

Analytic results for massive $2 \rightarrow 2$ processes

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We discuss recent (semi) analytic results for $2 \rightarrow 2$ processes with massive internal and external particles in various regions of phase space. In the physical applications we restrict ourselves to $gg \rightarrow HH$.

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1. Introduction

At this conference many results for multi-loop calculations have been reported. Often the internal particles are all massless which often also holds for most of the external lines [1]. However, there are also processes where it is important to keep the mass of the internal particles. A prime example is Higgs boson pair production in gluon fusion but also processes like $gg \rightarrow ZZ$, $gg \rightarrow ZH$ or Higgs plus jet production. Furthermore, once electroweak corrections are considered the gauge and Higgs boson masses occur in internal lines.

In general analytic results for massive $2 \rightarrow 2$ processes are rare and, if available (see, e.g., Refs. [2, 3]), they have a complicated analytic structure which makes them often difficult to handle and the numerical evaluation is not straightforward.

On the other hand, there are purely numerical approaches, which are often computationally expensive. Furthermore, numerical results are significantly less flexible, e.g., in connection to changes of renormalization schemes.

In this contribution we discuss analytic approximations. Individual results are valid in certain limits. However, their combination can cover the whole phase space. The approximations which are discussed in the following consist either of analytic expansions, which are composed of simple functions, or of power-log expansions with precise numerical coefficients. In either case, a fast numerical evaluation is guaranteed. Needless to say, analytic approximations allow for a flexible use, in particular in connection to renormalization scheme changes.

In this proceedings contribution we discuss several recent results for massive $2 \rightarrow 2$ processes at two and three loops. We concentrate on $gg \rightarrow HH$. The techniques can also be applied to other processes such as gauge and Higgs boson production in gluon fusion or Higgs plus jet production. We will not discuss the large- m_t expansion which for $gg \rightarrow HH$ is available up to NNLO [4–6]. Recently also the full electroweak corrections have been computed in this limit [7]. Exact NLO QCD results for $gg \rightarrow HH$ are available from Refs. [8–10].

2. High-energy limit

In Refs. [11, 12] analytic results for the NLO QCD corrections for $gg \rightarrow HH$ have been obtained in the limit $s, t, u \gg m_t^2 \gg m_H^2$. The second inequality sign leads to a simple Taylor expansion, which can be performed at the integrand level. This effectively eliminates the scale m_H . The remaining integrals depend on s, t and m_t^2 (with $u = -s - t$). An expansion for small top quark mass at the amplitude level is tedious. It is more convenient to perform a reduction and implement $s, t \gg m_t^2$ at the level of the two-loop master integrals.

The results of Refs. [11, 12] have been used in Ref. [13] to combine the high-energy expansion with the exact numerical calculations of Refs. [8, 9] such that the latter has to be used only in a restricted region of phase space. This significantly reduces the required CPU time. For this analysis an expansion up to m_t^3 was available.

In Ref. [15] the high-energy expansion has been refined and deep expansions in $m_t^2/\{s, t, u\}$ could be obtained. This leads to a significant qualitative improvement, in particular in combination with the use of Padé approximation which is applied to the m_t expansion. Using more than 100 expansion terms in m_t we can construct a large number of different Padé approximants together

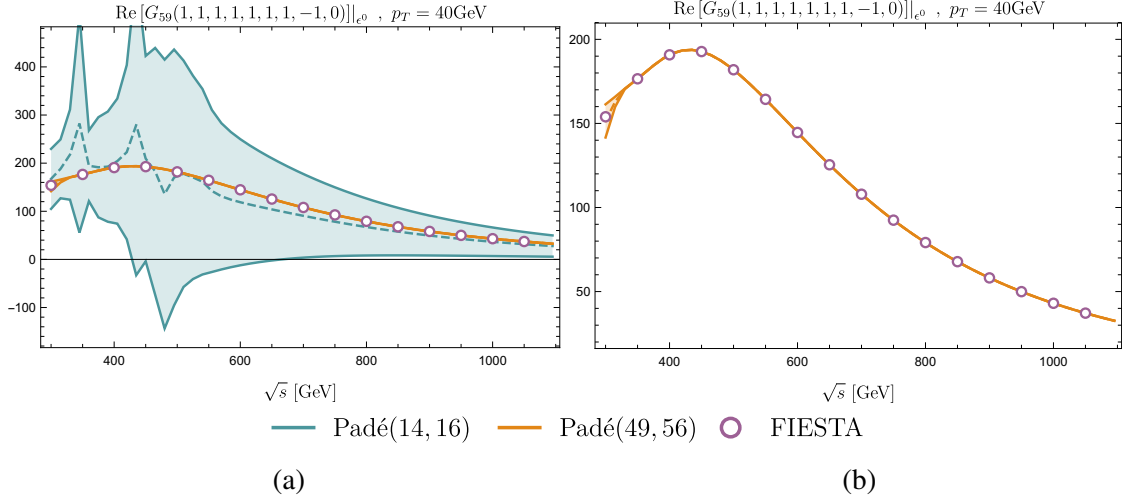


Figure 1: Comparison of Padé-based approximations constructed from different expansion depths ($N_{\text{low}}, N_{\text{high}}$) with numerical results obtained using FiestA, for a non-trivial non-planar master integral, see Ref. [14] for details.

with an estimate of the uncertainty [16]. The latter can be validated by comparing Padé results of selected scalar integrals to numerical results obtained with FiestA [17] or pySecDec [18]. An example is shown in Fig. 1. The left panel shows in green high-energy results based on 32 expansion terms and in orange results where 112 expansion terms are incorporated. In both cases the central values and the uncertainties are shown. Numerical results obtained with FiestA are shown as circles. The magnification on the right panel shows the impressive accuracy of the Padé method, even close to the two top quark threshold. A deep high-energy expansion accompanied with Padé approximation can thus be viewed as a precision tool for massive $2 \rightarrow 2$ Feynman integrals.

In Ref. [15] a first step towards the electroweak corrections has been taken and the high-energy expansion has been applied to the two-loop box diagrams where a Higgs boson is exchanged between the top quarks. This introduces an additional scale m_H^{int} , the mass of the internal Higgs boson. Exact analytic calculations are again most likely not possible or at least quite involved. On the other hand, one may consider in addition to the high-energy limit either of the two cases:

$$(A) \quad m_t^2 \gg (m_H^{\text{int}})^2 = (m_H^{\text{ext}})^2,$$

$$(B) \quad m_t^2 \approx (m_H^{\text{int}})^2 \gg (m_H^{\text{ext}})^2.$$

Here “ $m_t^2 \approx (m_H^{\text{int}})^2$ ” means a Taylor expansion in the mass difference. In (A) “ \gg ” requires the application of an asymptotic expansion in the large mass limit, which involves non-trivial subdiagrams. On the other hand, in (B) “ \gg ” leads to a Taylor expansion as above. It has been shown in Ref. [15] that both cases lead to good results as can be seen from Figs. 2(a) and (b) where results for the real part of $F_{\text{box}1}$ are shown for $p_T = 500$ GeV and $p_T = 200$ GeV, respectively. The colours correspond to approach (B) and the results from approach (A) are shown in gray and black. One observes a nice convergence when including higher order expansion terms and, furthermore, the results for the two approaches agree well at or even below the percent level.

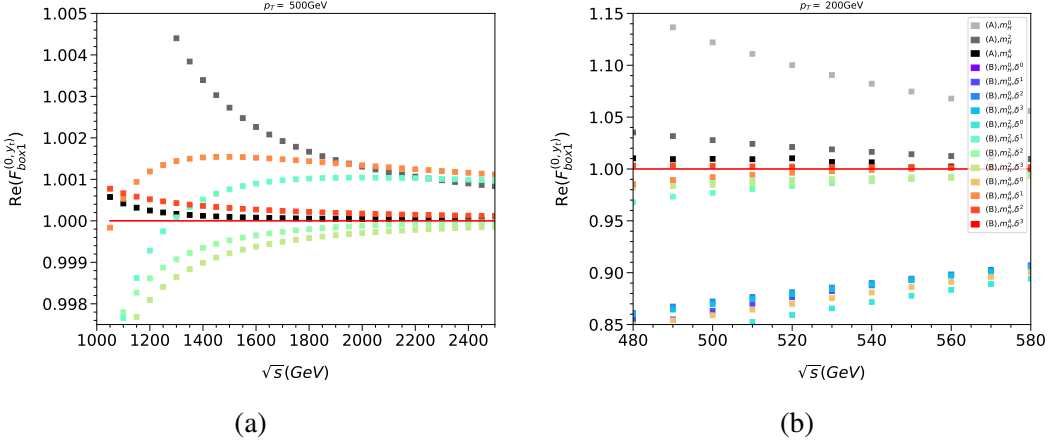


Figure 2: Real part of F_{box1} for different values of p_T and various expansion terms in m_H^{ext} and $\delta = 1 - m_H^{\text{int}}/m_t$.

3. $t \rightarrow 0$ expansion

In this section we still consider two-loop corrections to $gg \rightarrow HH$. However instead of high-energy expansions we discuss expansions around the forward scattering limit. This idea has been applied for the first time to $gg \rightarrow HH$ in Ref. [19] under the name “ p_T expansion”. Later the method has been refined in Refs. [20–22].

In our approach [14] we Taylor-expand in the Mandelstam variable t and in the final-state masses independently, whereas the p_T expansion provides a combination of both expansions. The two prescriptions are equivalent in the sense that once the final result is expressed in terms of the same variables and potential higher order terms are discarded one obtains identical expressions.

In Ref. [14] we have shown that the expansion in t converges very quickly for $p_T \lesssim 200$ GeV. Instead of order 100 terms, as for the high-energy expansion, only a few (we have computed six terms) are sufficient. An expansion up to order m_H^4 is sufficient to obtain results which show perfect agreement with the exact expressions and deviate at most at the percent level for small values of \sqrt{s} . Furthermore, in Ref. [14] we have shown that the combination of the high-energy and $t \rightarrow 0$ expansion covers the whole phase-space and thus no purely numerical approach is necessary (see also Ref. [21] where a similar approach has been proposed, though with less input from the high-energy and around the forward limit).

In Fig. 3 we show the results for the C_F part of one of the form factors for $p_T = 170$ GeV as a function of \sqrt{s} . For the small- t expansion (red and orange colour) terms up to t^5 are taken into account and the high-energy expansion (light and dark blue) includes Padé approximations with at least $(m_t^2)^{49}$ and at most $(m_t^2)^{56}$ terms. In all cases quartic terms in m_H are included. For this value of p_T we observe perfect agreement of the two expansions which can be seen in the black and gray curves and the scale on the right side of the plot. For smaller values of p_T the small- t expansion is even more reliable, and for larger values of p_T the high-energy expansion. This and similar plots for different values of p_T (see Ref. [14]) demonstrate that the combination of the two expansions cover the whole phase space.

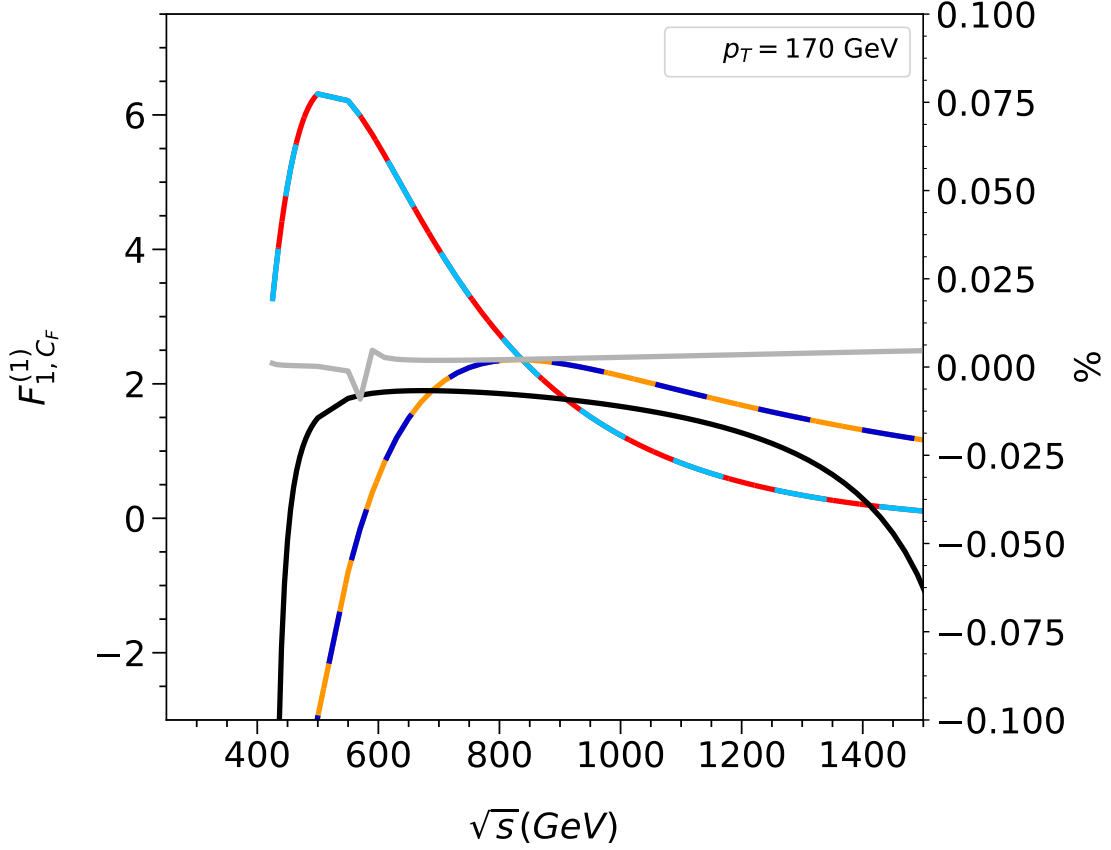


Figure 3: Real (red and light blue) and imaginary (orange and dark blue) parts of the C_F contribution to the two-loop form factor $F_{\text{box1}}^{(1)}$ (see Ref. [12] for a precise definition) as a function of \sqrt{s} for $p_T = 170$ GeV. The red and orange curves correspond to the expansion for $t \rightarrow 0$ and the light and dark blue curves to the high-energy expansion. The relative differences (see scale on the right side) are shown as black (real part) and gray (imaginary part) curves.

4. First steps to three loops: Fermionic contribution for $t = 0$

We have seen in the previous Section that the t expansion covers a large part of the phase space. For this reason, in Ref. [23] the limits $t = 0$ and $m_H = 0$ has been applied to three-loop $gg \rightarrow HH$ diagrams which contain a closed loop with massless fermions. Even for the leading expansion term the reduction problem is quite involved such that further improvements are necessary in order to obtain the complete result. Subleading expansion terms will be even significantly more involved.

The first step taken in Ref. [23] can be viewed as exploratory study which shows that it is in principle possible to apply the t expansion at three loops. For $t = 0$ and $m_H = 0$ we have performed the reduction [24] to master integrals and have established the corresponding differential equation. Using boundary conditions from the large- m_t limit, which can be computed analytically, we apply the "expand and match" approach to obtain semi-analytic results, which are valid for all values of s/m_t^2 . In Fig. 4 we show results for the n_l part of the form factor F_{box1} (see Ref. [23] for a precise definition) as a function of \sqrt{s} . We stress that the plotted expressions are stepwise defined

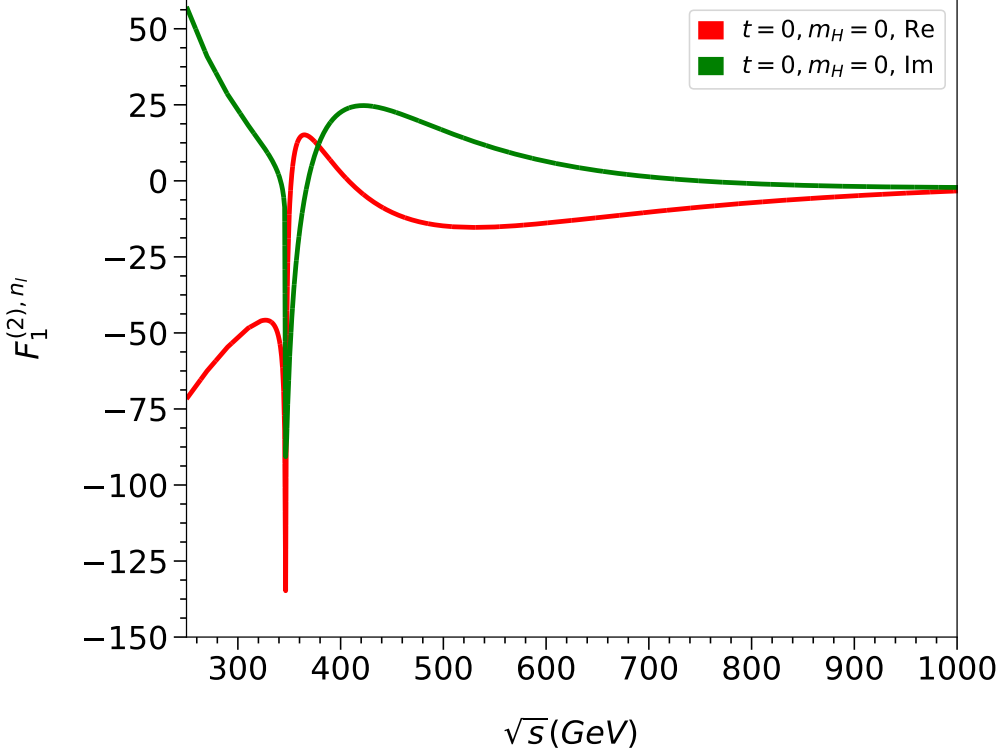


Figure 4: Real (red) and imaginary (green) parts of the n_l part of $F_{\text{box}1}$ at three loops.

functions where in each region we have a power-log expansion with precise numerical coefficients. The choice for the expansion variable depends on the respective region. For example, at threshold we have $v = \sqrt{1 - 4m_t^2/s}$ and at high energies we have $x = m_t^2/s$.

Although $t = 0$ and $m_H = 0$ is a very crude approximation, from considerations at one- and two-loop order it can be expected that for p_T in the vicinity of 100 GeV the (unknown) exact result is approximated at the level of 30%. This is confirmed by a comparison at three-loop order in the large- m_t limit, where the complete n_l terms are available [5].

At two loops the $t = 0$ calculation provided the initial condition for the differential equations which could then be used to obtain higher order expansion terms in t . At three loops this will not be possible since the reduction to master integrals of the box diagrams, which depend on s, t and m_t^2 , is currently out of reach. Thus, we have to expand at the integrand level.

5. Conclusions

In this proceedings contribution we discuss several analytic approximations for two- and three-loop corrections to $gg \rightarrow HH$, which is used as a template for a wider range of processes like $gg \rightarrow ZZ$, $gg \rightarrow ZH$ but also H plus jet production in gluon fusion. The analytic results in the high-energy region and the semi-analytic expression obtained for $t \rightarrow 0$ are sufficiently simple such that the fast numerical evaluation is possible. Furthermore, in the combination of both limits the whole phase space can be covered — at least for $gg \rightarrow HH$ at two loops.

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