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New Results from IR-Improved Amplitude-Based Resummation in Quantum Field Theory

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There is a continuing effort to support and prepare the precision physics programs for the present and planned future colliders such as HL-LHC, FCC, CLIC, CEPC, and CPPC. We discuss new results from IR-improved amplitude-based resummation in quantum field theory relevant to such support and preparation with some emphasis on the interplay between soft and collinear resummation algebras.

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1. In Memoriam

Sadly, my (BFLW) close friend and collaborator, Prof. Stanislaw Jadach, passed away suddenly on February 26, 2023. His *CERN Courier** obituary is reproduced here in Fig. 1. He was a

STANISŁAW JADACH 1947–2023

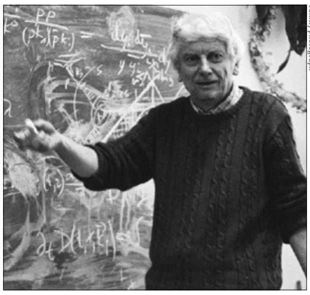
A leading light in radiative corrections

Stanisław Jadach, an outstanding theoretical physicist, died on 26 February at the age of 75. His foundational contributions to the physics programmes at LEP and the LHC, and for the proposed Future Circular Collider at CERN, have significantly helped to advance the field of elementary particle physics and its future aspirations.

Born in Czerteż, Poland, Jadach graduated in 1970 with a masters in physics from Jagiellonian University. There, he also defended his doctorate, received his habilitation degree and worked until 1992. During this period, whilst partly under martial law in Poland, Jadach took trips to Leiden, Paris, London, Stanford and Knoxville, and formed collaborations on precision theory calculations based on Monte Carlo event-generator methods. In 1992 he moved to the Institute of Nuclear Physics Polish Academy of Sciences (PAS) where, receiving the title of professor in 1994, he worked until his death.

Prior to LEP, all calculations of radiative corrections were based on first- and, later, partially second-order results. This limited the theoretical precision to the 1% level, which was unacceptable for experiment. In 1987 Jadach solved that problem in a single-author report, inspired by the classic work of Yennie, Frautschi and Suura, featuring a new calculational method for any number of photons. It was widely believed that soft-photon approximations were restricted to many photons with very low energies and that it was impossible to relate, consistently, the distributions of one or two energetic photons to those of any number of soft photons. Jadach and his colleagues solved this problem in their papers in 1989 for differential cross sections, and later in 1999 at the level of spin amplitudes. A long series of publications and computer programmes for re-summed perturbative Standard Model calculations ensued.

Most of the analysis of LEP data was based



Family photograph

Stanisław Jadach made major contributions to the physics programmes at LEP and the LHC.

exclusively on the novel calculations provided by Jadach and his colleagues. The most important concerned the LEP luminosity measurement via Bhabha scattering, the production of lepton and quark pairs, and the production and decay of W and Z boson pairs. For the W-pair results at LEP2, Jadach and co-workers intelligently combined separate first-order calculations for the production and decay processes to achieve the necessary 0.5% theoretical accuracy, bypassing the need for full first-order calculations for the four-fermion process, which were unfeasible at the time. Contrary to what was deemed possible, Jadach and his colleagues achieved calculations that simultaneously take into account QED radiative corrections and the complete spin-spin correlation effects in the production and decay of two tau leptons. He also had success in the 1970s in novel simulations of strong interaction processes.

After LEP, Jadach turned to LHC physics. Among other novel results, he and his collaborators developed a new constrained Markovian algorithm for parton cascades, with no need to use backward evolution and predefined parton distributions, and proposed a new method, using a "physical" factorisation scheme, for combining a hard process at next-to leading order with a parton cascade, much simpler and more efficient than alternative methods.

Jadach was already updating his LEP-era calculations and software towards the increased precision of FCC-ee, and is the co-editor and co-author of a major paper delineating the need for new theoretical calculations to meet the proposed collider's physics needs. He co-organised and participated in many physics workshops at CERN and in the preparation of comprehensive reports, starting with the famous 1989 LEP Yellow Reports.

Jadach, a member of the Polish Academy of Arts and Sciences (PAAS), received the most prestigious awards in physics in Poland: the Marie Skłodowska-Curie Prize (PAS), the Marian Mięsowicz Prize (PAAS), and the prize of the Minister of Science and Higher Education for lifetime scientific achievements. He was also a co-initiator and permanent member of the international advisory board of the RADCOR conference.

Stanisław (Staszek) was a wonderful man and mentor. Modest, gentle and sensitive, he did not judge or impose. He never refused requests and always had time for others. His professional knowledge was impressive. He knew almost everything about QED, and there were few other topics in which he was not at least knowledgeable. His erudition beyond physics was equally extensive. He is already profoundly and dearly missed.

Wiesław Płaczek *Jagiellonian University,* **Maciej Skrzypek** *and* **Zbigniew Was** *Institute of Nuclear Physics and* **Bennie Ward** *Baylor University.*

Figure 1: *CERN Courier* obituary for the late Prof. Stanislaw Jadach.

pioneering member of the RADCOR International Advisory Board and he was the Chair of the Local Organizing Committee when the Institute of Nuclear Physics hosted the 1996 RADCOR in Krakow, Poland. His many outstanding contributions to our field helped to keep our field alive. We all miss him dearly. This contribution is dedicated in memoriam to him.

2. Introduction

The future of precision quantum field theory is dictated by the planned future colliders: FCC [1], CLIC [2], ILC [3] CEPC [4], CPPC [5], ... For example, in the case of the FCC-ee, factors of improvement from ~ 5 to ~ 100 are needed from theory. Resummation is key to such improvements in many cases. In this context, we discuss here the amplitude-based resummation following from the methodology of Yennie, Frautschi, and Suura (YFS) in Ref. [6] as it has been realized Refs. [7–13]. This approach to resummation treats the resummation of infrared (IR) singularities to all orders in

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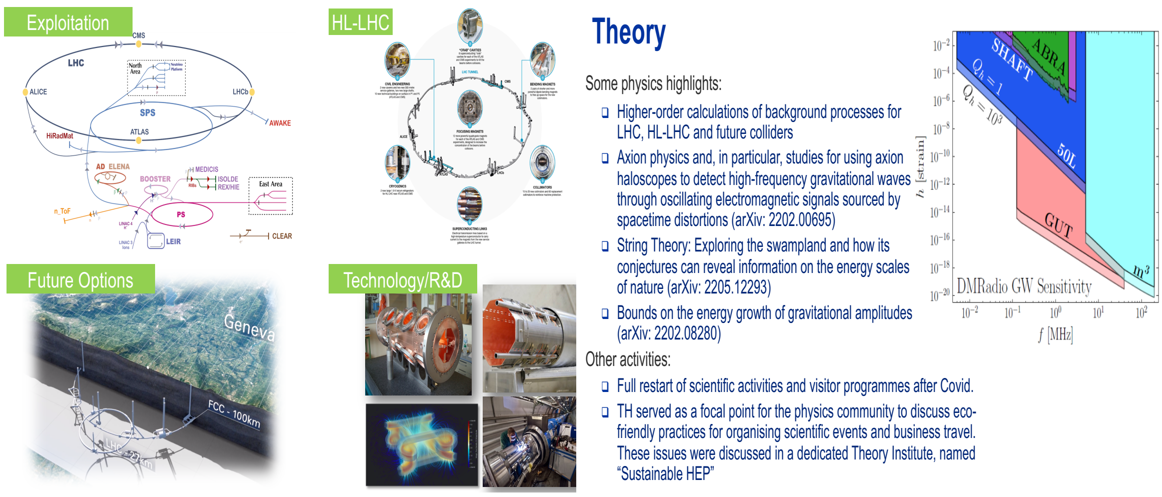
the loop expansion. One of us (BFLW) has extended [14] this methodology to non-Abelian gauge theories as well.

It is encouraging that the need for precision theory for the future collider physics objectives is appreciated at world-leading laboratories such as CERN, as we illustrate in Fig. 2 with excerpts from Ref. [15]. The future options for CERN are shown featuring the FCC and the important role of higher-order calculations for its background processes is a theory highlight. We can hope that the funding agencies take note of the implied connection.



(a) (b)

Figure 2: Excerpts from Ref. [15] on the state of CERN: (a), Future Options and R& D; (b), Theory highlights.

The fundamental attraction of the YFS approach is that there is no limit in principle to the precision one may achieve as long as one calculates the corresponding hard radiation residuals to the desired order in the respective coupling constant. This should be contrasted with other methods of resummation in which various degrees of freedom are integrated out to allow the resummation thereby engendering an intrinsic uncertainty in observables which require the reinstatement of those degrees of freedom; for, the reinstatement can only be done approximately.

More specifically, YFS methods in QED are exact to all orders in the infrared limit but treat collinear big logs perturbatively in the hard photon residuals. DGLAP-based collinear factorization as recently done at the next-leading log level in Refs. [16–19], for example, treats collinear logs to all orders but has a non-exact infrared limit. In what follows, after briefly reviewing in the next Section the elements of exact amplitude resummation theory which we employ, we first present (in Section 4) new results for precision collider physics and pioneering results in quantum gravity using the usual YFS methods. We then investigate in Section 5 improving the collinear limit of YFS theory. A key point is the following: Exact amplitude-based resummation realized on an event-by-event basis gives enhanced precision for a given level of exactness and this is essential for future precision physics as exemplified by CERN.

3. Brief Review of Exact Amplitude-Based Resummation Theory

As it is still not generally familiar, we include here a synopsis of exact amplitude-based resummation theory. The master formula that exhibits the theory is as follows:

$$
\begin{aligned}
d\bar{\sigma}_{\rm res} &= e^{\rm SUM_{IR}(QCED)} \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int \prod_{j_1=1}^{n} \frac{d^3 k_{j_1}}{k_{j_1}} \\
&\prod_{j_2=1}^{m} \frac{d^3 k'_{j_2}}{k'_{j_2}} \int \frac{d^4 y}{(2\pi)^4} e^{iy\cdot(p_1+q_1-p_2-q_2-\sum k_{j_1}-\sum k'_{j_2})+D_{\rm QCED}} \\
&\tilde{\bar{\beta}}_{n,m}(k_1,\ldots,k_n;k'_1,\ldots,k'_m)\frac{d^3 p_2}{p_2^0}\frac{d^3 q_2}{q_2^0},
\end{aligned} \tag{1}
$$

where the *new*[†] (YFS-style) residuals $\tilde{\bar{\beta}}_{n,m}(k_1,\ldots,k_n;k'_1,\ldots,k'_m)$ have n hard gluons and m hard photons. The new residuals and the infrared functions $\rm SUM_{IR}(QCED)$ and $D_{\rm QCED}$ are defined in Ref. [21, 22]. As explained in Ref. [21, 22], parton shower/ME matching engenders the replacements $\tilde{\bar{\beta}}_{n,m} \to \tilde{\hat{\bar{\beta}}}_{n,m}$, which allow us to connect with MC@NLO [23, 24], via the basic formula

$$
d\sigma = \sum_{i,j} \int dx_1 dx_2 F_i(x_1) F_j(x_2) d\hat{\sigma}_{\rm res}(x_1 x_2 s). \tag{2}
$$

New results in precision LHC and FCC physics have been obtained using Eq.(1). One of us (BFLW) has extended the latter equation to general relativity as an approach to quantum gravity. In each respective application, our new results are accompanied with new perspectives. We discuss such new results and perspectives in the next Section.

4. New Perspectives for Precision Physics: High Energy Colliders and Quantum Gravity

The realization of eq.(1) in the MC event generator $\mathcal{KK}$MC-hh [12] by four of us (SJ, BFLW, ZAW, SAY) allows a new perspective on the expectations for precision physics for the Standard Theory EW interactions at HL-LHC. This is illustrated by the plots in Fig. 3 in the ATLAS analysis [25] of $Z\gamma$ production at 8 TeV. The data are compared to the Powheg-Pythia8-Photos [26–

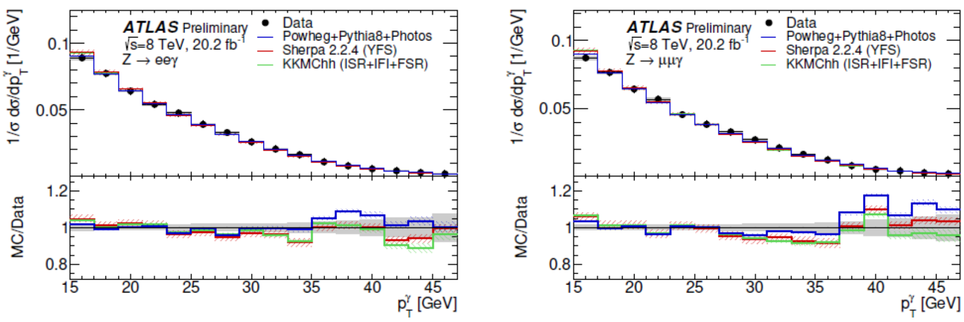


Figure 3: ATLAS analysis of Z/γ production at 8 TeV.

[†]The *non-Abelian* nature of QCD requires a new treatment of the corresponding part of the IR limit [20] so that we usually include in $\rm SUM_{IR}(QCED)$ only the leading term from the QCD exponent in Ref. [20] – the remainder is included in the residuals $\tilde{\bar{\beta}}_{n,m}$.

31], Sherpa2.2.4(YFS) [32, 33], and $\mathcal{KK}$MC-hh predictions for the γp_T spectrum. At this level of the uncertainties in the data at this point, all three predictions are in reasonable agreement with the data. At HL-LHC, with 10 times the current statistics, a precision test against the theories will obtain.

An important issue is the effect of QED contamination in non-QED PDFs. In a new perspective toward this issue [34–36] we use Negative ISR (NISR) evolution to address the size of this contamination directly. We have, using a standard notation for PDFs and cross sections, the cross section representation

$$\begin{aligned}\sigma(s) = \frac{3}{4}\pi\sigma_0(s) \sum_{q=u,d,s,c,b} \int d\hat{x}\; dz dr dt \int dx_q dx_{\bar{q}}\; \delta(\hat{x} - x_q x_{\bar{q}} z t) \\ \times f_q^{h_1}(s\hat{x}, x_q) f_{\bar{q}}^{h_2}(s\hat{x}, x_{\bar{q}})\; \rho_I^{(0)}(\gamma_{Iq}(s\hat{x}/m_q^2), z)\; \rho_I^{(2)}(-\gamma_{Iq}(Q_0^2/m_q^2), t) \\ \times \sigma_{q\bar{q}}^{Born}(s\hat{x}z)\; \langle W_{MC}\rangle,\end{aligned} \tag{3}$$

which includes an extra convolution with the well known second order exponentiated ISR "radiator function" $\rho_I^{(2)}$ with the negative evolution time argument $-\gamma_{Iq}(Q_0^2/m_q^2)$ defined in Ref. [34]. The QED below Q_0 is thus removed. We illustrate this in Fig. 4 from Ref. [35] for the P_{T_γ} for the

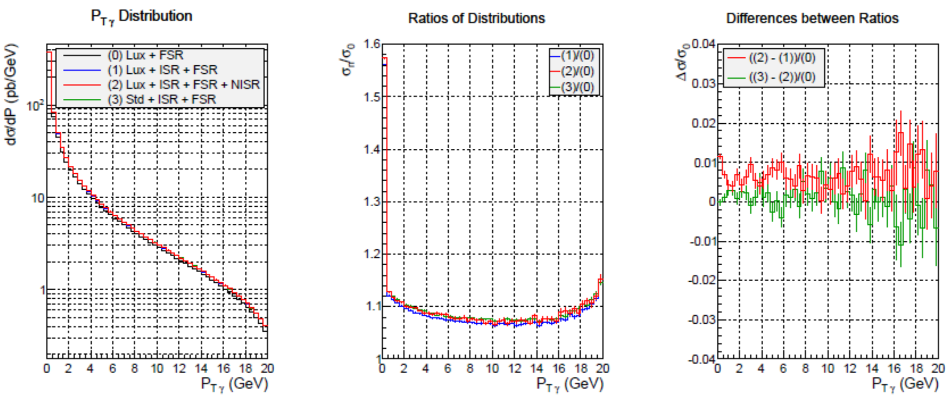


Figure 4: The distribution for P_{T_γ} of the photon for which it is greatest for events with at least one photon and each lepton having $p_{T\ell} > 25GeV, \eta_\ell < 2.5$ calculated with (0) FSR only (black). (1) FSR + ISR (blue). and (2) FSR + ISR with NISR (red) for NNPDF3.1-LuxQED NLO PDFs. For comparison, (3) shows FSR + ISR with ordinary NNPDF3.1 NLO PDFs (green). The center graph shows ISR on/off ratios (1)/(0) (blue),(2)/(0) (red) and (3)/(0) (green). The right-hand graph shows the fractional differences ((1)- (2))/(0) in red and ((2)-(3))/(0) in green.

photon for which it is the largest in $Z\gamma^*$ production and decay to lepton pairs at the LHC at 8 TeV for cuts as described in the figure. As we see in the figure, the results show that the effect of QED contamination in non-QED PDFs is below the errors on the PDFs in agreement with arguments in Ref. [37].

For the planned EW/Higgs factories, five of us (SJ, WP, MS, BFLW, SAY) have discussed in Refs. [38–40] the new perspectives for the BHLUMI [8] luminosity theory error. This new perspective is illustrated in Fig. 5 [40] wherein we show the current purview for the FCC-ee at M_Z and that for the proposed higher energy colliders. In addition to the improvements at M_Z shown in

Forecast study for FCCee$_{M_Z}$			
Type of correction / Error	Published [1]	Strict	Redone
(a) Photonic $\mathcal{O}(L_e^2\alpha^3)$	0.10×10^{-4}	0.10×10^{-4}	0.10×10^{-4}
(b) Photonic $\mathcal{O}(L_e^4\alpha^4)$	0.06×10^{-4}	0.06×10^{-4}	0.06×10^{-4}
(b') Photonic $\mathcal{O}(\alpha^2L^0)$		0.17×10^{-4}	0.17×10^{-4}
(c) Vacuum polariz.	0.6×10^{-4}	0.6×10^{-4}	0.6×10^{-4}
(d) Light pairs	0.5×10^{-4}	0.4×10^{-4}	0.27×10^{-4}
(e) Z and s-channel γ exch.	0.1×10^{-4}	0.1×10^{-4}	0.1×10^{-4}
(f) Up-down interference	0.1×10^{-4}	0.08×10^{-4}	0.08×10^{-4}
Total	1.0×10^{-4}	0.76×10^{-4}	0.70×10^{-4}

(a)

Forecast			
Type of correction / Error	ILC$_{500}$	ILC$_{1000}$	CLIC$_{3000}$
(a) Photonic $\mathcal{O}(L_e^2\alpha^3)$	0.13×10^{-4}	0.15×10^{-4}	0.20×10^{-4}
(b) Photonic $\mathcal{O}(L_e^4\alpha^4)$	0.27×10^{-4}	0.37×10^{-4}	0.63×10^{-4}
(c) Vacuum polariz.	1.1×10^{-4}	1.1×10^{-4}	1.2×10^{-4}
(d) Light pairs	0.4×10^{-4}	0.5×10^{-4}	0.7×10^{-4}
(e) Z and s-channel γ exch.	$1.0\times10^{-4(*)}$	2.4×10^{-4}	16×10^{-4}
(f) Up-down interference	$<0.1\times10^{-4}$	$<0.1\times10^{-4}$	0.1×10^{-4}
Total	1.6×10^{-4}	2.7×10^{-4}	16×10^{-4}

(b)

Figure 5: Current purview on luminosity theory errors: (a), FCC-ee at M_Z; (b), proposed higher energy colliders

Fig. 5(a) to 0.007%, the use of the results in Ref. [41] together with lattice methods [42, 43] opens the possibility that item (c) in Fig 5(a) could be reduced by a factor of 6[‡].

In Refs. [44, 45] one of us (BFLW) has shown that amplitude-based resummation applied to quantum gravity tames its UV divergences. One of the many consequences is, using a standard notation,

$$\begin{aligned}\rho_\Lambda(t_0) &\cong \frac{-M_{Pl}^4(1+c_{2,eff}k_{tr}^2/(360\pi M_{Pl}^2))^2}{64}\sum_j\frac{(-1)^F n_j}{\rho_j^2}\\ &\quad\times\frac{t_{tr}^2}{t_{eq}^2}\times(\frac{t_{eq}^{2/3}}{t_0^{2/3}})^3\\ &\cong\frac{-M_{Pl}^2(1.0362)^2(-9.194\times10^{-3})}{64}\frac{(25)^2}{t_0^2}\\ &\cong(2.4\times10^{-3}eV)^4.\end{aligned} \tag{4}$$

Here, t_0 is the age of the universe and we take it to be $t_0\cong13.7\times10^9$ yrs and $t_{tr}\sim25t_{Pl}$ [44–46] is the transition time between the Planck regime and the classical Friedmann-Robertson-Walker(FRW) regime. In the estimate in (4), the first factor in the second line comes from the radiation dominated period from t_{tr} to t_{eq} and the second factor comes from the matter dominated period from t_{eq} to t_0. The estimate in (4) is close to the experimental result [47][§] $\rho_\Lambda(t_0)|_{\rm expt}\cong((2.37\pm0.05)\times10^{-3}eV)^4$.

[‡]The formula to be studied is $\Delta\alpha_{had}(t)=\Delta\alpha_{had}(-Q_0^2)|_{lat}+[\Delta\alpha_{had}(t)-\Delta\alpha_{had}(-Q_0^2)]|_{pQCDAdler}$ with *lat* denoting the methods of Refs. [42, 43] and *pQCDAdler* denoting the methods of Ref. [41].

[§]See also Ref. [48, 49] for analyses that suggest a value for $\rho_\Lambda(t_0)$ that is qualitatively similar to this experimental result.

5. Improving the Collinear Limit in YFS Theory

It is known [50] that in the usual YFS theory the virtual infrared function B in the s-channel resums (exponentiates) the non-infrared term $\frac{1}{2}Q_e^2\frac{\alpha}{\pi}L$ in $e^+(p_2)\ e^-(p_1) \to \bar{f}(p_4)\ e^-(p_3)$ using an obvious notation where the respective big log is $L = \ln(s/m_e^2)$ when $s = (p_1 + p_2)^2$ is the center-of-mass energy squared. It is also known from Ref. [51] that the term $\frac{3}{2}Q_e^2\frac{\alpha}{\pi}L$ exponentiates – see also Refs. [16–19] for recent developments in attendant collinear factorization approach. Does the YFS theory allow an extension that would also exponentiate the latter term? In Ref. [50], three of us (SJ, BFLW, ZAW) have answered this question in the affirmative.

Specifically, we find that the virtual infrared function B in the s-channel can be extended to

$$\begin{aligned} B_{CL} &\equiv B + \mathbf{\Delta B} \\ &= \int \frac{d^4k}{k^2}\frac{i}{(2\pi)^3}\left[\left(\frac{2p-k}{2kp-k^2} - \frac{2q+k}{2kq+k^2}\right)^2 - \frac{\mathbf{4pk - 4qk}}{\mathbf{(2pk - k^2)(2qk + k^2)}}\right], \end{aligned} \tag{5}$$

while the real infrared function $\tilde{B}$ can be extended to

$$\begin{aligned} \tilde{B}_{CL} &\equiv \tilde{B} + \mathbf{\Delta\tilde{B}} \\ &= \frac{-1}{8\pi^2}\int \frac{d^3k}{k_0}\Big\{(\frac{p_1}{kp_1} - \frac{p_2}{kp_2})^2 + \frac{\mathbf{1}}{\mathbf{kp_1}}(\mathbf{2} - \frac{\mathbf{kp_2}}{\mathbf{p_1p_2}}) \\ &\quad + \frac{\mathbf{1}}{\mathbf{kp_2}}(\mathbf{2} - \frac{\mathbf{kp_1}}{\mathbf{p_1p_2}})\Big\}, \end{aligned} \tag{6}$$

where the extensions are indicated in boldface in an obvious notation. The YFS infrared algebra is unaffected by these extensions while the B_{CL} does exponentiate the entire $\frac{3}{2}Q_e^2\frac{\alpha}{\pi}L$ term and the $\tilde{B}_{CL}$ does carry the respective collinear big log of the exact result in Ref. [52] in the soft regime.

The corresponding collinear extension of the CEEX soft eikonal amplitude factor defined in Ref. [53] for the photon polarization σ and e^- helicity σ' is given by

$$\begin{aligned} \mathfrak{s}_{CL,\sigma}(k) = \sqrt{2}Q_e e\Bigg[&-\sqrt{\frac{p_1\zeta}{k\zeta}}\frac{< k\sigma|\hat{p}_1 - \sigma >}{2p_1k} + \delta_{\sigma'-\sigma}\sqrt{\frac{\mathbf{k\zeta}}{\mathbf{p_1\zeta}}}\frac{\mathbf{< k\sigma|\hat{p}_1\sigma' >}}{\mathbf{2p_1k}} \\ &+\sqrt{\frac{p_2\zeta}{k\zeta}}\frac{< k\sigma|\hat{p}_2 - \sigma >}{2p_2k} + \delta_{\sigma'\sigma}\sqrt{\frac{\mathbf{k\zeta}}{\mathbf{p_2\zeta}}}\frac{\mathbf{< \hat{p}_2\sigma'|k - \sigma >}}{\mathbf{2p_2k}}\Bigg], \end{aligned} \tag{7}$$

where from Ref. [53] $\zeta \equiv (1, 1, 0, 0)$ for our choice for the respective auxiliary vector in our Global Positioning of Spin (GPS) [54] spinor conventions with the consequent definition $\hat{p} = p - \zeta m^2/(2\zeta p)$ for any four vector p with $p^2 = m^2$. The collinear extension terms are again indicated in boldface.

We expect these extended infrared functions to give in general a higher precision for a given level of exactness [55].

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