s)$ are the light-like and angle-dependent cusp anomalous dimensions, respectively. $\gamma^{q(g)}$ and γ^Q denote anomalous dimensions for massless and massive partons, respectively. The first three lines in eq. (10) have been presented and calculated in [1, 8–10, 23]. $\mathcal{T}_{ijkl}$ has symmetry properties as

$$\mathcal{T}_{ijkl} = \mathcal{T}_{jilk} = -\mathcal{T}_{ikjl} = -\mathcal{T}_{ljki} = \mathcal{T}_{klij}\,. \tag{11}$$

Together with the Jacobi identity

$$\mathcal{T}_{iklj} + \mathcal{T}_{iljk} + \mathcal{T}_{ijkl} = 0\,, \tag{12}$$

the kinematic functions F_{h3} can be rewritten as an odd function, i.e. $F_{\text{h3}}(x,y,z,\alpha_s) = -F_{\text{h3}}(y,x,z,\alpha_s)$.

The kinematic functions F_4, F_{h2} and F_{h3} are constrained by two-particle collinear limits, and F_{h2} and F_{h3} are also constrained by small-mass limits. As shown in fig. 2, when two massless particles become collinear in an $n-$particle scattering amplitude, it can be factorized into the corresponding $(n-1)$-particle amplitude multiplied by the $1 \to 2$ splitting function, i.e.

$$|\mathcal{M}(\{p_1,p_2,\ldots,p_n\}\epsilon)\rangle \simeq \mathbf{Sp}(\{p_1,p_2\})|\mathcal{M}(\{p_a,\ldots,p_n\}\epsilon)\rangle\,. \tag{13}$$

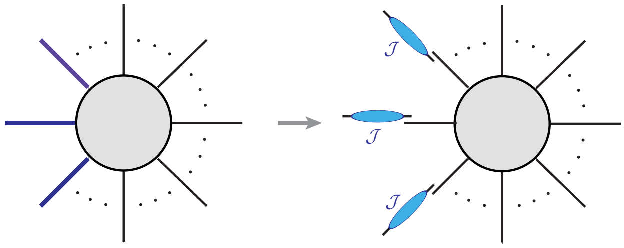

Figure 2: Small-mass limit.

where p_1 and p_2 are the momenta of any two of the particles becoming collinear, and $p_a = p_1 + p_2$. By using color conservation, we have

$$
\begin{aligned}
\mathbf{\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu) =& \mathbf{\Gamma}\left(\{p_1,p_2,\dots,p_n\},\{\underline{m}\},\mu\right) - \mathbf{\Gamma}\left(\{p_a,\dots,p_n\},\{\underline{m}\},\mu\right)\\
=& \Gamma_{\rm cusp}(\alpha_s)\left[\boldsymbol{T}_1\cdot\boldsymbol{T}_2\left(\ln\frac{\mu^2}{-s_{12}} + \ln[z(1-z)]\right) + C_{R_1}\ln z + C_{R_2}\ln(1-z)\right]\\
&+ \left[\gamma^1(\alpha_s) + \gamma^2(\alpha_s) - \gamma^a(\alpha_s)\right]\mathbf{1}\\
&+ \left[f(\alpha_s) + 4F_4(\omega_{ij},\omega_{ij},\alpha_s)\right]\left(-\frac{C_A^2}{4}\boldsymbol{T}_1\cdot\boldsymbol{T}_2 - 2\mathcal{T}_{1122}\right)\\
&+ 4\sum_{i\neq 1,2}\mathcal{T}_{12ii}\left[f(\alpha_s) - 2F_4(\omega_{ij},\omega_{ij},\alpha_s)\right]\\
&+ 2\sum_I \mathcal{T}_{12II}\left[F_{\rm h2}(0,\alpha_s) - f(\alpha_s) - 4F_4(\omega_{ij},\omega_{ij},\alpha_s)\right]\\
&+ 2\sum_I\sum_{i\neq 1,2}\left(\mathcal{T}_{12iI} + \mathcal{T}_{21iI}\right)\left[F_{\rm h3}(0,r_{1iI},r_{1iI},\alpha_s) - 4F_4(\omega_{ij},\omega_{ij},\alpha_s)\right] + \cdots\,,
\end{aligned}
\tag{14}
$$

Since the splitting function only depends on p_1 and p_2, the last three lines in eq. (14) have to vanish. Then we have

$$
\lim_{\omega\to-\infty} F_4(\omega,\omega,\alpha_s) = \frac{f(\alpha_s)}{2}\,, \qquad F_{\rm h2}(0,\alpha_s) = 3f(\alpha_s)\,, \qquad F_{\rm h3}(0,r,r,\alpha_s) = 2f(\alpha_s)\,, \tag{15}
$$

In addition, when the masses of the external particles are much smaller than the hard scales, the amplitudes factorize into the corresponding massless amplitude multiplied by jet functions [16, 17], which account for collinear singularities and only depend on the information of the corresponding heavy quarks. This allow us to constrain $F_{\rm h3}$ similarly as what we did in the collinear limit, e.g.

$$
\lim_{v_I^2\to 0} F_{\rm h3}(r_{ijI},r_{ikI},r_{jkI},\alpha_s) = 2f(\alpha_s) + 4F_4(\beta_{ijkI},\beta_{ijkI} - 2\beta_{kjiI},\alpha_s)\,. \tag{16}
$$

3. Calculation at three loops

The two new terms presented in the last line of eq. (10) describes the tripole and quadrupole correlations with a single massive parton for the first time. In this work, we calculate the tripole kinematic function

$$F_{\rm h2}(r,\alpha_s) = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathcal{F}_{\rm h2}(r) + O(\alpha_s^4) \tag{17}$$

analytically at three loops. It is more convenient to compute $\mathcal{F}_{\rm h2}(r)$ using soft matrix element because of the simplicity of Feynman integrals with linear propagators. In order to isolate the UV poles, a proper IR regulator has to be introduced. Though some regulators have been used to extract the UV poles in two-loop massive and three-loop massless scattering, the calculations of Feynman integrals are extremely complicated because extra scales are involved. Here we employ a novel regularization to obtain UV poles. We focus on the soft function from the threshold factorization for one-particle inclusive single top quark production associated with a color singlet in hadron collisions. It is defined as

$$\boldsymbol{S}(\omega) = \langle 0|\overline{\mathrm{T}}\left[\boldsymbol{Y}_{n_1}^\dagger \boldsymbol{Y}_{n_2}^\dagger \boldsymbol{Y}_v^\dagger\right]\delta(\omega - v\cdot\hat{p})\mathrm{T}\left[\boldsymbol{Y}_{n_1}\boldsymbol{Y}_{n_2}\boldsymbol{Y}_v\right]|0\rangle\,, \tag{18}$$

where $\boldsymbol{Y}_{n(v)} = \boldsymbol{Y}_{n(v)}(0)$ denotes semi-infinite soft Wilson lines along $n^\mu(v^\mu)$ direction, $\hat{p}^\mu$ is the momentum operator picking up the total momentum of all soft emissions in final states, and $\mathrm{T}(\overline{\mathrm{T}})$ indicates (anti-)time ordering. In SCET, the soft function defined at cross-section level are only UV divergent, because the IR poles cancel out after all the diagrams are added up. ω is the only dimensionful variable involved in eq. (18), so it can be factored out. In principle, the calculation of the soft function at three loops can be categorized into three parts: triple real emissions at tree level, double real emissions at one loop, and single real emission at two loops. However, there is a way to compute the soft function in terms of loop integrals by rewriting its definition in eq. (18) as [18]

$$\boldsymbol{S}(\omega) = \frac{1}{2\pi}\mathrm{Re}\left[\boldsymbol{\Sigma}(\omega + i0) - \boldsymbol{\Sigma}(\omega - i0)\right]\,, \tag{19}$$

with

$$\boldsymbol{\Sigma}(\omega) = \int_0^\infty dt\, e^{i\omega t}\langle 0|\mathrm{T}\left[\boldsymbol{Y}_{n_1}^\dagger(tv)\boldsymbol{Y}_{n_2}^\dagger(tv)\mathrm{P}\exp\left[ig\int_0^t ds\, v\cdot A^c(sv)\boldsymbol{T}_v^c\right]\boldsymbol{Y}_{n_1}(0)\boldsymbol{Y}_{n_2}(0)\right]|0\rangle\,. \tag{20}$$

This allowed us to straightforwardly take advantage of modern multi-loop technology.

To calculate $\mathcal{F}_{\rm h2}(r)$, only color-connected diagrams contributing to the color factors $\mathcal{T}_{iijI}$, $\mathcal{T}_{ijII}$ and $\mathcal{T}_{iiII}$ are taken into account. The diagrams are organised web by web, where mixing matrices can be obtained by the replica trick [19]. Though the replica trick was employed at amplitude level, it is still compatible at cross-section level. We show how to employ the replica trick on our soft function in fig. 3. The amplitude on the left hand side is the same as the eight subdiagram on right hand side, so we can use the same mixing matrix to project their maximally non-abelian part out.

The Feynman diagrams are calculated in general covariant gauge. After IBP reduction, there are 173 linear independent master integrals (MIs), which can be evaluated by using the method. of differential equation (DE). Then, the DE systems are transformed to an ε-form [20], and the MIs are solved iteratively order-by-order in ϵ with the alphabet $\{r, r-1, r-2, (r-1)\sqrt{r}, \sqrt{r(r-1)}\}$. The boundary conditions are chosen at $r = 1$, and we use dimensional recurrence relations to relate all the MI to a set of quasi-finite integrals in dimension $d = n - 2\epsilon$ $(n = 4, 6, 8, \dots)$, which can be evaluated by using `HyperInt` [21]. Finally, all the Feynman integrals are expressed by linear combinations of Goncharov Polylogarithms (GPLs) and the generalized harmonic polylogarithms (GHPLs).

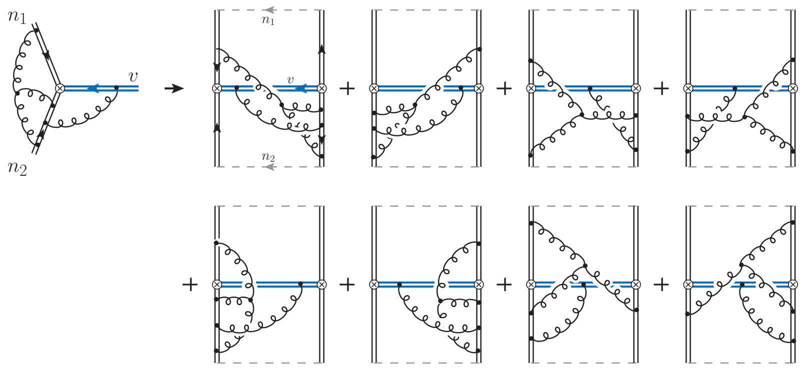

Figure 3: A sample for the comparison between diagrams of the soft correlator and the soft function at cross-section level in (19). For the subdiagrams on right hand side, the external and internal double lines denote the semi-infinite and finite-length soft Wilson lines in (20), respectively. Each diagram on the right hand side has the same color factor as the one on the left.

4. Result

After replacing the MIs in $\mathcal{F}_{h2}(r)$ with their analytical results, we find that all the ϵ poles drop out, and only GPLs appear in the final expression. The final expression can be remarkably simplified to

$$\begin{aligned}\mathcal{F}_{h2}(r) =& 128\Big[H_{-1,0,0,0} + H_{-1,1,0,0} + H_{1,-1,0,0} - H_{1,0,0,0}\Big] + 128\,(\zeta_2+\zeta_3)\,\Big[H_{1,0} - H_{-1,0}\Big] \\ &+ 96\,(\zeta_3+\zeta_4)\,\Big[H_{-1} - H_1\Big] + 128\zeta_2\Big[H_{-2,0} - H_{2,0} + H_{-1,0,0} - H_{1,0,0}\Big] \\ &+ 256\Big[H_{1,2,0,0} + H_{2,0,0,0} - H_{-2,0,0,0} + H_{-1,-2,0,0} - H_{-1,2,0,0} - H_{1,-2,0,0} \\ &\quad - H_{-1,0,0,0,0} + H_{1,0,0,0,0}\Big] + 48\,(2\zeta_2\zeta_3+\zeta_5)\;,\end{aligned} \tag{21}$$

where $H_{\vec{a}} \equiv H_{\vec{a}}(\sqrt{r})$ are the harmonic polylogarithms (HPLs) [22]. This result is consistent with the constraints of two-particle collinear limit and small-mass limit in eqs. (15) and (16).

5. Summary

In this work, we present the general structure of the soft anomalous dimension for QCD amplitudes with a single massive parton up to three loops. The color structures and corresponding kinematic dependences for tripole and quadrupole correlations are presented for the first time, and their behaviours in two-particle collinear limit and small-mass limit are discussed in details. To extract the UV poles of soft matrix elements, we employ a novel IR regulator at cross-section level. By using modern techniques for multi-loop calculations, e.g. differential equation method and dimensional recurrence relations, we compute the three-loop kinematic function for the tripole correlation, the analytical result of which can be remarkably simplified to a linear combination of HPLs with transcendental weight four and weight five.

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