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IBPs and differential equations in parameter space

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We present a projective framework for the construction of Integration by Parts (IBP) identities and differential equations for Feynman integrals, working in Feynman-parameter space. This framework originates with very early results which emerged long before modern techniques for loop calculations were developed [17–22]. Adapting and generalising these results to the modern language, we use simple tools of projective geometry to generate sets of IBP identities and differential equations in parameter space, with a technique applicable to any loop order. We test the viability of the method on simple diagrams at one and two loops, providing a unified viewpoint on several existing results.

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1. Introduction

The calculation of high-order Feynman integrals is crucial for the current and future precision physics programs at particle accelerators [1]: to this end, modern methods have evolved that go much beyond a direct evaluation. These developments began with the discovery of Integration-by-Parts (IBP) identities in dimensional regularization [2, 3], and the introduction of the method of differential equations [4–6]: calculations are now typically tackled with automatic algorithms combining these ideas [7]. Further developments have involved an enhanced understanding of the role of generalised unitarity, and of the linear functional spaces where classes of Feynman integrals reside [8–11], and an optimised use of dimensional regularization [12]. A vast amount of work along these lines has significantly expanded the range of processes for which high-order calculations are possible, and has much deepened our mathematical understanding of Feynman integrals [1, 13].

Interestingly, the historical exploration of Feynman integrals via IBPs and differential equations predates these recent developments. Indeed, the projective nature of Feynman parameter integrands and the monodromy properties of Feynman integrals attracted early attention from mathematicians and physicists already in the late 1960's [14–17]. Around that time, in particular, Tullio Regge and his collaborators explored the monodromy properties of various classes of Feynman integrals [17–20], offering several insights that align with (and predate) contemporary findings. For example Regge argued, already at the time [17], that Feynman integrals belong to a class of generalised hypergeometric functions, and proposed that these functions should satisfy differential equations of the Picard-Fuchs type.

Although actual computational algorithms didn't emerge from these studies, Barucchi and Ponzano [21, 22] constructed an explicit implementation of Regge's ideas, applicable to one-loop diagrams. The corresponding Feynman integrals, in parametric form, were organised into sets connected by difference equations, akin to currently used IBPs; linear systems of homogeneous differential equations in the Mandelstam invariants were then derived, mirroring the known one-loop monodromy structure.

This note, summarising the results presented in [24], builds on the work of Regge and collaborators, to propose a projective framework for deriving IBP identities and systems of linear differential equations for Feynman integrals, directly in parameter space. The framework accommodates dimensional regularisation, extends to infrared-divergent integrals, and generalises naturally beyond on loop. Interestingly, our results also underscore the role of boundary terms in IBP identities within the projective framework: unlike the momentum-space approach in dimensional regularization, these terms do not generally vanish in parameter space, and indeed they play a critical role in connecting complex integrals to simpler ones.

We begin our note by setting up conventions for Feynman integrals in parameter form, in Section 2. Next, in Section 3, we introduce projective forms, and we use their properties to show how one can construct systems of difference equations for generic projective integrals. In Section 4 we focus on Feynman integrals, and provide a general procedure to construct IBPs in this context, developing the one-loop case in some detail as an example. Explicit one-loop examples are presented in Section 5, and two-loop examples in Section 6. Finally, Section 7 briefly discusses perspectives for future work.

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2. Notations

Scalar Feynman integrals arise in loop-level perturbative calculations in any quantum filed theory. In a momentum space formulation they take the form

$$I_G\left(\nu_i, d\right) = (\mu^2)^{\nu - ld/2} \int \prod_{r=1}^{l} \frac{d^d k_r}{\mathrm{i}\pi^{d/2}} \prod_{i=1}^{n} \frac{1}{\left(-q_i^2 + m_i^2\right)^{\nu_i}} , \qquad q_i = \sum_{r=1}^{l} \alpha_{ir} k_r + \sum_{j=1}^{m} \beta_{ij} p_j , \tag{1}$$

where q_i are the momenta flowing in each propagator, k_r are the independent loop momenta, and p_j are the external momenta, while d is the space-time dimension, and the integer exponents ν_i satisfy $\sum_i \nu_i = \nu$. The integration over loop momenta in Eq. (1) can be performed in full generality by means of the Feynman parameter technique. Using the notations from Refs. [13, 25, 26], the integral becomes

$$I_G\left(\nu_i, d\right) = \frac{\Gamma(\nu - ld/2)}{\prod_{j=1}^{n} \Gamma(\nu_j)} \int_{z_j \geq 0} d^n z\, \delta\left(1 - \sum_{j=1}^{n} z_j\right) \left(\prod_{j=1}^{n} z_j^{\nu_j - 1}\right) \frac{\mathcal{U}^{\,\nu - (l+1)d/2}}{\mathcal{F}^{\,\nu - ld/2}} , \tag{2}$$

where the Symanzik polynomials $\mathcal{U}$ and $\mathcal{F}$,

$$\mathcal{U} = \sum_{\mathcal{T}_G} \prod_{i \in \mathcal{T}_G} z_i , \qquad \mathcal{F} = \sum_{C_G} \frac{\hat{s}\left(C_G\right)}{\mu^2} \prod_{i \in C_G} z_i - \mathcal{U} \sum_{i \in \mathcal{I}_G} \frac{m_i^2}{\mu^2} z_i , \tag{3}$$

can be defined purely from the graph properties. To this end, let us denote by $\mathcal{I}_G$ the set of the internal lines of G, each endowed with a Feynman parameter z_i. A *co-tree* $\mathcal{T}_G \subset \mathcal{I}_G$ is a set of internal lines of G such that that the lines in its complement $\overline{\mathcal{T}}_G \subset \mathcal{I}_G$ form a spanning tree. Similarly, consider subsets $C_G \subset \mathcal{I}_G$ with the property that, upon omitting the lines of C_G from G, the graph becomes a disjoint union of two connected subgraphs. Each subset C_G defines a *cut* of graph G, and contains $l + 1$ lines; an invariant mass $\hat{s}\left(C_G\right)$ can be associated with each cut, by squaring the sum of the momenta flowing in (or out) of one of the two subgraphs. The Symanzik polynomial $\mathcal{U}$ is homogeneous of degree l, while the Symanzik polynomial $\mathcal{F}$ is homogeneous of degree $l + 1$, so that the integrand (measure included) is homogeneous of degree 0.

3. A projective framework

A crucial mathematical property of Feynman integrals is that their integrands are projective forms in the space of Feynman parameters, which can be identified with $\mathbb{PC}^{n-1}$. This property is crucial for the characterisation of the function spaces to which the integrals belong, and to many techniques for their explicit evaluation. The relevance of projective invariance was understood since the earliest systematic studies of Feynman diagrams [17]. In this Section, we provide an extremely concise summary of the relevant ideas.

In order to introduce projective forms, begin by considering a generic subset A, $|A| = a$, of the set $D = \{1, \ldots, N\}$, and define the a-form

$$\omega_A = dz_{i_1} \wedge \ldots \wedge dz_{i_a} , \tag{4}$$

with $i_1 < \ldots < i_a$. One can show that ω_A integrates to the projective $(a-1)$-form

$$\eta_A = \sum_{i \in A} \epsilon_{i, A-i}\, z_i\, \omega_{A-i}\,, \qquad d\eta_A = a\omega_A\,, \tag{5}$$

where we introduced the signature factor

$$\epsilon_{k,B} = (-1)^{|B_k|}\,, \qquad B_k = \{i \in B, i < k\}\,. \tag{6}$$

As an example, for $A = \{1,2,3\}$ one finds

$$\eta_{\{1,2,3\}} = z_1\, dz_2 \wedge dz_3 - z_2\, dz_1 \wedge dz_3 + z_3\, dz_1 \wedge dz_2\,. \tag{7}$$

Projective forms such as η_A are homogeneous of degree 1 in each coordinate z_i, and they can serve as measures of integration for projective integrals. Indeed, parametric Feynman integrands can be represented in the general form

$$\alpha_{n-1} = \eta_{n-1}\, \frac{Q\big(\{z_i\}\big)}{D^P\big(\{z_i\}\big)}\,, \tag{8}$$

where $D\big(\{z_i\}\big)$ and $Q\big(\{z_i\}\big)$ are polynomials of degrees such that the form (measure included) is homogeneous of degree 0. A well-known example is the integrand for the massless one-loop box integral, which reads

$$\psi_3\,(\lambda, r) = \frac{(z_1 + z_2 + z_3 + z_4)^\lambda}{(r\, z_1 z_3 + z_2 z_4)^{2+\lambda/2}}\, \eta_{\{1,2,3,4\}}\,. \tag{9}$$

For finite integrals both P in Eq. (8) and ν in Eq. (2) are integers. In the presence of divergences, as is the case for the massless box, we can incorporate dimensional regularisation by allowing for general values of d, and thus of λ in Eq. (9).

Two theorems naturally emerge within this projective framework. First of all, an essential property of projective forms is the following [17].

Theorem 1. *The boundary of a projective form is itself projective.*

This theorem arises from the properties of the operator

$$p\;:\; \sum_{|A|=q} R_A(z_i)\, \omega_A \quad \to \quad \sum_{|A|=q} R_A(z_i)\, \eta_A\,, \tag{10}$$

mapping affine q-forms into projective $(q-1)$-forms. It can be shown to satisfy

$$p^2 = 0\,, \qquad d \circ p + p \circ d = 0\,. \tag{11}$$

Based on these properties, a proof of the theorem can be found in [24].

Next, it is possible to show that α_{n-1} is a closed form, while η_{n-1} is null on any surface defined by $z_i = 0$. A second theorem then follows

Theorem 2. *Given two integration domains, $O, O' \in \mathbb{C}^n$, if their image in $\mathbb{PC}^{n-1}$ is the same simplex, then $\int_O \alpha_{n-1} = \int_{O'} \alpha_{n-1}$.*

This theorem, also known as Cheng-Wu theorem [23], allows, in practice, to set to zero any subset of the n parameters z_i in the argument of the δ function in Eq. (2), providing a useful tool for concrete calculations.

4. Integration by parts in projective space

The correspondence between projective forms and parametric integrals is obtained from the usual choice of chart in projective space identifying a coordinate symplex in $\mathbb{R}^n$ with the choice $\sum_{i=1}^n z_i = 1$. With this choice one finds simply

$$\int_{S_{n-1}} \eta_{n-1} \, \frac{Q(z)}{D^P(z)} \; = \; \int_{z_i \geq 0} dz_1 \ldots dz_n \; \delta\left(1 - \sum_{i=1}^n z_i\right) \frac{Q(z)}{D^P(z)} \, . \tag{12}$$

We now show how the projective structure just introduced allows to easily construct sets of difference equations connecting families of Feynman integrals, which play the role of the conventional IBP identities usually derived in momentum space. To this end, consider the projective $(n-2)$-forms

$$\omega_{n-2} \; \equiv \; \sum_{i=1}^n (-1)^i \, \eta_{\{z\}-z_i} \, \frac{H_i(z)}{(P-1)\,\big(D(z)\big)^{P-1}} \, , \tag{13}$$

where $\eta_{\{z\}-z_i}$ denotes the projective volume form in $\mathbb{PC}^{n-2}$, obtained by omitting the coordinate z_i, and $H_i(z)$ are polynomials with a degree chosen (together with P) to ensure projectivity. Differentiating these forms generates (at the integrand level) a set of identities among parametric integrals, which correspond to those obtained via integration by parts. One finds

$$d\omega_{n-2} \; = \; \frac{1}{(P-1)\,\big(D(z)\big)^{P-1}} \, \eta_{\{z\}} \sum_{i=1}^n \frac{\partial H_i(z)}{\partial z_i} \; - \; \frac{\eta_{\{z\}}}{\big(D(z)\big)^P} \sum_{i=1}^n H_i \, \frac{\partial D(z)}{\partial z_i} \, . \tag{14}$$

Eq. (14) plays a central role in our method. By suitably choosing the polynomials $H_i(z)$, it allows to close systems of linear differential equations for Feynman integrals that can be used to compute them, just as usually done in the momentum space approach. We emphasise that these identities apply for any number of loops or external legs. In the remainder of this section, we will discuss a concrete implementation at the one-loop level developing the ideas of Ref. [21].

At one loop, parametric integrals have the general form

$$I_G(\nu_i, d) \; = \; \frac{\Gamma(\nu - d/2)}{\prod_{j=1}^n \Gamma(\nu_j)} \int_{z_j \geq 0} d^n z \, \delta\big(1 - z_{n+1}\big) \, \frac{\prod_{j=1}^{n+1} z_j^{\nu_j - 1}}{\left[\sum_{i=1}^{n+1} \sum_{j=1}^{i-1} s_{ij} z_i z_j\right]^{\nu - d/2}} \, , \tag{15}$$

where we introduced the notations

$$z_{n+1} \; \equiv \; \sum_{i=1}^n z_i \, , \qquad \qquad \nu_{n+1} \; \equiv \; \nu - d + 1 \, , \tag{16}$$

and for the Mandelstam invariants we use

$$s_{ij} \; = \; \frac{(q_j - q_i)^2}{\mu^2} \quad (i, j = 1, \ldots, n) \, , \qquad \qquad s_{i,n+1} \; = \; s_{n+1,i} \; \equiv \; -\frac{m_i^2}{\mu^2} \, . \tag{17}$$

We now make the simplest and natural choice in Eq. (14), picking the polynomials $H_i(z)$ to coincide with the numerator of the relevant integral, for each value of i. Thus we pick

$$H_i \; = \; \delta_{ih} \left(\prod_{j=1}^n z_j^{\nu_j - 1} \right) \left(\sum_{k=1}^n z_k \right)^{\nu - d} \; = \; \delta_{ih} \prod_{j=1}^{n+1} z_j^{\nu_j - 1} \, , \tag{18}$$

for $h = 1, \ldots, n$. Applying this choice produces a one-loop 'integration-by-parts' identity that can be written as follows [21]:

$$\begin{aligned} d\omega_{n-2} + \sum_{k=1}^{n+1} (s_{kh} + s_{k,n+1}) \, f\big(\{\mathcal{R} - k\}_0 \,, \{k\}_1\big) \; &= \; \frac{\nu_h - 1}{\nu - (d+1)/2} \, f\big(\{h\}_{-1} \,, \{\mathcal{R} - h\}_0\big) \\ &+ \; \frac{\nu - d}{\nu - (d+1)/2} \, f\big(\{n+1\}_{-1} \,, \{\mathcal{R} - \{n+1\}\}_0\big) \,, \end{aligned} \tag{19}$$

where, following Ref. [21], we introduced an index notation such that, for the function

$$f\big(\{\nu_1, \ldots, \nu_{n+1}\}\big) \; \equiv \; f\big(\{\mathcal{R}\}\big) \; = \; \eta_{\{z\}} \, \frac{\prod_{j=1}^{n+1} z_j^{\nu_j - 1}}{\left(\sum_{i=1}^{n+1} \sum_{j=1}^{i-1} s_{ij} z_i z_j\right)^{\nu - d/2}} \,, \tag{20}$$

we can raise or lower the exponents ν_i by adding $\{-1, 0, 1\}$ in the subsets $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ of set $\mathcal{R}$ respectively, and we denote the resulting function by

$$f\big(\{\mathcal{I}\}_{-1} \,, \{\mathcal{J}\}_0 \,, \{\mathcal{K}\}_1\big) \,. \tag{21}$$

Note that the exponent of the denominator in Eq. (20) is adjusted accordingly, to maintain projective properties. Note also that the action of raising and lowering exponents according to the convention in Eq. (21) is subject to a constraint, arising from the definition of $\mathcal{U}$ in Eq. (3). Specifically, the following sum rule holds

$$\sum_{i=1}^{n} f\big(\{\mathcal{R} - i\}_0 \,, \{i\}_1\big) \; = \; f\big(\{\mathcal{R} - \{n+1\}\}_0 \,, \{n+1\}_1\big) \,. \tag{22}$$

As we will see in the next section, Eq. (19) and Eq. (22) can be used to close systems of differential equations, leading to the determination of the one-loop Feynman integrals under study. Two-loop examples will be discussed in Section 6.

5. One-loop examples

In this section, we present two explicit examples of the use of Eq. (19). Consider first the massless one-loop box integral, setting $t/s \equiv r$ and with all momenta incoming. Using dimensional regularisation, we define

$$I_{\rm box} \; \equiv \; \Gamma(2+\epsilon) \int_{S_{n-1}} \eta_{\{z\}} \, \frac{(z_1 + z_2 + z_3 + z_4)^{2\epsilon}}{(r z_1 z_3 + z_2 z_4)^{2+\epsilon}} \; \equiv \; \Gamma(2+\epsilon) \, I(1,1,1,1;2\epsilon) \,, \tag{23}$$

where for the box family of integrals we use the notation $I(\nu_1, \nu_2, \nu_3, \nu_4; \nu_5)$. Differentiating with respect to r raises two indices by one unit, as in

$$\partial_r I(1,1,1,1;2\epsilon) \; = \; -(2+\epsilon) \, I(2,1,2,1;2\epsilon) \tag{24}$$

$$\partial_r I(2,1,2,1;2\epsilon) \; = \; -(3+\epsilon) \, I(3,1,3,1;2\epsilon) \,. \tag{25}$$

According to a theorem by Barucchi and Ponzano [21], for any one-loop diagram a system of differential equation can be set up, involving the desired integral, plus the ones obtained by lifting an even number of propagators by 1. For the massless box, following this construction we find that a closed system of differential equations can be obtained for the integrals[1]

$$\Big\{I(1,1,1,1;2\epsilon), I(2,1,2,1;2\epsilon), I(1,2,1,2;2\epsilon), I(2,2,2,2;2\epsilon)\Big\}\,. \tag{26}$$

The system is obtained by using identities generated by Eq. (19), such as, for example,

$$r I(3,1,3,1;2\epsilon) + \int d\omega_{n-2} \;=\; \frac{2}{3+\epsilon} I(2,1,2,1;2\epsilon) + \frac{2\epsilon}{3+\epsilon} I(3,1,3,1;-1+2\epsilon)\,. \tag{27}$$

where the integral of $d\omega_{n-2}$ gives a vanishing boundary term, since

$$\frac{z_1^2 z_3\,(z_1+z_2+z_3+z_4)^{2\epsilon}}{(3+\epsilon)(r z_1 z_3 + z_2 z_4)^{3+\epsilon}}\big(z_2 dz_3\wedge dz_4 - z_3 dz_2 \wedge dz_4 + z_4 dz_2\wedge dz_3\big)\Big|_{\partial S_{n-1}} \;=\; 0\,. \tag{28}$$

The system of differential equations obtained in this way can be written as

$$\partial_r \mathbf{b} \;\equiv\; \partial_r \begin{pmatrix} I(1,1,1,1;2\epsilon) \\ I(2,1,2,1;2\epsilon) \\ I(1,2,1,2;2\epsilon) \\ I(2,2,2,2;2\epsilon) \end{pmatrix} = \begin{pmatrix} 0 & -(2+\epsilon) & 0 & 0 \\ 0 & -\frac{3+\epsilon}{r} & 0 & -\frac{3+\epsilon}{r} \\ 0 & 0 & 0 & -(3+\epsilon) \\ 0 & -\frac{1}{(3+\epsilon)r(1+r)} & \frac{1}{(3+\epsilon)r(1+r)} & -\frac{1+\epsilon+3r}{(3+\epsilon)r(1+r)} \end{pmatrix} \mathbf{b}\,. \tag{29}$$

This system can be brought to canonical form by using (for example) the technique of Magnus exponentiation [31]. It can then be solved by iteration, and the solution, consistently with the literature [27], is given by

$$\begin{aligned} I_{\rm box} \;=\; \frac{k(\epsilon)}{r}\bigg[\frac{1}{\epsilon^2} - \frac{\log r}{2\epsilon} - \frac{\pi^2}{4} + \epsilon\bigg(\frac{1}{2}{\rm Li}_3(-r) - \frac{1}{2}{\rm Li}_2(-r)\log r + \frac{1}{12}\log^3 r \\ - \frac{1}{4}\log(1+r)\left(\log^2 r + \pi^2\right) + \frac{1}{4}\pi^2\log r + \frac{1}{2}\zeta(3)\bigg) + O(\epsilon^2)\bigg]\,, \end{aligned} \tag{30}$$

with $k(\epsilon) = 4 - \frac{\pi^2}{3}\epsilon^2 - \frac{40\zeta(3)}{3}\epsilon^3$.

The difference equations generated in parameter space by Eq. (19) effectively include also dimensional-shift identities, and they connect the desired integrals to lower-point integrals through non-vanishing boundary terms. As an example, consider the following identity for five-point integrals:

$$\int_{S_{\{1,2,3,4,5\}}} d\omega_3 + s_{13}\, I(1,1,2,1,1;2\epsilon) + s_{14}\, I(1,1,1,2,1;2\epsilon) \;=\; \frac{2\epsilon}{2+\epsilon} I(1,1,1,1,1;-1+2\epsilon)\,, \tag{31}$$

with

$$d\omega_3 \;=\; d\left[-\eta_{\{2,3,4,5\}}\,\frac{(z_1+z_2+z_3+z_4+z_5)^{2\epsilon}}{(2+\epsilon)\,(s_{13}z_1z_3 + s_{14}z_1z_4 + s_{24}z_2z_4 + s_{25}z_2z_5 + s_{35}z_3z_5)^{2+\epsilon}}\right]\,. \tag{32}$$

[1] It is well-known that a basis of master integrals for the massless box requires only three integrals. Here we are simply illustrating the Barucchi-Ponzano construction, which in this case yields an over-complete basis, and we have not attempted optimisations. On the other hand, the method correctly predicts the size of the basis for the most general one-loop diagram, as recently confirmed by Refs. [28–30].

The integration of this form using Stokes theorem produces a non-vanishing boundary term, corresponding to the one-loop box integral with one external leg off-shell. Specifically, one finds

$$\int_{S_{\{2,3,4,5\}}} \eta_{\{2,3,4,5\}} \, \frac{(z_2+z_3+z_4+z_5)^{2\epsilon}}{(s_{24}z_2z_4+s_{25}z_2z_5+s_{35}z_3z_5)^{2+\epsilon}} = I^{(1)}_{\text{box}}(s_{25}) \, , \tag{33}$$

where in this case s_{25} is the mass of the off-shell leg. Using similar identities, dimensional-shift relations for the one-loop pentagon [32] can easily be reproduced. One finds

$$\begin{aligned} 2(2+\epsilon)\, I(1,1,1,1,1;1+2\epsilon) \;=\; & \left\{ \frac{s_{13}s_{24}-s_{13}s_{25}-s_{14}s_{25}+s_{14}s_{35}-s_{24}s_{35}}{s_{13}s_{14}s_{25}} \, I^{(1)}_{\text{box}}(s_{25}) \right. \\ & - \frac{s_{13}s_{24}+s_{13}s_{25}-s_{14}s_{25}+s_{14}s_{35}-s_{24}s_{35}}{s_{13}s_{24}s_{25}} \, I^{(2)}_{\text{box}}(s_{13}) \\ & - \frac{s_{13}s_{24}-s_{13}s_{25}+s_{14}s_{25}-s_{14}s_{35}+s_{24}s_{35}}{s_{13}s_{24}s_{35}} \, I^{(3)}_{\text{box}}(s_{24}) \\ & + \frac{s_{13}s_{24}-s_{13}s_{25}+s_{14}s_{25}-s_{14}s_{35}-s_{24}s_{35}}{s_{14}s_{24}s_{35}} \, I^{(4)}_{\text{box}}(s_{35}) \\ & \left. - \frac{s_{13}s_{24}-s_{13}s_{25}+s_{14}s_{25}+s_{14}s_{35}-s_{24}s_{35})}{s_{14}s_{25}s_{35}} \, I^{(5)}_{\text{box}}(s_{14}) \right\} \\ & + 2\epsilon\, I(1,1,1,1,1;-1+2\epsilon) \, . \end{aligned} \tag{34}$$

Since the integral in the last line is finite in $d = 4$, this gives the well-known result stating that the massless pentagon integral is given by a liner combination of box integrals, up to corrections vanishing in four dimensions.

6. Two-loop examples

We now very briefly discuss the application of the method beyond one loop. A first interesting case is given by the family of l-loop sunrise diagrams, *i.e.* diagrams contributing to a two-point function, with two vertices connected by $l+1$ propagators, illustrated in Fig. 1. These integrals have been extensively studied in recent years, since they provide a natural laboratory for multi-loop calculation, and in particular, with massive legs, provide the simplest example of integrals involving elliptic curves, and thus yielding functions beyond polylogarithms (see, for example, [33–43] and references therein).

The first Symanzik polynomial for l-loop sunrise integrals is given by

$$\mathcal{U}_l = \sum_{i=1}^{l+1} z_1 \ldots \hat{z}_i \ldots z_{l+1} \, , \tag{35}$$

where $\hat{z}_i$ denotes the omission of z_i. Eq. (35) displays the high degree of symmetry of the graph, while the second Symanzik polynomial $\mathcal{F}$ depends on the configuration of masses on the internal legs. In the specific case of $l = 2$ and equal internal masses, the Feynman parametric integral is

$$I(\nu_1,\nu_2,\nu_3;\lambda_4) = \int_{S_{\{1,2,3\}}} \frac{\eta_3 \, z_1^{\nu_1-1} z_2^{\nu_2-1} z_3^{\nu_3-1} \left(z_1z_2+z_2z_3+z_3z_1\right)^{\lambda_4}}{\Big[r\, z_1z_2z_3 - (z_1+z_2+z_3)\left(z_1z_2+z_2z_3+z_3z_1\right)\Big]^{\frac{2\lambda_4+\nu}{3}}} \, . \tag{36}$$

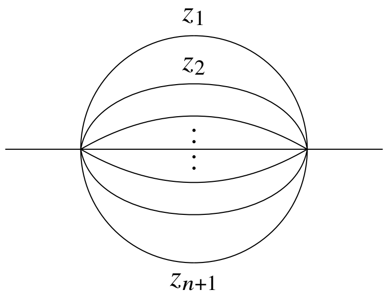

Figure 1: Sunrise diagram at l loops.

By choosing a suitable numerator in our master identity, Eq. (14),

$$H(z) \; = \; z_1^{\nu_1 - 1} z_2^{\nu_2 - 1} z_3^{\nu_3 - 1} \big(z_1 z_2 + z_2 z_3 + z_3 z_1 \big)^{\lambda_4} \, , \tag{37}$$

one easily derives integration by parts identities, and one can build a linear system of differential equations that closes (as expected) on the three master integrals, $I(1, 1, 1; 3\epsilon)$, $I(2, 1, 1; 1 + 3\epsilon)$, and the one-loop tadpole integral $I(2, 2; 1 + 3\epsilon)$. Interestingly, also in this case the non-vanishing boundary term provides an inhomogeneous contribution to the system. It arises form the basis integral

$$\int d\omega_1 \; = \; \frac{1}{2(1+\epsilon)} \int_{S_{\{1,2\}}} \eta_{\{1,2\}} \, \frac{(z_1 z_2)^\epsilon}{\big[- (z_1 + z_2) \big]^{2 + 2\epsilon}} \; = \; \frac{(-1)^{2\epsilon}}{2 + 2\epsilon} \, \frac{\Gamma^2(1+\epsilon)}{\Gamma(2 + 2\epsilon)} \, , \tag{38}$$

corresponding to the massive one-loop tadpole. In two space-time dimensions, the sunrise integral is finite and the linear system can be analysed for $\epsilon = 0$. More precisely, as discussed in more detail in [24], the first-order differential equations can be combined into a single second-order equation for the equal-mass sunrise, which has long been known to be of elliptic type [33, 44, 45]. We find

$$\begin{aligned} \frac{r}{3} \, \frac{d^2}{dr^2} I(1,1,1;0) \; &+ \; \left(\frac{1}{3} + \frac{3}{r - 9} + \frac{1}{3(r-1)} \right) \frac{d}{dr} I(1,1,1;0) \\ &- \left(\frac{1}{4(r-9)} + \frac{1}{12(r-1)} \right) I(1,1,1;0) \; = \; \frac{2}{(r-1)(r-9)} \, . \end{aligned} \tag{39}$$

It is important to note that the procedure we followed is not expected to generalise smoothly to the two-loop sunrise diagram with unequal masses, since a straightforward application of Stokes' theorem in that case must take into account the presence of singularities at the simplex boundaries: the difference between the two cases is discussed in detail in Ref. [46]. We leave the analysis of the general case to future work. On the other hand, we note that our method readily reproduces the classic results of Ref. [3] for two-point, five-propagator integrals, which can be systematically reduced to four-propagator integrals yielding simple combinations of Γ functions. Once again, boundary terms play a distinctive role in parameter space, as discussed in detail in [24].

7. Perspectives

In this note, we have summarised the results of Ref. [24], where we introduced a projective framework for deriving IBP identities and differential equations for Feynman integrals directly in parameter space, building upon very early work by Tullio Regge and collaborators [17–22]. Specifically, we showed how these early techniques can be adapted to include dimensional regularization, and how they can be generalised beyond one loop.

Comparing the parameter-space method to momentum-space approaches, it's clear that the organisation of calculations differs significantly. The integral bases and the resulting differential equations generated by the projective framework are in general distinct from the conventional ones. One notable aspect of this framework is the role played by boundary terms, which vanish in the momentum-space approach. In this case, instead, they play a crucial role, linking complex integrals to simpler ones. We note also that parameter-space integrands closely mirror the graph symmetries, and circumvent issues related to loop-momentum routing and irreducible numerators, which can complicate momentum-space algorithms. Importantly, the projective framework aligns closely with the algebraic structures underpinning Feynman integrals, which may provide direction for future progress.

The present work is largely a feasibility study: for the future, the goal is clearly to extend these techniques to more complex integrals, including higher-loop and multi-scale examples, possibly developing automated tools. Besides the obvious interest in direct physics applications, this will allow for a necessary detailed comparison of parameter-space and momentum-space approaches, including computational aspects.

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