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Gradient-flow renormalon subtraction and the hadronic tau decay series

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The inconsistency between the fixed-order (FO) and contour-improved (CI) representation of the QCD corrections to the inclusive hadronic tau decay width limits the precision to which the strong coupling can be determined from this process. It has recently been shown that subtracting the infrared renormalon divergence related to the gluon condensate resolves the discrepancy. Here we suggest to employ the gradient flow to define gauge-invariant regularized operators and to use the corresponding condensates in the operator product expansion. The associated rearrangement of the perturbative series results in automatic renormalon subtraction without the need to determine explicitly the Stokes constants that normalize the divergent asymptotic series. Applying this method to the gluon condensate, we find that the CI series is modified and now agrees with the (unmodified) FO series. This conclusively demonstrates the preference for the fixed-order approach, as has been advocated long ago.

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1. Introduction

The inclusive hadronic decay width of the τ-lepton supplies an important measurement of the strong coupling $\alpha_s(\mu)$ (in the $\overline{\text{MS}}$ scheme) at the relatively low scale $\mu = m_\tau = 1.777$ GeV. It can be computed systematically in perturbation theory in terms of the operator product expansion (OPE) of the Adler function $D(Q^2)$ [1]. The perturbative coefficients are known to the five-loop $O(\alpha_s^4)$ order [2] and the vacuum condensate and other power-suppressed corrections amount to less than 5% of the perturbative correction. Making use of analyticity of the Adler function, the inclusive hadronic τ decay width is computed from a weighted integral of the Adler function over a circle in the complex Q^2-plane. The result can be expanded in $\alpha_s(m_\tau)$, a representation known as fixed-order perturbation theory (FOPT). Alternatively, the contour integral of $\alpha_s(Q^2)^n$ in the Adler function expansion is computed exactly without re-expansion in $\alpha_s(m_\tau)$. At first sight, this contour-improved representation (CIPT) [3, 4] appears to be the method of choice, since it sums potentially large "π^2-terms" from the analytic continuation of logarithms into the complex plane.

However, the authors of Ref. [5] noted that contrary to expectation, the numerical difference between the FO and CI approximations does not decrease as one adds successively more orders, and traced the origin of the problem to the fact that QCD perturbative expansions are only asymptotic series, specifically due the so-called renormalon divergence [6, 7]. Based on a plausible ansatz for the Borel transform of the series expansion, it was shown that the FOPT series approaches the Borel sum within its ambiguity, while CIPT does not, with the conclusion that CIPT should be abandoned. (A mathematical analysis that explains this behaviour of CIPT has appeared shortly before this talk [8].) Further analysis [9] showed that the numerical difference between FOPT and CIPT is caused by the infrared (IR) renormalon pole related to the gluon condensate in the OPE despite the fact that its contribution to the τ decay width is suppressed by the weight function that relates the τ width to the Adler function.

The discrepancy between the FOPT and CIPT predictions for the hadronic τ width has been a major limitation for the strong coupling determination, since a definitive resolution of the problem should be such that both methods approach asymptotically the same value of the (appropriately defined) Borel sum. A solution that meets this requirement was proposed recently [10, 11]. The idea is to subtract the leading IR renormalon divergence from the Adler function series, very similar to what is routinely done for the pole mass renormalon, where one defines (leading) renormalon-free quark mass schemes (see the review [12]). The proposed subtraction scheme requires that one first determines the normalization (Stokes constant) of the infrared renormalon, which can only be done approximately in practice, since the Stokes constant is non-perturbative. The authors of Refs. [10, 11] demonstrated that the subtraction has little effect on the FOPT series (due to the above-mentioned suppression of the gluon condensate and the associated renormalon series), but brings the CIPT series in line with the FOPT one asymptotically, in agreement with the earlier interpretation [5] of the FOPT-CIPT discrepancy. In this method, one then combines the subtracted series with the ill-defined $\overline{\text{MS}}$-scheme gluon condensate to yield a (leading) renormalon-free Adler function series and an unambiguous gluon condensate. Obtaining a non-perturbatively defined gluon condensate by such a combination has been attempted earlier [13] but turned out to be difficult in practice.

In this proceedings, we describe a renormalon subtraction procedure based on the gradient flow [14, 15] that circumvents the need to obtain the Stokes constant and simultaneously provides a non-perturbative definition of the gluon condensate that can be implemented with existing lattice QCD technology. We also report first results from our study [16] of the FO and CI hadronic τ decay series in this method.

2. Gradient-flow renormalon subtraction and definition of condensates

The connection between the gluon condensate and the leading IR renormalon divergence in the $\overline{\text{MS}}$ scheme series of the Adler function has been known for a long time [17, 18]. It is also well-known that if the OPE was implemented with an explicit momentum cut-off, the IR renormalons in the short-distance coefficients would disappear and the ambiguity in subtracting the power divergence of the condensates would also be removed [19]. The reason why such procedures have not been employed in practice is that simple cut-off definitions of the gluon condensate are not gauge-invariant and further it would not be possible to compute the perturbative series (for the Adler function) to the five-loop order in the presence of a cut-off.

We propose to define the gradient-flow regularized gluon condensate as the matrix element of the product of the corresponding fields at finite flow time t. The basic idea is general, and applies to other matrix elements of local operators appearing in the OPE. It works because the gradient-flowed fields are smeared fields and composite operators of flowed fields do not need operator renormalization. The power-divergences of local higher-dimension operators reappear as singular terms as $t \to 0$. Thus, gradient-flowed local operators are naturally defined with a cut-off or order $1/\sqrt{t}$ with the crucial advantage that gauge-invariance is preserved. The gradient-flowed gluon condensate is identical to the so-called action density

$$E(t) \equiv \frac{g_s^2}{4}\langle 0|\widetilde{G}^A_{\mu\nu}(t)\widetilde{G}^{A\mu\nu}(t)|0\rangle\,, \tag{1}$$

where the matrix-valued field strength is defined in terms of the flowed gluon field [14]

$$\widetilde{G}_{\mu\nu}(t) = \partial_\mu B_\nu(t) - \partial_\nu B_\mu(t) - ig_s\left[B_\mu(t), B_\nu(t)\right] \tag{2}$$

by the standard expression. The action density has been studied non-perturbatively on the lattice as a function of the flow parameter t [14]. For small flow time, its OPE is given by

$$E(t) = \pi^2\left(\frac{\widetilde{C}_1(t)}{t^2} + \widetilde{C}_{G^2}(t)\langle\frac{\alpha_s}{\pi}G^2\rangle + O(t)\right). \tag{3}$$

The most singular term $1/t^2$ corresponds to the quartic power divergence of the local operator $G^2 = G^A_{\mu\nu}G^{A\mu\nu}$ and its mixing into the unit operator with coefficient $\widetilde{C}_1(t)$, which can be computed perturbatively in the strong coupling. The next term involves the usual local gluon condensate, which we define as the scale-invariant gluon condensate

$$\langle\frac{\alpha_s}{\pi}G^2\rangle \equiv \frac{\beta(\alpha_s)}{\pi\beta_0\alpha_s}\langle 0|G^A_{\mu\nu}G^{A\mu\nu}|0\rangle\,. \tag{4}$$

The coefficient $\widetilde{C}_1(t)$ has been computed to $O(\alpha_s^2)$ in [14] and $O(\alpha_s^3)$ in [20], the coefficient $\widetilde{C}_{G^2}(t)$ is known to next-to-next-to-leading order (NNLO) $O(\alpha_s^2)$ from [21].

3. Gradient-flow renormalon-subtracted Adler function

The Adler function is defined in terms of the vector-current two-point function, and expanded as

$$D(s) = -s\frac{d\Pi(s)}{ds} = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{n,k}\, \ln^{k-1}\frac{-s}{\mu^2} = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} c_{n,1}\, a_Q^n$$

$$= \frac{N_c}{12\pi^2}\left[1 + a_Q + 1.64a_Q^2 + 6.37a_Q^3 + 49.08a_Q^4 + \ldots\right]. \tag{5}$$

Here $s = -Q^2$ and $a_\mu = \alpha_s(\mu)/\pi$ ($c_{n,n+1} = 0$ except for $n = 0$). The series is known up to the five-loop order [2]. The numerical values refer to the relevant case with $n_f = 3$ massless quark flavours. Including the leading gluon condensate correction in the OPE, we may write

$$D(Q^2) = \frac{N_c}{12\pi^2}\left(C_1(Q^2) + \frac{C_{G^2}(Q^2)}{Q^4}\langle\frac{\alpha_s}{\pi}G^2\rangle + O(1/Q^6)\right). \tag{6}$$

The coefficient function $C_G^2(Q^2)$ of the gluon operator is known to $O(\alpha_s)$ [22] and NNLO $O(\alpha_s^2)$ [23]. The expansion of $C_1(Q^2)$ given by the second line of (5) is an asymptotic series. The leading divergence arises from an ultraviolet (UV) renormalon. Its normalization (Stokes constant) turns out to be numerically small. At intermediate perturbative orders, the dominant component of the asymptotic expansion is of IR origin. The corresponding IR renormalon series takes the form

$$C_1^{\rm IR}(Q^2) = C_{G^2}(Q^2)\,\frac{\mu^4}{Q^4}\sum_n \alpha_s^{n+1}(\mu)K\left(-\frac{\beta_0}{a}\right)^n n!\, n^b\left(1 + \frac{s_1}{n} + O(1/n^2)\right), \tag{7}$$

where $a = 2$ is related to the dimension of the gluon condensate operator, b, s_1 to the QCD beta function, and K is the unknown Stokes constant. The relation is a consequence of the fact that the Adler function is an observable, hence the ambiguity in defining the sum of the asymptotic series must be fixed by whatever scheme one chooses to define the gluon condensate non-perturbatively.

Since the action density provides such a definition, the IR renormalon divergence of $\widetilde{C}_1(t)$ must have exactly the same form as (7) with $C_{G^2}(Q^2)$ replaced by $\widetilde{C}_{G^2}(Q^2)$. We therefore solve (3) for $\langle\frac{\alpha_s}{\pi}G^2\rangle$, and eliminate it from (6) to obtain

$$D(Q^2) = \frac{N_c}{12\pi^2}\Bigg(\underbrace{\left[C_1(Q^2) - \frac{r}{t^2Q^4}\widetilde{C}_1(t)\right]}_{\text{renormalon cancels}} + \underbrace{\frac{r}{Q^4}\frac{E(t)}{\pi^2}}_{\text{nonpert. defined}} + O(1/Q^6)\Bigg) \tag{8}$$

with

$$r = \frac{C_{G^2}(Q^2)}{\widetilde{C}_{G^2}(t)} = \frac{2\pi^2}{3}\left(\frac{1}{6} - \frac{35}{24}\frac{\alpha_s(\mu)}{\pi} + O(\alpha_s^2)\right). \tag{9}$$

The key advantage of (8) over (6) is that the subtraction solves both problems of the standard OPE: the perturbative series in square brackets is free of the leading IR renormalon divergence, while the leading non-perturbative correction is now well-defined and can be computed directly on the lattice.

In the following, we discuss the perturbative expansion of the subtracted Adler function. Both, the Adler function and the subtraction term are known to rather high orders from the perspective of multi-loop computations, but there is a slight mismatch as the subtraction term is available only to $O(\alpha_s^3)$. In previous studies of the behaviour of the perturbative expansion [5], it proved instructive to merge the exactly known low-order coefficients with the asymptotic behaviour to model the series expansion to all orders. For the Adler function, an estimate of the $O(\alpha_s^5)$ coefficient is included, and then $c_1, \ldots c_5$ are employed to determine the five unknown parameters of the ansatz

$$B[D](u) \;=\; B[D_1^{\rm UV}](u) + B[D_2^{\rm IR}](u) + B[D_3^{\rm IR}](u) + d_0^{\rm PO} + d_1^{\rm PO} u \tag{10}$$

for the Borel transform of the Adler function. The first three terms on the right-hand side incorporate the first UV renormalon and first two IR renormalon singularities with unknown Stokes constants, which together with $d_0^{\rm PO}$, $d_1^{\rm PO}$ are chosen such that $c_1, \ldots c_5$ are reproduced exactly from the expansion of the Borel transform in u. We refer to [5] for the explicit expressions and details.

For the subtraction term, three low-order terms e_1, e_2, e_3 in the expansion of E are available. However, the series expansion of E has no UV renormalons, while the Stokes constant of the gluon condensate renormalon series $E_2^{\rm IR}$ is tied to $D_2^{\rm IR}$ to effect the renormalon cancellation related to the universal gluon condensate. Hence no further information is required to determine the three parameters of the ansatz

$$B[E](u) \;=\; B[E_2^{\rm IR}](u) + B[E_3^{\rm IR}](u) + e_0^{\rm PO} + e_1^{\rm PO} u \tag{11}$$

for the Borel transform of the series expansion of the subtraction term in terms of the three exactly known coefficients e_1, e_2, e_3.

For the following analysis we implement the above-described Borel transform model of [5], generalized to 5-loop accuracy for the QCD beta-function and including the $O(\alpha_s^2)$ contribution to $C_{GG}(Q^2)$ [23]. The expression for r is included in the present preliminary analysis only to $O(\alpha_s)$. We set $\mu = m_\tau$, $\alpha_s(m_\tau) = 0.34$, and the gradient-flow time to $8t = 20/m_\tau^2$, which corresponds to a low UV cut-off on the gradient-flow regularized gluon condensate at the limit of perturbativity.

In Figure 1 we show the cumulative partial sum to order n of the series expansion of the perturbative correction to the Adler function, more precisely to Δ_D, defined as

$$D(s) = \frac{N_c}{12\pi^2}\,[1 + \Delta_D]\,. \tag{12}$$

The upper (blue) points show the unsubtracted Adler function, which increases slowly with order until after the 10th order the sign of the added terms begins to alternate as a consequence of the dominance of the first UV renormalon. The sign alternation becomes visible only at such high orders as the normalization of UV renormalons is suppressed in the $\overline{\rm MS}$ scheme [24] as can be checked explicitly in the so-called large-β_0 approximation [25–27]. The renormalon-subtracted Adler function (lower, orange points) hares this feature, as the subtraction term does not affect the UV renormalon behaviour. We then observe that the main difference to the standard unsubtracted series (5) is that in intermediate orders the subtracted series terms are very small and the partial sum up to the 8th order almost coincides with the subtracted Adler functions at $O(\alpha_s^2)$.

The Adler function including condensate corrections should be independent of the subtraction term, which suggests that the gradient-flow regularized gluon condensate differs significantly from the standard one. The standard definition is ambiguous, but comparison with Figure 6 of [5] shows

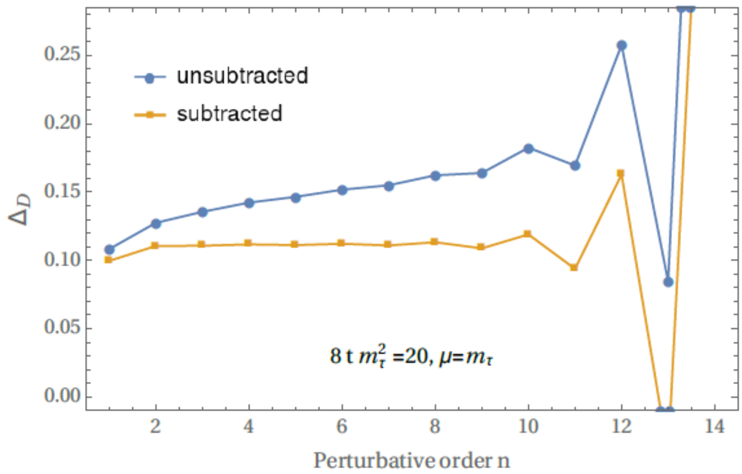

Figure 1: Unsubtracted and gradient-flow subtracted Adler-function series, summed to perturbative order n.

that the ambiguity is smaller than the difference between the two sets of points in Figure 1 near the minimal term of the unsubtracted series at the fifth perturbative order. Nevertheless, for the following analysis of the hadronic τ lepton width in the FO and CI expansion, the value of the gradient-flow regularized gluon condensate is not important, since its contribution is suppressed by two powers of α_s relative to the Adler function.

4. Hadronic tau decay series

Turning to the hadronic τ lepton width, we write the QCD contribution in the standard form

$$R_\tau = -i\pi \oint_{|x|=1} \frac{dx}{x}\,(1-x)^3\, 3\,(1+x)\, 2D(M_\tau^2 x) = N_c \left[1 + \delta^{(0)} + \text{power corrections} \right]. \tag{13}$$

The perturbative correction to the parton-model prediction $R_\tau = 3$ is quantified by $\delta^{(0)}$. From (5) one finds

$$\delta^{(0)}_{\rm FO} = \sum_{n=1}^{\infty} a(M_\tau^2)^n \sum_{k=1}^{n} k\, c_{n,k}\, J_{k-1} \quad \text{with} \quad J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x}\,(1-x)^3\,(1+x)\ln^l(-x) \tag{14}$$

in FOPT, and

$$\delta^{(0)}_{\rm CI} = \sum_{n=1}^{\infty} c_{n,1}\, J_n^a(M_\tau^2) \quad \text{with} \quad J_n^a(M_\tau^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x}\,(1-x)^3\,(1+x)\, a^n(-M_\tau^2 x) \tag{15}$$

in CIPT, resulting in the following expansions for the first four exactly known (in brackets: plus the estimate of the fifth) terms:

$$\delta^{(0)}_{\rm FO} = 0.1082 + 0.0609 + 0.0334 + 0.0174\,(+\,0.0088\,) = 0.2200\ (0.2288) \tag{16}$$

$$\delta^{(0)}_{\rm CI} = 0.1479 + 0.0297 + 0.0122 + 0.0086\,(+\,0.0038\,) = 0.1984\ (0.2021) \tag{17}$$

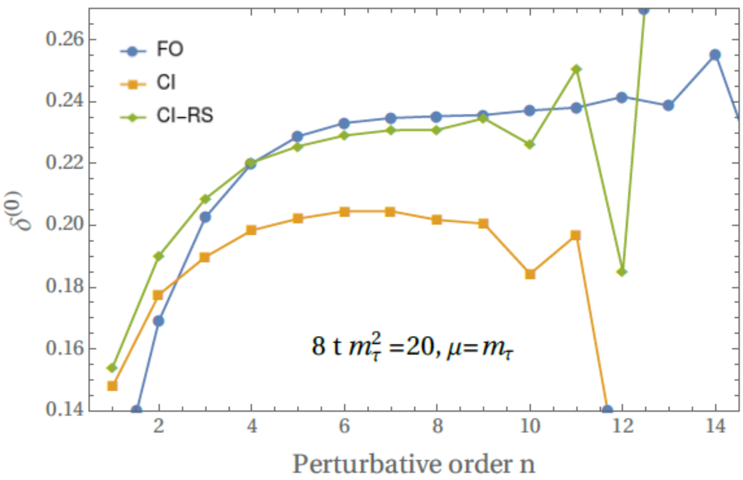

Figure 2: Perturbative correction $\delta^{(0)}$ to the inclusive hadronic τ lepton width summed to order α_s^n in FOPT (blue), CIPT (orange) and renormalon-subtracted CIPT (green).

The observation that the partial sums do not approach each other upon adding successive terms illustrates the FOPT-CIPT discrepancy.

We extend (14), (15) to include the subtraction term of the Adler function. It is important to note that the subtraction term contributes to the FO coefficient only when it contains a $\ln Q^2/\mu^2$, which can appear only through C_{G^2} in r. As can be seen from (9), this happens only from $O(\alpha_s^2)$. Hence, to the NLO r-order employed here, the unsubtracted and gradient-flow renormalon-subtracted FO series for the hadronic τ decay width are identical. This reflects the mentioned suppression of the gluon condensate contribution to inclusive tau decay. As a consequence, the power correction stemming from the gluon condensate is a sub-percent effect numerically. To the contrary, the gradient-flow subtraction contributes to the CI series from the first order in α_s, which demonstrates explicitly that the CI prescription is incompatible with the OPE. Numerically, the renormalon-subtracted CI series reads in the first five orders:

$$\delta^{(0)}_{\rm CI,RS} \;=\; 0.1542 + 0.0362 + 0.0185 + 0.0118\,(+\,0.0053\,) \;=\; 0.2207\;(0.2259) \qquad (18)$$

Comparing this to the FO series (16), we observe that the two series are now close to each other from the fourth order. Figure 2 shows the FO, CI and subtracted CI-RS series in higher orders adopting the Borel function ansatz discussed above. This demonstrates conclusively that the renormalon-subtraction via gradient flow solves the original FOPT-CIPT discrepancy in favour of FOPT as advocated in [5]

5. Conclusion

The discrepancy between the fixed-order and contour-improved computation of the perturbative QCD corrections to the inclusive hadronic τ-lepton decay width [5, 9] has limited the precision to which the strong coupling can be determined from this process. The issue was recently resolved by gluon-condensate renormalon subtraction [10, 11]. In the present work we suggested a new method to perform renormalon subtraction by defining the local operators in the OPE by their gradient-flow representation at finite flow time t. The flow time acts as a gauge-invariant UV cut-off. The rearrangement of perturbative corrections from the $\overline{\text{MS}}$ perturbative series into the gradient-flow regularized operator automatically subtracts the IR renormalon divergence of the series associated with the corresponding operator. The advantage of this method over previous ones is that it avoids the determination of the normalization (Stokes) constant of the renormalon series while simultaneously providing a non-perturbatively valid definition of the condensates, allowing for a consistent addition of power corrections to the perturbative series. The flowed operators at the required value of t can be computed on the lattice. The gradient flow separates the continuum limit $a \to 0$ on the lattice from the cut-off scale $1/\sqrt{t}$ defining the renormalon subtraction, and allows one to define cut-off condensates in the continuum.

Applying this method to the gluon condensate, we find that the CI series for the hadronic tau width is significantly modified despite the fact that the gluon condensate makes a numerically negligible correction to the OPE for inclusive τ-lepton decay. Within the approximations employed in the present study, the FO series remains unaltered by the subtraction due to the inclusive spectral weight function that suppresses the gluon condensate by two powers of α_s. Fig. 2 shows the three series expansions (FO, CI, CI-renormalon subtracted) and conclusively demonstrates the preference for the FO approach (as advocated in [5]), since the FO and subtracted CI series are in full agreement.

The present analysis should be refined by considering the dependence of the subtraction on the renormalization scale μ and flow time t. In principle, r from (9) is already available to $O(\alpha_s^2)$. Including this term causes a technical complication, since the subtraction term acquires a logarithmic Q-dependence at this order, but has little numerical effect on the final result. The details of this analysis will be presented in [16].

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