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Introduction to higher-spin theories

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These lecture notes are based on a six-hour lecture given at the 2022 edition of the Modave Summer School in Mathematical Physics. They are meant to be an introductory course to higher-spin theories. We start with the basics of group theory allowing one to define higher-spin fields, then discuss the free theory in the metric-like and the frame-like formulations. We end with the construction of higher-spin symmetry algebras and emphasise the link between them and the free theory, as well as the considerations that allow one to construct the former from the symmetries of the latter.

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1. Introduction

These lecture notes are the summary of a course on higher-spin theories. The goal was to present, in a self-contained way, extensions of General Relativity including fields of arbitrary integer spin, which describe unitary unitary, irreducible representations of the Poincaré or (A)dS group with arbitrary spin, that we will refer to as higher-spin gravity, and revolving around the following question: *is there an interacting theory of higher-spin gravity?* Already at the classical level, this proves to be a challenging task, but with potentially promising rewards: understanding higher-spin symmetry could open a new path to quantising gravity and would provide the missing link between ordinary lower-spin theories and String Theory.

Crossing the spin-two barrier (to borrow the terminology of [1]) is a notoriously difficult task, met with many famous no-go theorems, some of them presented in appendix A. These theorems shed doubt on the viability of any higher-spin theory, but mostly rely on S-matrix arguments. Crucially, it was realised that going to curved space-time instead of a flat one would allow to bypass many no-gos, as first suggested by Fradkin and Vasiliev [2–4], and confirmed within the framework of the higher-spin AdS/CFT correspondence [5, 6].

Starting within the well-known framework of perturbative classical gauge theory, we try to follow the historical way in which the story of higher-spin gravity unfolded. The free theory, which can be seen as a generalisation of linearised General Relativity, was built by Fronsdal in any space-time of constant curvature [7, 8] and soon after, the program of completing the free theory by adding interactions perturbatively was initiated. The early yes-go results on the existence of cubic vertices [9] in flat and curved background, culminating in the construction of an interacting theory in AdS_4 [10] and $AdS_{d\geq 4}$ [11] was a confirmation that, although more exotic than their lower-spin cousins, interacting higher-spin theories do exist.

Already at the level of the cubic theory, it was realised that the non-Abelian higher-spin algebra encoding the symmetry was a crucial ingredient. A fully interacting theory in AdS_4 was then built [10], proposing a completion to the program of obtaining non-linear higher-spin theories through the gauging of a higher-spin algebra extending the AdS_4 isometry algebra and mirroring Cartan's construction of General Relativity through the gauging of the Poincaré or (A)dS algebras. Its conjectured holographic dual was proposed in [5, 6] and further developed, e.g., in [12, 13].

In these notes, we put the accent on the link between the symmetries of the free theory, the higher-spin generalisation of linearised diffeomorphisms, the frame-like formulation and the construction of a higher-spin algebra. We will make some reminders on lower-spin (i.e. spin less or equal to two) theories along the way, both as a warm-up for the more technical parts and also to make the case that, despite their rough history, higher-spin theories share many similarities with the gauge theory of gravity.

The higher-spin algebra in any dimension is then constructed, and we examine the case of AdS_3 separately and its connection with higher-spin gravity in the Chern-Simons formulation. Some elements of higher-spin holography [14, 15] are also flashed, with a highlight on the the dual role played by the higher-spin algebra of Fradkin, Vasiliev (in $d = 4$, [3]) and Eastwood (generalisation to all dimensions [16]), serving the dual role of an algebra of higher endomorphisms of the singleton module on the boundary and a higher-spin algebra in the bulk.

The material presented in these notes is by no means new, and we refer to [1, 17–31] for a complete overview of the subject, ranging from non-technical presentations to more in-depth reviews.

1.1 Motivations

We begin by giving some motivation as to why we want to study such theories. Although it constitutes an interesting mathematical and field-theoretic problem on its own, the higher-spin program may find motivations in String Theory, quantum gravity and holography, of which we sketch some elements.

From String Theory

In String Theory, excited states are built as tensor products of oscillators. As an example, the first excited state of bosonic closed String Theory in $d = 26$ contains a rank-two tensor which can be split into a rank-two symmetric traceless tensor $G_{(\mu\nu)}$ (the graviton), a two-form $B_{[\mu\nu]}$ (the Kalb-Ramond field) and a scalar ϕ (the dilaton). The second excited state is built as the product of up to 4 oscillators and has a much more complicated spectrum, involving higher representations of the Lorentz group. Higher-spin excitations appear naturally in String Theory (also in fermionic theories with both open and closed strings), as the decomposition in irreducible components of the Lorentz group of higher excited states

$$|N; p\rangle = \varepsilon_{\mu_1\cdots\mu_{N_L}\nu_1\cdots\nu_{N_R}}\, \tilde{\alpha}^{\mu_1}_{-m_1} \cdots \tilde{\alpha}^{\mu_{N_L}}_{-m_{N_L}}\, \alpha^{\nu_1}_{-n_1} \cdots \alpha^{\nu_{N_R}}_{-n_{N_R}} |0; p\rangle\,, \tag{1.1}$$

with $\varepsilon_{\mu_1\cdots\mu_{N_L}\nu_1\cdots\nu_{N_R}}$ a polarisation tensor and $N = \sum_{k=1}^{N_L} m_k = \sum_{k=1}^{N_R} n_k$ due to level-matching. The $N > 1$ states are usually disconsidered in the low-energy regime, because of their mass

$$M^2 = \frac{4}{\alpha'}(N-1)\,, \tag{1.2}$$

which becomes very large[1] when $\alpha' \to 0$, as opposed to the (super-)gravity multiplet $N = 1$ which remains massless.

Nevertheless, higher-spin modes are relevant for the UV-complete character of String Theory. In fact, string scattering amplitudes possess a hidden symmetry relating scattering amplitudes of states with different spins, as was first observed by Gross in [32]. Moreover, if one probes the tensionless limit of String Theory $\alpha' \to \infty$, higher-spin excitations become massless. This observationhas led to the conjecture that String Theory could be described as the broken phase of a higher-spin gauge theory, in which all massless higher-spin excitations are treated equally. Concerning the treatment of higher-spin excitations in String Theory, and the relation between String Theory and higher-spin gauge theories, one may refer to [22, 27, 33].

From quantising gravity

The old attempt of quantising gravity using techniques of Quantum Field Theory turned out to be inconsistent because of divergences requiring the introduction of infinitely many counter-terms [34], thereby losing predictive power. One way out of this would be to require that a powerful

[1]Strictly speaking, one needs to compare the string scale $\sqrt{\alpha'}$ to another quantity with dimension of length R, furnished by e.g. a curvature. In string compactification, such a length scale is naturally there.

symmetry fixes the form of all counter-terms, making the theory UV-finite or at least renormalisable. One may wonder if higher-spin fields have any place within this program: is higher-spin symmetry powerful enough to cure the UV divergences of gravity, or is it simply making the problem of quantising gravity more difficult?

Such a procedure indeed finds a realisation in String Theory, where higher-spin excitations of the string arise naturally as a way to keep the UV behaviour of the string under control. Starting from General Relativity, one may hope that the introduction of inifinitely many higher-spin fields would help 'soften' the bad UV behaviour of gravity. The problem is therefore to define a theory extending General Relativity and with manifest higher-spin gauge symmetry.

Such a theory would not only serve as a toy model for the tensionless limit of String Theory but also as a candidate to quantise gravity from the bottom up (no strings attached, but the end goal would be to relate this construction to String Theory). The candidate we will study is one of the simplest as it involves only symmetric fields with arbitrary spin. While this setup qualitatively differs from String Theory (as the latter also involves fields with mixed symmetry), we expect that we can learn a lot on the latter by the study of the former.

From holography

One of the distinguishing features of String Theory is its holographic character [35, 36]. Holographic duals of String Theories (e.g. the original IIB / $\mathcal{N} = 4$ SYM duality) usually involve some large-N regime of lower-spin theories such as (super-)Yang-Mills, but in accordance with the intuition on the tensionless limit of String Theory, the putative holographic dual of the tensionless limit of type IIB String Theory is conjectured to be a theory of higher-spin gravity on $\text{AdS}_5 \times S^5$ [37].

While a proof of this conjecture seems still far away, there is evidence that Vasiliev's theory of massless symmetric gauge fields in AdS has a very simple holographic dual [5, 6], namely a free or critical (Wilson-Fischer) $O(N)$ vector model at large N, making it an ideal toy model to study the holographic correspondence. These models also have interesting phenomenological applications in condensed-matter physics (the Ising model).

In these notes, we will briefly touch on the holographic dual of AdS higher-spin theory, explaining in particular the link between the symmetries in the bulk and the boundary. Finally, let us mention the Gaberdiel-Gopakumar duality between three-dimensional higher-spin gravity and two-dimensional $\mathcal{W}_N$-minimal models built as Wess-Zumino-Witten models [38, 39].

1.2 Plan

In section 2 we recall the necessary tools from group theory in order to define higher-spin fields as unitary irreducible representations of the vacuum isometry group of maximally symmetric space-times. In section 3 we present the metric-like formulation of Fronsdal, which is the simplest way of formulating the dynamics, and present the general construction of interactions in a Lagrangian theory. In section 4 we move on to the frame-like formulation, which makes full use of the gauge symmetries found in the previous sections. This paves the way to section 5 where we explain the construction of higher-spin symmetry algebras and explain the link between a non-Abelian algebra and cubic vertices in AdS. Finally, we provide some perspective on the higher-spin program in section 6. In appendix A, we recall some facts about no-go theorems for interacting higher-spin

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theories, as well as possible ways out, while B.2 and C are technical appendices concerning Young tableaux manipulation and the realisation of the higher-spin algebra as the enveloping algebra of conformal isometries, realised on the singleton module.

1.3 Conventions

The dimension of space-time is $d \geq 3$, we will mostly be interested on the three maximally symmetric space-times $\mathbb{M}_d \simeq \mathbb{R}^{d-1,1}$ (Minkowski), AdS$_d$ (Anti-de Sitter) or dS$_d$ (de Sitter), with isometry groups $ISO(d-1,1)$, $SO(d-1,2)$ or $SO(d,1)$. In these notes, d-dimensional flat (Minkowski) space-time $\mathbb{M}_d$ has mostly-plus metric $\eta_{ab} = \text{diag}(-,+,\ldots,+)$. The *ambient space* $\mathbb{R}^{d,1}$, of which the previous space-times are different sections, has dimension $d+1$. The conformal boundary of AdS$_d$ has dimension $d-1$. Indices in $\mathbb{M}_d$ or (A)dS$_d$ space-time will be denoted by Greek indices μ, ν if they are space-time indices or lowercase Latin letters a, b if they are fibre indices. In ambient space they will be denoted by uppercase Latin letters A, B. We will work in the symmetric convention: groups of indices separated by a comma are symmetrised. To ease the notation, we will refrain from using the comma for the generator of Lorentz transformations $J_{a,b} = J_{ab} = J_{[ab]}$ and $J_{A,B} = J_{AB} = J_{[AB]}$.

We use the Einstein summation convention for a repeated index appearing once in the covariant form and once in the contravariant form. We also use the convention that a set of symmetrised indices $(a_1 a_2 \ldots a_k)$ are represented by the shorter notation $a(k)$. In other words, repeated indices that are not summed (either all covariant or all contravariant) are symmetrised, dividing by the minimum number of terms entering the symmetrisation. More details on our conventions and some reminders of tensor calculus can be found in appendix B.1.

2. Higher-spin particles as unitary irreducible representations of vacuum isometry algebras

In this part, we will do some brief reminders of group theory that will allow us to define massless higher-spin particles as irreducible representations of the Poincaré or AdS algebra. Although we will not focus on the case of dS, most of the results concerning higher-spin representations in AdS can be extrapolated to dS.

2.1 Minkowski

We will follow Wigner's induced representations method [40]. We first fix the action of the generator of translations of $ISO(d-1,1)$. For massless representations, the "little group" which stabilises this action is given by the group of Euclidean isometries $ISO(d-2)$. Restricting to its Lorentz subgroup by imposing a trivial action of the $(d-2)$-dimensional translations means that we are dealing with *truly massless* particles (as opposed to continuous spin for example) which are characterised by symmetric and traceless tensors under $SO(d-2)$. For this part, we will mainly follow [41].

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2.1.1 Poincaré algebra

The Poincaré algebra $\mathfrak{iso}(d-1,1)$ reads

$$i[J_{ab}, J_{cd}] = \eta_{bc}\, J_{ad} - \eta_{ac}\, J_{bd} - \eta_{bd}\, J_{ac} + \eta_{ad}\, J_{bc}\,, \tag{2.1a}$$

$$i[J_{ab}, P_c] = \eta_{bc}\, P_a - \eta_{ac}\, P_b\,, \tag{2.1b}$$

$$i[P_a, P_b] = 0\,, \tag{2.1c}$$

where J_{ab} are Lorentz transformations and P_a translations. They correspond to the infinitesimal transformations of the coordinates $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ and $\mathfrak{iso}(d-1,1)$ is the Lie algebra of the group $ISO(d-1,1) \simeq SO(d-1,1) \ltimes \mathbb{R}^{d-1,1}$. In the following, we will sometimes consider a version of the algebra where the structure constants on the right-hand side of the Lie bracket $[\cdot,\cdot]$ are real. This can be achieved by sending the generators to $-i$ times themselves.

2.1.2 Casimirs of the Poincaré algebra

Elements in the centre of the Universal Envelopping Algebra (more details in section 5.1.1) are known as Casimir invariants and are used to characterise representations. The algebras of isometries of maximally symmetric space-times, $\mathfrak{so}(d-1,2)$, $\mathfrak{so}(d,1)$ and $\mathfrak{iso}(d-1,1)$, all have the same number of independent Casimirs [42]. For even $d = 2k$, there are k Casimirs of orders 2, 4, ..., $2k$, while for odd $d = 2k+1$, there are $k+1$ Casimirs of orders 2, 4, ..., $2k$, $k+1$.

In $d = 3$ Both Casimirs are quadratic and are given by $P^2 = P_a P^a$ and $W = \epsilon^{abc} J_{ab} P_c$.

In $d = 4$ Classes of irreducible representations of the Poincaré algebra are characterised by its two Casimirs, the mass-squared and the square of the Pauli-Lubanski pseudo-vector

$$C_2 = -P^2 = -P_a P^a\,, \quad C_4 = W^2 = W_a W^a\,, \tag{2.2}$$

where $W^a = \frac{1}{2}\epsilon^{abcd} J_{bc} P_d$. The eigenvalues of C_2 and C_4 are usually parameterised as $P^2 = -m^2$ and $W^2 = m^2 s(s+1)$.

Arbitrary $d \geq 5$ We have more than two Casimirs, given by the mass-squared $P^2 = P_a P^a$ and higher products built from the generalisation of the Pauli-Lubanski pseudo-vector [43, 44]

$$\frac{1}{2} W^{a_4 \cdots a_d} = \epsilon^{a_1 \cdots a_d} J_{a_1 a_2} P_{a_3}\,, \quad \frac{1}{4} W^{a_6 \cdots a_d} = \epsilon^{a_1 \cdots a_d} J_{a_1 a_2} J_{a_3 a_4} P_{a_5}\,, \quad \cdots\,. \tag{2.3}$$

If the dimension is even, then the squares of the previous tensors are Casimir invariants, if the dimension is odd, then one of them is already a scalar and is an invariant (the one where the Levi-Civita tensor saturates the indices of the generators).

2.1.3 Wigner's classification

We follow the method of induced representations by Wigner [40]. We determine the action of the translations

$$P_a |p\rangle = p_a |p\rangle\,, \tag{2.4}$$

and look at the subgroup of Lorentz transformations stabilising the vector p_a, called the little group. The rank of the little group depends on the value of the mass-squared.

Massless For the massless case, the little group is $ISO(d-2)$. One can repeat the Wigner classification for this little group by first specifying the action of the $(d-2)$-dimensional translations. Here, because of the Euclidean signature, there are only two cases: non-trivial (given by a vector of non-zero norm) or trivial. Discarding the case of non-trivial action which gives rise to continuous (or infinite) spin representations, the ingredient needed to complete the Wigner classification in this case is to specify the action of the Lorentz subgroup $SO(d-2)$. In general, it is given by a tensor $T_{a_1 \cdots a_{n_1}, b_1 \cdots b_{n_2}, \cdots, c_1 \cdots c_{n_k}} = T_{a(n_1), b(n_2), \cdots, c(n_k)}$ characterised by its symmetry in the indices, given by the Young tableau

$$\mathbb{Y}_{d-2}(n_1, n_2, \cdots, n_k) = \begin{array}{|l}\hline \quad\quad\quad n_1 \quad\quad\quad \\ \hline \quad\quad n_2 \quad\quad \\ \hline \quad \cdots \quad \\ \hline n_k \\ \hline \end{array}_{SO(d-2)} \,, \tag{2.5}$$

where $0 < n_k \leq \cdots \leq n_1$ and $0 \leq k \leq \left\lfloor \frac{d-2}{2} \right\rfloor$ (any tableau with more rows can be dualised using the Levi-Civita tensor). The number of components of this tableau will be denoted with absolute value. The conventions are as follows: the indices are symmetrised in each row, traceless inside of each row and between rows, and the compatibility condition imposes that symmetrisation of the indices in a row with an index from the following row gives zero. As an example,

$$T_{a(n_1), ab(n_2-1), c(n_3)} = 0\,, \quad T_{a(n_1), b(n_2), bc(n_3-1)} = 0\,. \tag{2.6}$$

In $d = 4$, the non-zero tensors are parameterised by an irreducible representation of $SO(2)$, which are all given by Young tableaux of the type

$$\mathbb{Y}_2(s) = \begin{array}{|c|}\hline \quad s \quad \\ \hline \end{array}_{SO(2)} \qquad \text{for } s \geq 0\,. \tag{2.7}$$

Indeed, the only non-vanishing tableau with more than one row, $\mathbb{Y}_2(1,1)$, can be dualised to a scalar using the two-dimensional Levi-Civita tensor. All in all, we found that only symmetric fields propagate in $d = 4$. We can also evaluate the number of propagating degrees of freedom using the formula

$$|\mathbb{Y}_{d-2}(s)| = \binom{d-3+s}{s} - \binom{d-5+s}{s-2} = \frac{(d+s-5)!}{s!(d-4)!}(d+2s-4)\,. \tag{2.8}$$

For $d = 4$ and $s \geq 1$, this gives $|\mathbb{Y}_2(s)| = 2$, meaning that massless symmetric higher-spin fields in $d = 4$ always carry 2 degrees of freedom.

Massive For massive particles, the little group is $SO(d-1)$. In $d = 4$, the numbers of propagating degrees of freedom of a massive symmetric higher-spin field is $|\mathbb{Y}_3(s)| = 2s + 1$.

Other Apart from the vacuum, or zero-momentum representation, there are also "exotic" representations of the Poincaré group such as tachyons or continuous spins, which we will not discuss [40, 41]. For a review of continuous spin representations, and their connection with higher-spin representations, see [45].

2.2 AdS space-time

Representation theory of AdS is widely different from the one of Minkowski, since the AdS algebra is semi-simple. However, some representations are present in both classifications, in particular massless fields of arbitrary spin.

2.2.1 AdS as a maximally symmetric space-time

Similarly to d-dimensional Minkowski space-time, AdS_d is a space-time of maximal symmetry (it has $\frac{d(d+1)}{2}$ Killing vectors). We first review its construction as a solution of Einstein's equations, as well as a more geometric construction relying on ambient space. Then, we classify the UIR of the AdS algebra.

Solutions of Einstein's equations Although Minkowski space $\mathbb{M}_d$ is straightforward to define, this is not so much the case of AdS_d space. Let us, for the sake of completeness, define AdS_d as the metric $\bar{g}$ solution to Einstein's vacuum equations with cosmological constant

$$G_{\mu\nu}[g] + \Lambda\, g = 0\,. \tag{2.9}$$

It is a space-time of constant curvature, so the only non-vanishing component of the Riemann tensor is its trace

$$R_{[\mu\nu][\rho\sigma]} = -\frac{1}{\ell^2}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\nu\rho}\bar{g}_{\mu\sigma})\,, \tag{2.10}$$

where the AdS radius ℓ is related to the cosmological constant as

$$\Lambda = -\frac{(d-1)(d-2)}{2\ell^2}\,. \tag{2.11}$$

Ambient space construction It is also convenient to define AdS_d as a section of $d+1$-dimensional ambient space, given by the constraint

$$-X_0^2 - X_d^2 + \sum_{i=1}^{d-1} X_i^2 = -\ell^2\,. \tag{2.12}$$

We can view AdS as a d-dimensional hyperboloid embedded into $d+1$-dimensional flat ambient space, enjoying $SO(d-1,2)$ global symmetry with algebra

$$i[J_{AB}, J_{CD}] = \eta_{BC}\, J_{AD} - \eta_{AC}\, J_{BD} - \eta_{BD}\, J_{AC} + \eta_{AD}\, J_{BC}\,, \tag{2.13}$$

where

$$J_{AB} = i(X_A\, \partial_B - X_B\, \partial_A)\,. \tag{2.14}$$

By taking $a \in \{0, \ldots, d-1\}$ and posing $P_a = \ell^{-1}\, J_{ad}$, we recover the more familiar form of the AdS algebra

$$i[J_{ab}, J_{cd}] = \eta_{bc}\, J_{ad} - \eta_{ac}\, J_{bd} - \eta_{bd}\, J_{ac} + \eta_{ad}\, J_{bc}\,, \tag{2.15a}$$

$$i[J_{ab}, P_c] = \eta_{bc}\, P_a - \eta_{ac}\, P_b\,, \tag{2.15b}$$

$$i[P_a, P_b] = \frac{1}{\ell^2} J_{ab}\,. \tag{2.15c}$$

Note that one can recover Poincaré as a limit of flat curvature[2] (or large radius) of AdS through an İnönü-Wigner contraction [46] leaving the Lorentz subalgebra untouched and Abelianising transvections.

[2]Formally, one needs to consider the regime of large radius with respect to another dimensionful scale, e.g., the Planck mass. This limit taken at the level of the Einstein-Hilbert action admits Minkowski space as a solution. In the following, we will assume that such a length scale is always present (i.e. gravity exists in the bulk).

2.2.2 Unitary irreducible representations of the AdS algebra

In the following, we will focus on lowest-weight irreducible representations of AdS. Let's take $\mathfrak{so}(2) \oplus \mathfrak{so}(d-1) \subset \mathfrak{so}(d-1,2)$ as a maximally compact subalgebra, with the $\mathfrak{so}(2)$ part generated by E such that $E^\dagger = E$ (real energy) and $\mathfrak{so}(d-1)$ generated by J_{ij} such that $J_{ij}^\dagger = -J_{ij}$. The generators in the non-compact directions will be called $J_i^\pm$ with $\left(J_i^\pm\right)^\dagger = J_i^\mp$. We have the splitting

$$\mathfrak{so}(d-1,2) \simeq \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_- \,, \tag{2.16}$$

where $\mathfrak{g}_0$ is spanned by E and J_{ij} (the Cartan generators associated to this decomposition) and $\mathfrak{g}_\pm$ by $J_i^\pm$. Irreducible representations are defined by their eigenvalues on the maximally compact subalgebra $\oplus_{n\in N}|E_n, \mathbb{Y}_n\rangle$, with the set N possibly being infinite, such that $E_0 \leq E_1 \leq \ldots$ are ordered eigenvalues of E and the $\mathbb{Y}^n_{d-1}$ characterise the transformation under J_{ij} (the analogue of the little group in flat space). We define the state with the lowest eigenvalue of E to be the *lowest weight* (or vacuum in the language of Fock states), i.e. it is annihilated by the lowering operator J_i^-

$$J_i^-|E_0, \mathbb{Y}_0\rangle = 0\,. \tag{2.17}$$

Then, the descendants are

$$J^+_{i_1} \ldots J^+_{i_m}|E_0, \mathbb{Y}_0\rangle\,. \tag{2.18}$$

Unitarity imposes that the squared norm of the state and all such descendants are definite positive. We find that this condition, for symmetric fields $\mathbb{Y}_0 = \mathbb{Y}_{d-1}(s)$ with $s > 0$ reads

$$E_0 \geq s + d - 3\,. \tag{2.19}$$

On the other hand, for a scalar $\mathbb{Y}_0 = \bullet$,

$$E_0 \geq \frac{d-3}{2}\,. \tag{2.20}$$

Note that P^2 is no longer a Casimir of the AdS algebra since it does not commute with transvections. The quadratic Casimir is instead

$$C_2|E_0, \mathbb{Y}_0\rangle = \frac{1}{2} J_{AB} J^{AB}|E_0, \mathbb{Y}_0\rangle = \left[E_0(E_0 - d + 1) + C_2(\mathfrak{so}(d-1))\right]|E_0, \mathbb{Y}_0\rangle\,. \tag{2.21}$$

where $C_2(\mathfrak{so}(d-1))$ is the quadratic Casimir of the rotation subalgebra. For totally symmetric tensors of rank s,

$$C_2(\mathfrak{so}(d-1))|E_0, \mathbb{Y}_{d-1}(s)\rangle = s(s + d - 3)|E_0, \mathbb{Y}_{d-1}(s)\rangle\,. \tag{2.22}$$

The analogue of a massless spinning field in AdS is one that has the same number of propagating degrees of freedom as in Minkowski space where the energy saturates the unitarity bound [47].

3. Metric-like formulation

In this section, we will derive the first – and simplest – formulation of the dynamics of free massless symmetric higher-spin fields for integer spin. It is the Fronsdal formulation [7], which generalises the Fierz-Pauli action [48]. In the following, indices are raised and lowered with respect to a background metric $\bar{g}_{\mu\nu}$, and we have a metric-compatible connection ∇_μ.

The advantage of the Fronsdal formulation is that it is a clear generalisation of the lower-spin cases (Maxwell's theory of electromagnetism and the linearised theory of General Relativity of Fierz and Pauli). There are however a few inconvenients with this formulation, for instance the fact that we have to deal with doubly-traceless fields (see section 3.2).

An action principle is also established. We will first have a look at lower-spin theories, namely Maxwell's and Einstein's, before generalising the equations of motion and action to higher-spin, both in flat and (A)dS space-time. We identify the reducibility parameters of the theory, that is, the gauge transformations that preserve the vacuum solution. These will play a crucial role in the algebraic approach of sections 4 and 5. We then move on to the classification of interaction vertices in a weak-field expansion, within the Noether procedure. At the cubic level, there are some non-Abelian vertices hinting at a higher-spin algebra, that is constrained by the consistency of quartic vertices. Finally, we provide some comments on quartic interactions.

3.1 Low-spin examples

We begin by recalling some well-known examples of gauge theories, starting with Maxwell and linearised General Relativity. We will look at the case of flat space first, then present the generalisation to curved (AdS) space.

3.1.1 Flat space

We choose the flat Minkowski metric as a background $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ (and therefore $\nabla_\mu = \partial_\mu$).

Spin-1 Maxwell's equations are formulated in terms of the $\mathfrak{u}(1)$ gauge potential A_μ with

$$\partial^\mu F_{\mu\nu}[A] = \partial^2 A_\nu - \partial_\mu\, \partial \cdot A = 0\,, \tag{3.1}$$

where $\partial \cdot A = \partial^\mu A_\mu$ and we used the curvature/field-strength tensor

$$F_{\mu\nu}[A] \equiv 2\, \partial_{[\mu}\, A_{\nu]}\ . \tag{3.2}$$

One can see directly that it is invariant under the following gauge variation

$$\delta A_\mu = \partial_\mu \alpha\ . \tag{3.3}$$

In $d \geq 2$ dimensions, we can find the propagating degrees of freedom by reducing the system to a bunch of Klein-Gordon equations. One can start by imposing the Lorenz[3] gauge $\partial \cdot A = 0$ and fix the residual gauge on the parameter α preserving this gauge $\partial^2 \alpha = 0$. There are d components in A_μ and we imposed two equations by fixing the gauge so the number of propagating degrees of freedom is $d - 2$.

[3]Other choices of gauge are useful, e.g. the Coulomb gauge $\partial_i A^i = 0$ when studying semi-classical or non-relativistic regimes of Maxwell's theory. In the following, we will always remain manifestly Lorentz-covariant.

Spin-2 The equations of motion for a linearised metric tensor $h_{\mu\nu}$ (Fierz-Pauli) are

$$R_{\mu\nu}[h] = 0\,, \tag{3.4}$$

by means of the linearised Ricci tensor, i.e. the trace of the linearised Riemann tensor

$$R_{\mu\nu}[h] = \partial^2 h_{\mu\nu} - 2\,\partial_{(\mu}\partial\cdot h_{\nu)} + \partial_\mu\partial_\nu h' = \partial^2 h_{\mu\nu} - 2\,\partial_{(\mu}D_{\nu)}[h]\,, \tag{3.5}$$

where $\partial^2 = \partial_\mu\partial^\mu$, $h' = \eta^{\mu\nu}\,h_{\mu\nu}$ and

$$D_\mu[h] \equiv \partial\cdot h_\mu - \frac{1}{2}\partial_\mu\, h'\,. \tag{3.6}$$

Alternatively, if one knows nothing about General Relativity, one can re-derive this equation of motion by starting from the two-derivative term $\partial^2 h_{\mu\nu}$ and adding traces and divergences such that this curvature becomes gauge-invariant under

$$\delta h_{\mu\nu} = \partial_{(\mu}\xi_{\nu)}\,. \tag{3.7}$$

Exercise: prove that the tensor (3.5) is invariant under (3.7).

Note that this is the linearisation of Einstein's vacuum equation if the background metric is already a solution of the full equations, which is the case for Minkowski space-time.

We can find the number of propagating degrees of freedom by imposing the De Donder gauge $D_\mu[h] = 0$ and find the residual gauge transformations (i.e. the subset of linearised diffeomorphisms) that preserve this gauge. They are given by the harmonic gauge parameters, i.e. satisfying $\partial^2\,\xi_\mu = 0$. All in all, we started with $\frac{d(d+1)}{2}$ and imposed $2d$ conditions to go back to a harmonic system so we have $\frac{(d-1)(d-2)}{2} - 1$, in accordance with $|\mathbb{Y}_{d-2}(2)|$.

One can fix the gauge even further because both $D_\mu[h] = 0$ and $\partial^2\xi_\mu = 0$ carry a Lorentz index (this does not affect the counting of degrees of freedom, but allows to recast the system in a more symmetric way). Indeed, on-shell one can impose separately $\partial\cdot h_\mu = 0$ and $h' = 0$, which is preserved by parameters satisfying $\partial\cdot\xi = 0$. Note that one does the same in $d = 4$ by imposing a transverse-traceless gauge.

The completely reduced system reads

$$\partial^2 h_{\mu\nu} = 0\,, \quad \partial\cdot h_\mu = 0\,, \quad h' = 0\,, \tag{3.8a}$$

$$\partial^2\xi_\mu = 0\,, \quad \partial\cdot\xi = 0\;. \tag{3.8b}$$

3.1.2 (A)dS space

We switch now to space-time of constant curvature, with metric $\bar{g}$ defined in section 2.2.1 (the de Sitter case can be recovered by formally sending $\ell^2 \to -\ell^2$ in all formulae) and associated Levi-Civita connection ∇. The previous analysis remains largely unchanged, with the only modification being the appearance of terms proportional to the cosmological constant. These arise from the commutator of two covariant derivatives being non-zero, e.g. for a vector V^ρ,

$$[\nabla_\mu, \nabla_\nu]V^\rho = -\frac{2}{\ell^2}\delta^\rho{}_{[\mu}\,V_{\nu]}\,, \tag{3.9}$$

due to the Riemann tensor taking the expression of eq. (2.10)

Spin-1 Maxwell's equations in curved space can be written

$$0 = \nabla^\mu F_{\mu\nu}[A] = \nabla^2 A_\nu - \nabla_\nu D[A] - [\nabla^\mu, \nabla_\nu] A_\mu = \nabla^2 A_\nu - \nabla_\nu D[A] + \frac{d-1}{\ell^2} A_\nu \,, \tag{3.10}$$

in terms of

$$F_{\mu\nu}[A] = 2\,\nabla_{[\mu} A_{\nu]} \,, \tag{3.11}$$

and $D[A] = \nabla \cdot A$. Upon gauge fixing, the equations of motion becomes

$$\nabla^2 A_\mu + \frac{d-1}{\ell^2} A_\mu = 0 \,, \quad \nabla \cdot A = 0 \,, \quad \nabla^2 \alpha = 0 \,, \tag{3.12}$$

with $\delta A_\mu = \nabla_\mu \alpha$.

Spin-2 The equation of motion is the vanishing of the linearised Ricci tensor

$$0 = R_{\mu\nu}[h] \equiv \nabla^2 h_{\mu\nu} - 2\,\nabla^\alpha \nabla_{(\mu} h_{\nu)\alpha} + \nabla_\mu \nabla_\nu h' - 2\frac{d-1}{\ell^2} h_{\mu\nu} \,. \tag{3.13}$$

This can be recast as

$$0 = \nabla^2 h_{\mu\nu} - 2\,\nabla_{(\mu} D_{\nu)}[h] + \frac{2}{\ell^2}\left(h_{\mu\nu} - \bar{g}_{\mu\nu} h'\right) , \tag{3.14}$$

with the De Donder tensor $D_\mu[h] = \nabla \cdot h_\mu - \frac{1}{2}\nabla_\mu h'$ (note the differences in the mass terms between this form and eq. (3.13), this is due to the commutation of covariant derivatives). Upon gauge fixing, the equations of motion read

$$\nabla^2 h_{\mu\nu} + \frac{2}{\ell^2} h_{\mu\nu} = 0 \,, \quad \nabla \cdot h_\mu = 0 \,, \quad h' = 0 \,, \tag{3.15a}$$

$$\nabla^2 \xi_\mu - \frac{d-1}{\ell^2} \xi_\mu = 0 \,, \quad \nabla \cdot \xi = 0 \,, \tag{3.15b}$$

with $\delta h_{\mu\nu} = \nabla_{(\mu}\, \xi_{\nu)}$. Note the appearance of the 'mass-like' term in eq. (3.13), which is necessary to guarantee the invariance of the equation of motion under linearised diffeomorphisms.

Exercise: recover the mass-like term in (3.13) by the requirement of gauge-invariance.

3.2 Fronsdal's equations of motion

After having reviewed in details the spin-1 and 2 cases, one can try to generalise to higher-spins. In order to remain as close as possible to the case of linearised gravity and write an action principle, one can start from a completely symmetric, *doubly traceless* field $\phi_{\mu(s)}$.

3.2.1 Flat space

For a doubly-traceless field $\phi_{\mu(s)}$, one defines the Fronsdal tensor [7, 8]

$$\begin{aligned} F_{\mu(s)}[\phi] &= \partial^2 \phi_{\mu(s)} - s\,\partial_\mu \partial \cdot \phi_{\mu(s-1)} + \frac{s(s-1)}{2} \partial_\mu \partial_\mu \phi'{}_{\mu(s-2)} \\ &= \partial^2 \phi_{\mu(s)} - s\,\partial_\mu D_{\mu(s-1)}[\phi] \,, \end{aligned} \tag{3.16}$$

with the De Donder tensor naturally generalising eq. (3.6)

$$D_{\mu(s-1)}[\phi] = \partial \cdot \phi_{\mu(s-1)} - \frac{s-1}{2}\partial_\mu \phi'_{\mu(s-2)}\,. \tag{3.17}$$

Fronsdal's equations in flat space are given by

$$F_{\mu(s)}[\phi] = 0 \qquad \text{(e.o.m.)}\,, \tag{3.18}$$

which are invariant under $\delta\phi_{\mu(s)} = \partial_\mu \xi_{\mu(s-1)}$, with $\xi'_{\mu(s-3)} = 0$. Both the condition that the field is doubly-traceless and the gauge parameter is traceless are new compared to the usual lower-spin examples.

> *Exercise:* show that the Fronsdal tensor $F_{\mu(s)}[\phi]$ is invariant under the gauge transformations $\delta\phi_{\mu(s)} = \partial_\mu \xi_{\mu(s-1)}$ if and only if the gauge parameter $\xi_{\mu(s-1)}$ is traceless.

We want to show that Fronsdal equations of motion propagate the right number of degrees of freedom, by going completely on-shell. To that end, we proceed like for the linearised spin-2 theory and fix the gauge by requiring

$$D_{\mu(s-1)}[\phi] = 0\,, \tag{3.19}$$

which is preserved by

$$\partial^2 \xi_{\mu(s-1)} = 0\,. \tag{3.20}$$

Furthermore, on-shell we can set the trace of $\phi_{\mu(s)}$ to zero by virtue of $\delta\phi'_{\mu(s-2)} = \partial \cdot \xi_{\mu(s-2)}$, by imposing that ξ is divergence-free. One obtains the set of differential equations known as a Fierz system

$$\partial^2 \phi_{\mu(s)} = 0\,, \qquad \partial \cdot \phi_{\mu(s-1)} = 0\,, \qquad \phi'_{\mu(s-2)} = 0\,, \tag{3.21a}$$

$$\partial^2 \xi_{\mu(s-1)} = 0\,, \qquad \partial \cdot \xi_{\mu(s-2)} = 0\,, \qquad \xi'_{\mu(s-3)} = 0\,, \tag{3.21b}$$

with $\delta\phi_{\mu(s)} = \partial_\mu \xi_{\mu(s-1)}$.

Let us quickly count the number of propagating degrees of freedom and show it matches those of a massless field. The Fronsdal tensor has the same symmetries as the field, i.e. it is a rank s symmetric doubly traceless tensor with

$$|\mathbb{Y}_d(s)| + |\mathbb{Y}_d(s-2)| = \binom{d-1+s}{s} - \binom{d-5+s}{s-4}\,, \tag{3.22}$$

while both the gauge fixing using the De Donder tensor $D_{\mu(s-1)}[\phi]$ and the parameters verifying the residual gauge condition $\partial^2 \xi_{\mu(s-1)} = 0$ are traceless

$$|\mathbb{Y}_d(s-1)| = \binom{d-2+s}{s-1} - \binom{d-4+s}{s-3}\,. \tag{3.23}$$

In total, we find that there are $|\mathbb{Y}_d(s)| + |\mathbb{Y}_d(s-2)| - 2|\mathbb{Y}_d(s-1)|$ propagating degrees of freedom.

> *Exercise:* check that this agrees with the expectation of eq. (2.8).

3.2.2 (A)dS space

As with the case of linearised gravity, the basic form of the equations of motion does not change, except for the appearance of "mass-like" terms. In AdS, the Fronsdal tensor becomes

$$F_{\mu(s)}[\phi] = \nabla^2\phi_{\mu(s)} - s\,\nabla_\mu D_{\mu(s-1)}[\phi] - \frac{1}{\ell^2}\left((\ell m_s)^2\,\phi_{\mu(s)} + s(s-1)\,\bar{g}_{\mu(2)}\,\phi'_{\mu(s-2)}\right). \qquad (3.24)$$

with

$$D_{\mu(s-1)}[\phi] = \nabla\cdot\phi_{\mu(s-1)} - \frac{s-1}{2}\nabla_\mu\phi'_{\mu(s-2)}\,, \qquad (3.25)$$

together with $\delta\phi_{\mu(s)} = \nabla_\mu\xi_{\mu(s-1)}$ and $\xi'_{\mu(s-3)} = 0$. There is a precise non-zero value of ${m_s}^2$ such that gauge symmetry and the number of propagating d.o.f. are the same as in flat space. This is given by

$$(\ell m_s)^2 = (s-2)(s+d-3) - s\,. \qquad (3.26)$$

This mass term can be seen to emerge from the representation theory of the (A)dS$_d$ in the case of completely symmetric bosonic fields [49]

$$(\ell m_s)^2 = E(E-d+1) - s = (s-2)(s+d-3) - s\,, \qquad (3.27)$$

for $E = s+d-3$.

By completely fixing the gauge, we obtain the (A)dS Fierz system

$$\nabla^2\phi_{\mu(s)} - {m_s}^2\phi_{\mu(s)} = 0\,, \qquad \nabla\cdot\phi_{\mu(s-1)} = 0\,, \qquad \phi'_{\mu(s-2)} = 0\,, \qquad (3.28a)$$

$$\nabla^2\xi_{\mu(s-1)} - {m'_s}^2\xi_{\mu(s-1)} = 0\,, \qquad \nabla\cdot\xi_{\mu(s-2)} = 0\,, \qquad \xi'_{\mu(s-3)} = 0\,, \qquad (3.28b)$$

with $\delta\phi_{\mu(s)} = \nabla_\mu\xi_{\mu(s-1)}$ and

$$(\ell m'_s)^2 = (s-1)(s+d-3)\,. \qquad (3.29)$$

Note that, contrary to flat space, when $\Lambda \neq 0$, there exist other discrete values of the "mass-like" parameters such that the number of propagating degrees of freedom lies between that of a massless field and that of a massive field. These points are called partially-massless [50, 51] and for spin s there are $s-1$ of them numbered $m_{s,t}$ for $t \in \{1\ldots s-1\}$, with $m_{s,0} = m_s$ corresponding to the massless case. One famous example is partially-massless gravity corresponding to $s=2$ and $t=1$, with $(\ell m_{2,1})^2 = -d$ [52–54].

3.3 Reducibility parameters

One can compute the reducibility parameters of the theory, i.e. the gauge variations preserving the vacuum solution $\phi_{\mu(s)} = 0$ for $s \geq 1$ (to be interpreted in the spin-2 case as the isometries of the backgroun). These are given in the Fronsdal formulation on a maximally symmetric background by the solutions of the equation

$$\nabla_\mu\xi_{\mu(s-1)} = 0\,, \qquad (3.30)$$

with traceless gauge parameter $\xi_{\mu(s-1)}$. Eq. (3.30) is a tensorial Killing equation, generalising the usual vectorial[4] Killing equation for $s=2$. In flat space, its solutions are simply given by

$$\xi_{\mu(s-1)} = \sum_{t=0}^{s-1} M_{\mu(s-1),\nu(t)}x^\nu\cdots x^\nu\,, \qquad (3.31)$$

[4] In maximally symmetric space-times, traceless Killing tensors can always be written as symmetrised traceless products of Killing vectors.

for constant tensors $M_{\mu(s-1),\nu(t)}$ which are in irreducible representations of the Lorentz group

$$\eta^{\mu\mu} M_{\mu(s-1),\nu(t)} = 0\,, \quad \eta^{\mu\nu} M_{\mu(s-1),\nu(t)} = 0\,, \quad \eta^{\nu\nu} M_{\mu(s-1),\nu(t)} = 0\,, \tag{3.32}$$

and the compatibility condition

$$M_{\mu(s-1),\mu\nu(t-1)} = 0\,. \tag{3.33}$$

One can see that the solutions to the Killing equation (3.30) are tensors with the symmetries (this is also true in (A)dS space-time) [55, 56]

$$\mathbb{Y}_d(s-1)\,, \quad \mathbb{Y}_d(s-1,1)\,, \quad \ldots\,, \quad \mathbb{Y}_d(s-1,s-1)\,. \tag{3.34}$$

In the $s = 1$ case, this gives rise to a single (scalar) generator of $\mathfrak{u}(1)$ transformations, while in the $s = 2$ case we recover the canonical realisation of translations and Lorentz transformations

$$\xi^\mu \partial_\mu = a^\mu \partial_\mu + \omega^{\mu,\nu} x_\nu \partial_\mu\,, \tag{3.35}$$

for $t = 0$ and $t = 1$ respectively. For higher-spins $s \geq 3$, they will be interpreted later as being the complete set of isometry generators of a spin-s field, in the frame-like formulation.

Note that these are the reducibility parameters of the *Fronsdal theory*. As we will see later, there exist also alternative formulations of the dynamics. However, the traceless tensorial Killing equation (3.30) always seems to pop up in any manifestly Lorentz-covariant formulation, so we will refer to these symmetries as higher-spin symmetries.

3.4 Fronsdal Lagrangian

For the equations of motion to have a chance of being Lagrangian, it is usually necessary that they have the same symmetries as the field $\phi_{\mu(s)}$ itself. This is the case in the Fronsdal formulation. Let the Einstein-like action

$$S = \frac{1}{2} \int d^d x \sqrt{-\bar{g}}\, \phi^{\mu(s)} G_{\mu(s)}[\phi]\,, \tag{3.36}$$

where the Einstein-like tensor is

$$G_{\mu(s)}[\phi] = F_{\mu(s)}[\phi] - \frac{s(s-1)}{4} \bar{g}_{\mu(2)} F'{}_{\mu(s-2)}[\phi]\,. \tag{3.37}$$

Upon variation of the action (3.36) with respect to ϕ, one recovers Fronsdal's equations modified by a trace term, but essentially

$$G_{\mu(s)}[\phi] = 0 \quad \Leftrightarrow \quad F_{\mu(s)}[\phi] = 0\,, \tag{3.38}$$

as long as $d \geq 3$, because $G''{}_{\mu(s-4)}[\phi] = F''{}_{\mu(s-4)}[\phi] = 0$. One could wonder why considering this complicated Lagrangian and not directly $\phi^{\mu(s)} F_{\mu(s)}[\phi]$? This is because of gauge invariance. Indeed,

$$\delta_\xi S = -\int d^d x \sqrt{-\bar{g}}\, \xi^{\mu(s-1)} B_{\mu(s-1)}[\phi]\,, \tag{3.39}$$

where the Bianchi tensor

$$B_{\mu(s-1)}[\phi] = \nabla \cdot G_{\mu(s-1)}[\phi] = \nabla \cdot F_{\mu(s-1)}[\phi] - \frac{s(s-1)}{4} \nabla_\mu F'{}_{\mu(s-2)}[\phi] \tag{3.40}$$

is a pure trace when $\phi''_{\mu(s-4)} = 0$ (remember that, for a symmetric tensor, the double trace can be calculated by contracting any two disjoint pairs of indices, e.g. $\phi^{\nu}{}_{\nu}{}^{\rho}{}_{\rho}{}^{\mu(s-4)}$). This is enough to ensure that the action (3.36) is gauge invariant, since in eq. (3.39) the gauge parameter $\xi^{\mu(s-1)}$ is traceless, so its contraction with a pure trace gives zero.

> *Exercise:* verify that this is indeed the Bianchi tensor is a pure trace if an only if $\phi_{\mu(s)}$ is doubly-traceless.

3.5 Alternative metric-like formulations

While the Fronsdal formulation is rather natural, there exist other formulations, each coming with their own set of benefits and disadvantages.

Maxwell-like One can find another formulation of spin $s \geq 2$ that generalises the Maxwell action with traceless fields (instead of doubly traceless fields for (3.16) and (3.36)) [57, 58]. The number of propagating degrees of freedom of the Maxwell-like theory is the same, although the Lagrangian and the gauge transformations take a different form

$$M_{\mu(s)}[\phi] = \nabla^2 \phi_{\mu(s)} - s\, \nabla_\mu \nabla \cdot \phi_{\mu(s-1)} - m_s^2\, \phi_{\mu(s)}\,, \tag{3.41}$$

with $\phi'_{\mu(s-2)} = 0$, $\delta\phi_{\mu(s)} = \nabla_\mu \xi_{\mu(s-1)}$ and $\xi'_{\mu(s-3)} = \nabla \cdot \xi_{\mu(s-2)} = 0$. This theory can be viewed as a partial gauge fixing of the Einstein-like theory. The differential constraint on the gauge parameter $\xi_{\mu(s-1)}$ being divergence-free is here to ensure that the field $\phi_{\mu(s)}$ remains traceless upon gauge variation. There is a Lagrangian formulation for the Maxwell-like theory

$$S = \frac{1}{2} \int \sqrt{-g}\, \phi^{\mu(s)} M_{\mu(s)}[\phi]\,. \tag{3.42}$$

Maxwell-like Lagrangians for theories with mixed-symmetry fields were built in [58], while cubic interactions were constructed in [59].

The unifying language of de Wit-Friedman curvatures One can recast the previous two formulations (Fronsdal or Einstein-like and Maxwell-like) into a unified setup by defining generalised curvatures. The de Wit-Friedman curvatures [60] are tensors with an increasing number of derivatives on the field. In the Fierz-Pauli theory for the field $h_{\mu(2)}$, we have

$$\Gamma^0{}_{\mu(2)} = h_{\mu(2)}\,, \tag{3.43a}$$

$$\Gamma^1{}_{\mu(2),\nu} = \partial_\nu h_{\mu(2)} - 2\, \partial_\mu h_{\mu\nu}\,, \tag{3.43b}$$

$$\Gamma^2{}_{\mu(2),\nu(2)} = 2\left(\partial_\nu \partial_\nu h_{\mu(2)} - 2\partial_\mu \partial_\nu h_{\mu\nu} + \partial_\mu \partial_\mu h_{\nu(2)}\right), \tag{3.43c}$$

where $\Gamma^0{}_{\mu(2)}$ is the field $h_{\mu(2)}$ itself, $\Gamma^1{}_{\mu(2),\nu}$ is proportional to the linearised Christoffel symbol and $\Gamma^2{}_{\mu(2),\nu(2)}$ is proportional to the linearised Riemann tensor. The latter is invariant under $\delta h_{\mu(2)} = \partial_\mu \xi_\mu$. The higher-spin generalisation of this is given by

$$\Gamma^0{}_{\mu(s)} = \phi_{\mu(s)}\,, \quad \Gamma^m{}_{\mu(s),\nu(m)} = m\, \partial_\nu \Gamma^{m-1}_{\mu(s),\nu(m-1)} - s\, \partial_\mu \Gamma^{m-1}{}_{\mu(s-1)\nu,\nu(m-1)}\,, \tag{3.44}$$

with the last element of the hierarchy, i.e. for $m = s$, which plays the role of a gauge-invariant curvature. The variation of the curvatures under linearised diffeomorphisms is given by

$$\delta_\xi \Gamma^m{}_{\mu(s),\nu(m)} \propto \partial_\mu \ldots \partial_\mu \xi_{\mu(s-m-1)\nu(m)} \tag{3.45}$$

and it can be seen that for $m = s$, the curvature $\Gamma^s{}_{\mu(s),\nu(s)}$ is automatically invariant. The Fronsdal tensor is built out of the second curvature

$$F_{\mu(s)} = \eta^{\nu(2)} \Gamma^2{}_{\mu(s),\nu(2)} , \tag{3.46}$$

while the Maxwell tensor out of the first curvature

$$M_{\mu(s)} = \partial^\nu \Gamma^1{}_{\mu(s),\nu} . \tag{3.47}$$

Theories built out of higher-order curvatures were formulated in [61]. They are higher-derivative and have in general a more complicated spectrum than a single massless fields, possibly with ghosts [62]. They can be recast in a two-derivative form by multiplying with inverse powers of the Laplacian $\frac{1}{\partial^2}$ at the expense of locality.

Compensator form One might wonder what happens if we relax the doubly-traceless condition. This is known as the unconstrained formulation, and in this case the equations of motion are non-Lagrangian. Consider a Fronsdal equation for a traceful $\phi_{\mu(s)}$ and gauge parameter $\xi_{\mu(s-1)}$

$$F_{\mu(s)}[\phi] = 3\, \partial_\mu \partial_\mu \partial_\mu \alpha_{\mu(s-3)} , \tag{3.48}$$

where $\alpha_{\mu(s-3)}$ is called a compensator field such that

$$\delta \alpha_{\mu(s-3)} = \xi'{}_{\mu(s-3)} . \tag{3.49}$$

The equations of motion for this system reproduce the same dynamics upon gauge fixing since the gauge transformation of α is algebraic [63–65].

Dual formulations If one performs a dualisation (contraction with a Levi-Civita tensor) of the field strengths defined in 3.5, one obtains one of the *dual* formulations of the dynamics, in terms of fields with mixed symmetry. For gravity, this was studied in [66, 67]. For higher-spin fields, see [65, 68–70]. The interest in studying these seemingly exotic (and more complicated) formulations relies on the difficulty of constructing interactions (see the next section). While cubic interactions can be constructed without problems from the Fronsdal Lagrangian. Even if it has been shown that for gravity, no quartic interactions can be constructed in this form [71], it might be that quartic interactions are easier to construct (or only exist in a local form) in the dual formulation of higher-spin fields. The dual formulation also has the advantage of making some symmetries more visible.

3.6 Constructing interactions

Now that the free theory is identified, we turn to the task of constructing interactions order by order, perturbatively in a small coupling constant. We will follow the method of [72], which reproduces the full Yang-Mills and gravity theory from the free action.

3.6.1 Noether procedure

In this section, we follow the review [73]. Let us start with a Lagrangian depending on a set of fields ψ (Maxwell, linearised graviton, Fronsdal etc.) and try to add interaction terms

$$S[\psi] = S_2[\psi] + g\, S_3[\psi] + O(g^2)\,, \tag{3.50}$$

where $S_2[\psi]$ is quadratic in ψ, $S_3[\psi]$ cubic, etc. The action should be invariant under the gauge transformations

$$\delta\psi = \delta_0\psi + g\,\delta_1\psi + O(g^2)\,, \tag{3.51}$$

where $\delta_0\psi$ is field-independent, $\delta_1\psi$ is linear in ψ etc. We expand $\delta S[\psi] = 0$ order-by-order to obtain equations of the form

$$\delta_0 S_2[\psi] = 0\,, \quad \delta_1 S_2[\psi] + \delta_0 S_3[\psi] = 0\,, \quad \ldots\,. \tag{3.52}$$

In the case of Fronsdal theory, $S_2[\psi]$ is given by the Fronsdal Lagrangian (3.36), and we have at lowest order

$$\delta_0\phi_{\mu(s)} = \nabla_\mu \xi_{\mu(s-1)}\,, \tag{3.53}$$

generating Abelian gauge transformations (more on this in the next section). The goal is to find next δ_1 and S_3 that verify these conditions, and possibly higher order conditions, known as quartic consistency conditions.

Note that as we go on, we have more and more complicated equations with more and more unknowns (actions and gauge transformations) that are subject to field-redefinition ambiguities (see, e.g., the discussion in [31]).

3.6.2 Abelian vs non-Abelian gauge transformations

There is a natural associative structure associated to gauge transformations that gives rise to a bracket. For two gauge transformations δ_α and δ_β acting on a field ψ, we define

$$[\delta_\alpha, \delta_\beta]\psi = (\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi\,. \tag{3.54}$$

It will prove interesting to distinguish between two classes of gauge transformations for higher-spin fields: Abelian and non-Abelian. Abelian gauge transformation δ_ξ are such that

$$[\delta_\xi, \delta_{\xi'}]\phi_{\mu(s)} = 0\,. \tag{3.55}$$

It is trivial to see that the gauge transformations δ_0 given in (3.53) are Abelian, since they do not depend on the field. On the other hand, non-Abelian gauge transformations δ_χ are such that

$$[\delta_\chi, \delta_{\chi'}]\phi_{\mu(s)} \neq 0\,. \tag{3.56}$$

We need at least gauge transformations that are linear in ϕ for the right-hand side to be non-zero, so non-Abelian gauge transformations are generated by δ_1 at least. At lowest order, the non-Abelian gauge transformations evaluated on reducibility parameters are field-independent, and so they generate a Lie algebra, provided that all necessary fields and gauge parameters are introduced.[5]

[5] For higher orders, we will have in general an algebroid since the structure constants are field-dependent.

In practice, higher-spin self-interaction can only be made gauge invariant if one introduces fields of all spin [72] (this also emerges as a consistency requirement at the quartic order), pointing to an infinite-dimensional Lie algebra (see, however [74] for obstructions in the case of spin-3 self-interactions).

Furthermore, the action of gauge transformation is associative on the space of fields, so the algebra generated by non-Abelian gauge transformations must satisfy the Jacobi identity

$$[[\delta_\chi, \delta_{\chi'}], \delta_{\chi''}]\phi_{\mu(s)} + \text{cyclic} = 0\,. \tag{3.57}$$

At lowest order, this means simply that the algebra is Lie. However, if we write $[\delta_\chi, \delta_{\chi'}]\phi = \delta_\alpha\phi$ with α a function of ϕ, χ and χ' as it is the case for higher-order structure constant, this turns into a non-trivial statement to be verified at each order for the consistency of the theory. A summary of the procedure is displayed in table 1.

Order	Theory	Gauge symmetries	Algebraic quantities
Quadratic	S_2 given in eq. (3.36)	$\delta_0\phi = \nabla\xi$	Spectrum of generators
Cubic	$S_3 = \mathcal{O}(\phi^3)$	$\delta_1\phi = \mathcal{O}(\chi, \phi)$	Structure constants
Quartic	$S_4 = \mathcal{O}(\phi^4)$	$\delta_2\phi = \mathcal{O}(\chi, \phi^2)$	Jacobi identity, ?...

Table 1: Strategy for the Noether procedure and translation in terms of the underlying algebraic structure.

The hope is that the identification of a non-Abelian Lie algebra emerging from the non-Abelian cubic gauge transformations in the case of higher-spin would be enough to completely determine the fully interacting theory, as is the case in the gauge (or Cartan) formulation of General Relativity. In the following, we will see that, although a gauge algebra can be defined (only in AdS space-time within this procedure), some obstructions arise at the level of quartic interactions.

3.6.3 Cubic vertices

We apply the previous procedure to the case of Yang-Mills, gravitational and higher-spin self-interactions. In the case of higher-spin, the algebra does not close already at the level of spin-3 self-interactions. We then discuss the coupling of higher-spin fields to gravity.

Yang-Mills In the case of Yang-Mills theory (one needs a non-Abelian gauge group for odd spin self-interactions), one finds that the only cubic terms allowed by gauge invariance and that are not subject to field redefinition contain either one or three derivatives. The terms with three derivatives are of Born-Infeld type (constructed from curvatures) and do not deform the algebra of gauge transformations, while the terms with only one derivative can be shown to generate Yang-Mills couplings

$$S_3[A] = \int d^dx\, f_{abc} A_\mu{}^a \partial^\mu A_\nu{}^b A^{\nu c}\,, \tag{3.58}$$

for some structure constants f_{abc} that are antisymmetric in the last two indices and

$$\delta_1 A_\mu{}^a = -f^a{}_{bc}\xi^b A_\mu{}^c\,. \tag{3.59}$$

By pushing the procedure to the next order, we find the full Yang-Mills theory with a quartic term, together with the requirement that the structure constants f_{abc} verify the Jacobi identity. The deformation procedure stops (i.e. $\delta_2 = 0$ and $S_{n\geq 5}[A] = 0$) at this order.

Gravity For spin-2 self-interaction, the work of deriving the spin-2 self-interaction was first done in [75–79]. It was shown that the Noether procedure indeed reproduces the weak-field expansion of Einstein's General theory of Relativity, together with the correct linearisation of general diffeomorphism transformations, up to cubic order.

Higher-spin The spin-three self-coupling term constructed through the Noether procedure [80] was shown [72] to generate non-Abelian gauge transformations that do not close (i.e. they do not form an algebra). This was the first hint that any candidate higher-spin algebra should be infinite dimensional.

The classification of all cubic vertices $s_1 - s_2 - s_3$ for higher-spin fields of spin s_1, s_2 and s_3 was performed in [9, 74, 81–92], see also [22, 59]. One remarkable feature is that, in flat space the number of derivatives for the cubic coupling to gravity (i.e. of the form $s - s - 2$) is bounded by below. Indeed one can try an Ansatz of the form

$$S_2[h] + S_2[\phi] + g \int d^d x \, W_{\mu(2),\nu(2)} \beta^{\mu(2),\nu(2)}(\phi, \phi) \,, \tag{3.60}$$

where $S_2[h]$ and $S_2[\phi]$ are the Fierz-Pauli and Fronsdal actions respectively, $W_{\mu(2),\nu(2)}$ is the linearised (spin-2) Weyl tensor and $\beta^{\mu(2),\nu(2)}$ is a tensorial expression quadratic in higher-spin field $\phi_{\mu(s)}$. It turns out that this expression is the most general one for a $s - s - 2$ cubic coupling (since the spin-2 free equation of motion sets the linearised Ricci to zero, only the Weyl can contribute) and can never be gauge invariant, for any choice of $\beta^{\mu(2),\nu(2)}$.

In AdS instead, this type of cubic vertex can be made gauge invariant by adding terms with higher derivatives. This is known as the Fradkin-Vasiliev mechanism (see [4] or [1] for a review, while the uniqueness of the Fradkin-Vasiliev mechanism was proven in [88]). The structure of cubic vertices with gravity is the following

# of derivatives	2	4	$\cdots$	$2s-4$	$2s-2$	$2s$	$2s+2$
Flat	×	×	$\cdots$	×	✓	✓	✓
(A)dS	✓	✓	$\cdots$	✓	✓	✓	✓

Table 2: Cubic vertices for the coupling of a spin $s > 2$ field with gravity. The $2s$ and $2s + 2$-derivative vertices are of Born-Infeld type. Only the $2s - 2$ vertex is non-Abelian and starting from it, one is able to reconstruct the tail of lower-derivative terms in (A)dS from gauge invariance, while in flat space it is automatically gauge-invariant. For $s > 2$ in flat space, there is no minimal (2-derivative) deformation of the Fronsdal Lagrangian coupled to gravity, in agreement with the Weinberg low-energy theorem A.1, while such a 2-derivative coupling exists when $\Lambda \neq 0$ as a consequence of $[\nabla, \nabla] \neq 0$. In flat space and in the light-cone formulation, a 2-derivative vertex has been found [81, 93, 94] and is at the centre of the light-cone formulation of higher-spin gravity [95].

The non-Abelian cubic vertex with $(2s - 2)$ derivatives hints at a (non-Abelian) algebra of gauge transformations. This algebra will play a central role in the frame-like formulation of the dynamics, see section 4. The full set of cubic vertices arising from the gauging of the (AdS) higher-spin algebra are presented in [2, 4], while their relations with CFT 3-point functions in the holographic setup was summarised in [96].

3.6.4 Beyond cubic vertices

While the cubic story is well under control, some obstructions arise at the level of quartic coupling (see [23, 97–99] for a summary). The difficulty is four-fold:

- first, there is a technical difficulty due to the increasing number of terms that we have to consider for quartic vertices,
- secondly, quartic vertices are subject to field-redefinition ambiguities,
- thirdly, there is a difficulty to construct perturbatively local quartic vertices within the Noether procedure,
- lastly, as we will briefly discuss below, there is evidence supporting the non-locality of the theory from CFT arguments.

The CFT non-locality argument [98, 99] is as follows: within the AdS/CFT correspondence, higher-spin bulk n-point interactions can be matched with CFT n-point correlation functions. The conjectured CFT dual of Vasiliev theory being particularly simple [5, 6], correlators are easily computed. One can then compare, in a certain regime of the kinematical data, the CFT 4-point function and AdS quartic amplitudes arising from tree-level exchange diagrams in the s, t and u channels. The difference between the two should correspond to the 'truly quartic' contribution, i.e. the contact term. Surprisingly, one finds that the contribution of the contact term to the total amplitude is proportional to the contribution of the exchange diagrams, which is a non-local object since it involves interactions computed at two points in AdS with separation $\gg \ell$). Whether this indicates a complete breakdown of the local character of the higher-spin theory, or that a different formulation should be found is still not completely clear.

Note that there are also different approaches to writing down interactions, for instance the BV-BRST one (see [100–106]).

4. Frame-like formulation

Having presented the metric-like formulation, and highlighted the existence of an underlying non-Abelian gauge algebra responsible for the non-Abelian cubic vertices coupling higher-spin fields to gravity, we now begin our quest for an independent definition of a higher-spin algebra that would capture the known non-Abelian cubic vertices and help towards the definition of a fully interacting theory.

We begin with some reminders of Einstein-Cartan theory with and without a cosmological constant, putting emphasis on the linearised theory. Then, we explain how to recover the metric-like Fronsdal theory from a generalisation of the Cartan approach to gravity setup, using gauge potentials with higher indices. Contrary to the linearised spin-2 case, more fields are necessary to make the formulation gauge-invariant and we explain how the extra fields relate to the Fronsdal field, as well as their role in respect to the symmetry generators of the free theory found in section 3.3. We conclude by giving the initial data for a putative higher-spin algebra as well as a tentative construction.

In this section, we will deal with differential forms. For example, a 1-form $M = M_\mu dx^\mu$ has exterior derivative $dM = \partial_{[\mu} M_{\nu]}\, dx^\mu \wedge dx^\nu$.

4.1 Reminder: (linearised) gravity *à la* Cartan

The Cartan view of General Relativity is that of local observers with their own coordinate frames. In mathematical terms, gravity is formulated geometrically as a theory over a principal bundle, whose base manifold is space-time and whose fibre is Lie-algebra-valued. Space-time is a model (homogeneous) coset space, and the Einstein-Cartan theory is built using the curvature of the Ehresmann connection associated to Lorentz transformations.

In physical terms, we write gravity as a gauge theory based on the Poincaré (or (A)dS) algebra, and gauge symmetry is given by local Lorentz transformations. From the connection one-form known as the vielbein $e_\mu{}^a$ and spin connection $\omega_\mu{}^{ab}$, associated to translations and Lorentz transformations respectively, one can build the quantities $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ and $e^\mu{}_a e^\nu{}_b R_{\mu\nu}{}^{ab}$, invariant under local Lorentz transformations and interpreted as the metric and the Riemann tensor.

4.1.1 Flat space

Let's start again with a real form of the Poincaré algebra, where we eliminated the factor of i in (2.1). The indices of the generators whose indices will now be named as fibre indices.

$$[J_{ab}, J_{cd}] = \eta_{bc}\, J_{ad} - \eta_{ac}\, J_{bd} - \eta_{bd}\, J_{ac} + \eta_{ad}\, J_{bc}\,, \tag{4.1a}$$

$$[J_{ab}, P_c] = \eta_{bc}\, P_a - \eta_{ac}\, P_b\,, \tag{4.1b}$$

$$[P_a, P_b] = 0\,, \tag{4.1c}$$

with flat Minkowski metric η_{ab}. We introduce a gauge potential taking values in this algebra

$$A_\mu = e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab} J_{ab}\,, \tag{4.2}$$

where we require that $e_\mu{}^a$ be non-degenerate (local frame), and gauge parameters

$$\xi = \xi^a P_a + \frac{1}{2}\lambda^{ab} J_{ab}\,, \tag{4.3}$$

such that $\delta A_\mu = D_\mu \xi = \partial_\mu \xi + [A_\mu, \xi]$. The curvature of the gauge potential is $F = DA = dA + \frac{1}{2}[A, A]$, or in components

$$F_{\mu\nu} = 2\,\partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu] = T_{\mu\nu}{}^a P_a + \frac{1}{2} R_{\mu\nu}{}^{ab} J_{ab}\,, \tag{4.4}$$

with

$$T_{\mu\nu}{}^a = 2\, D_{[\mu}\, e_{\nu]}{}^a = 2\,\partial_{[\mu}\, e_{\nu]}{}^a + 2\,\omega_{[\mu}{}^{ab}\, e_{\nu]b}\,, \tag{4.5a}$$

$$R_{\mu\nu}{}^{ab} = 2\, D_{[\mu}\, \omega_{\nu]}{}^{ab} = 2\,\partial_{[\mu}\, \omega_{\nu]}{}^{ab} + 2\,\omega_{[\mu}{}^{ca}\, \omega_{\nu]c}{}^b\,, \tag{4.5b}$$

and

$$\delta e_\mu{}^a = \partial_\mu\, \xi^a + \omega_\mu{}^a{}_b\, \xi^b - \lambda^a{}_b\, e_\mu{}^b\,, \tag{4.6a}$$

$$\delta\omega_\mu{}^{ab} = \partial_\mu\, \lambda^{ab} + 2\,\omega_\mu{}^{c[a}\, \lambda^{b]}{}_c\,. \tag{4.6b}$$

In the case of gravity, it is standard to eliminate the spin connection while keeping the vielbein (representing the metric). This can be achieved by setting $T_{\mu\nu}{}^a = 0$ (torsionless) and build an action which is invariant under local Lorentz transformations (and general coordinate redefinition). This is the Einstein-Cartan action

$$S_{\text{EC}} = \frac{1}{16\pi G}\int d^dx\,\det(e)\, e^\mu{}_a e^\nu{}_b R_{\mu\nu}{}^{ab}\,, \tag{4.7a}$$
$$= \frac{1}{(d-2)!}\frac{1}{16\pi G}\int \epsilon_{a_1\cdots a_D}\left(d\omega^{a_1a_2} + \omega^{ba_1}\wedge\omega_b{}^{a_2}\right)\wedge e^{a_3}\wedge\cdots\wedge e^{a_D}\,. \tag{4.7b}$$

To prove that this action is equivalent to the Einstein-Hilbert action, we vary with respect to $\omega_\mu{}^{ab}$ and take the wedge product with one extra vielbein to recover the torsion constraint

$$T_{\mu\nu}{}^a = 0 \qquad \text{(e.o.m.)}\,, \tag{4.8}$$

which we solve as

$$\omega_\mu{}^{ab} = 2\, e^{\rho[a}\partial_{[\mu}e_{\rho]}{}^{b]} - e_{\mu c}e^{\rho a}e^{\nu b}\partial_{[\rho}e_{\nu]}{}^c\,. \tag{4.9}$$

Let us stress that the torsionless condition is an equation of motion obtained from the Einstein-Cartan action. Defining $g_{\mu\nu} = e_\mu{}^a\, e_\nu{}^b\,\eta_{ab}$, it then becomes an interesting exercise to plug back the expression for $\omega_\mu{}^{ab}$ inside the action to find the Einstein-Hilbert action.

Exercise: check that the metric $g_{\mu\nu} = e_\mu{}^a\, e_\nu{}^b\,\eta_{ab}$ and the action (4.7) are invariant under the gauge transformations (4.6) with parameter λ^{ab} (prove first that the determinant $\det(e)$ is gauge invariant, and that $R_{\mu\nu}{}^{ab}$ transforms as a tensor in its fibre indices).

Linearised gravity Let us delve further on the linearised case. Let's write

$$e^a = h^a + \tilde{e}^a\,, \qquad \omega^{ab} = 0 + \tilde{\omega}^{ab}\,, \tag{4.10}$$

with $h_\mu{}^a = \delta_\mu{}^a$ and forget about the tilde. The expansion of the action (4.7) up to quadratic order in the fields e and ω is, up to a pre-factor,

$$\int \epsilon_{a_1\cdots a_d}\left(de^{a_1} + \frac{1}{2}h_b\wedge\omega^{ba_1}\right)\wedge\omega^{a_2a_3}\wedge h^{a_4}\wedge\cdots h^{a_d}\,. \tag{4.11}$$

Varying it with respect to ω yields the equation of motion

$$de^a + h_b\wedge\omega^{ba} = 0 \qquad \text{(e.o.m.)}\,, \tag{4.12}$$

which is equivalent to the vanishing of the linearised torsion. This action has an obvious gauge symmetry given by $\delta e^a = d\xi^a$ that was absent in the non-linear action. In other words, linearised diffeomorphisms are a gauge symmetry of the linearised Einstein Cartan action. Since there is an algebraic gauge transformation

$$\delta e_\mu{}^a = -\lambda^a{}_b\, h_\mu{}^b = \lambda_\mu{}^a\,, \tag{4.13}$$

we can exploit completely local Lorentz invariance to choose the metric gauge

$$e^{[b;a]} = 0 \qquad \text{(gauge fixing)}\,, \tag{4.14}$$

where the notation $e^{b;a} = \eta^{\mu b}\, e_\mu{}^a$ is here to suggest that, in the linearised theory, we can treat the frame and the world indices on the same footing, but that $e_\mu{}^a$ has no particular symmetry. Therefore, the degrees of freedom of the linearised metric are already accessible[6] in $e_\mu{}^a$ and it should come as no surprise that, upon imposing the vanishing of the torsion obtained by varying with respect to $\omega_\mu{}^{ab}$, we obtain the Fierz-Pauli action for the linearised spin-2 fluctuation $h_{\mu\nu} = 2\, h_{(\mu}{}^a\, e_{\nu)a}$.

Note also that studying the integrability of the torsion equation (4.12) gives the Bianchi identity

$$0 = d\left(de^a + h_b \wedge \omega^{ba}\right) = -h_b \wedge d\omega^{ba} = \frac{1}{2} h_b \wedge R^{ab}\,, \qquad (4.15)$$

where the linearised curvature is $R^{ab} = 2\, d\omega^{ab}$ and we used $d^2 = 0$ as well as $dh_b = 0$. The equation $h_b \wedge R^{ab} = 0$ is also equivalent, upon expressing ω^{ab} as a function of the linearised vielbein, to setting to zero the trace of the linearised Riemann tensor, which is the equation of motion imposed by the Fierz-Pauli action.

4.1.2 (A)dS space-time

We now turn to the case of (A)dS space-time. The previous results will be largely unaffected by the presence of the cosmological constant. Indeed, gauging the AdS algebra instead of the Poincaré one only introduces an extra term appearing in the curvature $R_{\mu\nu}{}^{ab} \rightarrow R_{\mu\nu}{}^{ab} + \frac{2}{\ell^2} e_{[\mu}{}^a e_{\nu]}{}^b$. Local Lorentz transformations are the same. One can then reconstruct the Einstein-Hilbert action with cosmological constant by the requirement that the equations of motion reproduce Einstein's equations with a cosmological constant.

In the linearised case, one can choose a background h^a and ϖ^{ab} (we will again drop the tilde)

$$e^a = h^a + \tilde{e}^a\,, \quad \omega^{ab} = \varpi^{ab} + \tilde{\omega}^{ab}\,, \qquad (4.16)$$

where the background vielbein h^a and covariant derivative ∇ verify

$$\nabla h^a = 0\,, \quad \nabla^2 s^{a_1\cdots a_s} = -\frac{1}{\ell^2}\left(h^{a_1} \wedge h_b \wedge s^{b a_2 \cdots a_s} + \cdots + h^{a_s} \wedge h_b \wedge s^{a_1\cdots a_{s-1} b}\right), \qquad (4.17)$$

for any p-form $s^{a_1\cdots a_s}$ (no particular symmetry on the fibre indices is assumed), where the linearised covariant derivative is defined as $\nabla = d + \varpi$.[7]

As an example, let us write the vanishing of the linearised torsion

$$\nabla e^a + h_b \wedge \omega^{ba} = 0 \quad \text{(e.o.m.)}\,. \qquad (4.18)$$

By studying the integrability of the previous equation under ∇, we get the Bianchi identity

$$\nabla T^a = 2\left(\nabla^2 e^a - h_b \wedge \nabla\omega^{ba}\right) = h_b \wedge \left(R^{ab} + \frac{4}{\ell^2} h^a \wedge e^b\right), \qquad (4.19)$$

where $R^{ab} = 2\,\nabla\omega^{ab} = 2\left(d\omega^{ab} + 2\,\varpi^{c[a} \wedge \omega_c{}^{b]}\right)$. For $T^a = 0$, this gives a second equation of motion, which is the consequence of the zero torsion condition, and is precisely the linearised Fierz-Pauli equation of motion upon replacing ω by its expression in terms of e.

[6]The linearised metric reads $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} = (h_\mu{}^a + e_\mu{}^a)\,(h_\nu{}^b + e_\nu{}^b)\,\eta_{ab} - \eta_{\mu\nu} = 2\, h_{(\mu}{}^a\, e_{\nu)a} + O(e^2)$.

[7]What we mean with this notation is that there is a non-zero spin connection piece in the covariant derivative, acting on the fibre indices, e.g., $\nabla T^a{}_b = dT^a{}_b + \varpi_c{}^a \wedge T^c{}_b - \varpi^c{}_b \wedge T^a{}_c$.

On the other hand, if one wants to generalise the action (4.11), one needs to be more careful. Upon replacing the exterior derivative d with a covariant exterior derivative ∇, one can see that the action is no longer invariant under the transformations $\delta\omega^{ab} = \nabla\lambda^{ab}$. This can be cured by introducing terms proportional to the cosmological constant of the form

$$\frac{1}{\ell^2}\int \epsilon_{a_1\dots a_d} e^{a_1} \wedge e^{a_2} \wedge h^{a_3} \wedge \cdots \wedge h^{a_d} \,, \tag{4.20}$$

which do not affect the torsionless condition, since they are independent of ω^{ab}.

> *Exercise:* find the proportionality constant in front of the gauge-restoring term (4.20) such that the total action is gauge-invariant.

4.2 Fronsdal's equations from vielbein and spin connection

The vielbein and spin-connection are the bread and butter of gravity in the Cartan form. The former contains the physical degrees of freedom while the latter is purely auxiliary and is related to the former via the torsion constraint. The equations of motion are expressed in terms of covariant quantities (while the action is gauge-invariant), and the curvature of the spin-connection is conveniently two-derivative in terms of the vielbein. We will now explain how to extend this two-field construction to higher-spins. While the spirit of the construction remains the same, contrary to gravity the higher-spin connection is not completely determined in terms of the higher vielbein via a torsion constraint, but the remaining portion is pure gauge.

4.2.1 Flat space

One rather unnatural feature of the Fronsdal equation is the fact that we have to deal with fields which are doubly traceless. This has however a natural interpretation in the frame formulation. We can guess that the correct embedding of a Fronsdal field inside of a generalised frame has one space-time index and is symmetric and traceless in its fibre indices. By converting space-time indices to fibre via the background vielbein $h_\mu{}^a = \delta_\mu{}^a$, the frame field $e^{b;a(s-1)}$ has the symmetry of the diagrams given by (see also (B.15))

$$e^{b;a(s-1)} \sim \Box \otimes \boxed{\quad s-1 \quad} = \boxed{\quad\; s \;\quad} \oplus \boxed{\quad s-2 \quad} \oplus \begin{array}{l}\boxed{\qquad s-1 \qquad}\\[-3pt] \Box\end{array} \,, \tag{4.21}$$

that is to say, it contains a traceless symmetric part of rank s, a traceless symmetric part of rank $s-2$ and a traceless hook component. The two symmetric parts can be combined to form a Fronsdal field $\phi^{a(s)} = e^{a;a(s-1)}$, which is doubly-traceless, since any double trace will necessarily involve (at least) a single trace in the fibre indices, which vanishes by hypothesis. The field $\phi^{a(s)}$ indeed transforms in the expected way, with a symmetric and traceless gauge parameter $\xi_{a(s-1)}$ such that

$$\delta e_{a;a(s-1)} = \partial_a \xi_{a(s-1)} \,. \tag{4.22}$$

Yet, the field $e^{b;a(s-1)}$ contains more components than just a Fronsdal field. Let us write an action which generalises the linearised Einstein-Cartan action of eq. (4.11)

$$\int \epsilon_{a_1\dots a_d}\left(de^{a_1 b(s-2)} - \frac{1}{2}h_c \wedge \omega^{a_1 b(s-2),c}\right) \wedge \omega_{b(s-2)}{}^{a_2,a_3} \wedge h^{a_4} \wedge \cdots \wedge h^{a_d} \,, \tag{4.23}$$

formulated in terms of the generalised vielbein $e^{a(s-1)}$ and a 'spin connection' $\omega^{a(s-1),b}$. This action actually does not depend on the hook component of $e^{b;a(s-1)}$, which is manifest in the fact that it is gauge invariant under the transformations

$$\delta e^{a(s-1)} = d\xi^{a(s-1)} - h_b \lambda^{a(s-1),b}\,, \quad \delta\omega^{a(s-1),b} = d\lambda^{a(s-1),b}\,. \tag{4.24}$$

This allows one to set to zero the unwanted degrees of freedom by imposing a metric-like gauge,

$$e^{[b;a]a(s-2)} = 0 \qquad \text{(gauge fixing)}\,. \tag{4.25}$$

Varying the action with respect to ω, we find the following torsion constraint

$$T^{a(s-1)} \equiv de^{a(s-1)} - h_b \wedge \omega^{a(s-1),b} = 0 \qquad \text{(e.o.m.)}\,, \tag{4.26}$$

which allows to express *some* components of ω as a function of the derivative of e. The field $\omega^{a(s-1),b}$ has actually more components than the ones fixed by the vanishing of the torsion, but they do not enter the action (we will come back to this shortly). When putting the expression for ω back inside of the action, we should recover the Fronsdal action. Alternatively, one can check the integrability of eq. (4.26)

$$0 = dT^{a(s-1)} = d\left(de^{a(s-1)} - h_b \wedge \omega^{a(s-1),b}\right) = h_b \wedge d\omega^{a(s-1),b}\,, \tag{4.27}$$

and one should obtain the Fronsdal equation of motion. This can either be verified directly, or can be argued on more general grounds, as follows:

- the field $e^{a(s-1)}$ (after gauge fixing the hook component to zero) has the same symmetries and gauge transformations as a Fronsdal field $\phi^{a(s)}$,
- the components of $\omega^{a(s-1),b}$ that are expressed in terms of $\phi^{a(s)}$ are first-derivative,
- the action is quadratic in $\phi^{a(s)}$, contains two derivative and it is gauge invariant,

so it must be the Fronsdal action (or vanish identically).

Exercise (harder): show that the equations of motion are indeed those of Fronsdal.

4.2.2 (A)dS space

As was the case for gravity, if one replaces the exterior derivative with a covariant one, the action

$$\int \epsilon_{a_1\dots a_d}\left(\nabla e^{a_1 b(s-2)} - \frac{1}{2}h_c \wedge \omega^{a_1 b(s-2),c}\right) \wedge \omega_{b(s-2)}{}^{a_2,a_3} \wedge h^{a_4} \wedge \cdots \wedge h^{a_d} \tag{4.28}$$

is no longer gauge invariant under variations involving $\lambda^{a(s-1),b}$. This can be corrected by adding terms of the form

$$\frac{1}{\ell^2}\int \epsilon_{a_1\dots a_d} e^{a_1 b(s-2)} \wedge e^{a_2}{}_{b(s-2)} \wedge h^{a_3} \wedge \cdots \wedge h^{a_d}\,, \tag{4.29}$$

which do not affect the torsionless condition, since they do not involve $\omega^{a(s-1),b}$. The variation with respect to the latter yields the usual equation of motion[8]

$$T^{a(s-1)} \equiv \nabla e^{a(s-1)} - h_b \wedge \omega^{a(s-1),b} = 0 \qquad \text{(e.o.m.)}\,. \tag{4.30}$$

Its integrability under ∇ gives

$$0 = \nabla T^{a(s-1)} = \nabla^2 e^{a(s-1)} + h_b \wedge \nabla\omega^{a(s-1),b} = h_b \wedge T^{a(s-1),b}\,, \tag{4.31}$$

where the second torsion $T^{a(s-1),b}$ is defined as

$$T^{a(s-1),b} = \nabla\omega^{a(s-1),b} + \frac{1}{\ell^2}\sigma_+\left(h^b \wedge e^{a(s-1)}\right)\,, \tag{4.32}$$

and where $\sigma_+(\dots)$ implements the projection of the indices to match the symmetry of the left-hand side. From there, one can plug back in the action and verify that it is equivalent to the Fronsdal Lagrangian in (A)dS.

4.3 Complete set of auxiliary fields from the reducibility parameters

As advertised, the equations (4.26) and (4.27) are actually the first of a long series, which is due to the fact that the spin connection has more components than the ones fixed by the torsion constraint. This is also closely linked to the *reducibility parameters*, aka the vacuum-preserving symmetries of the theory, described in section 3.3.

4.3.1 Flat space

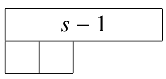
In order to remove the extra component of $e^{b;a(s-1)}$, we introduced a new gauge parameter $\lambda^{a(s-1),b}$ which precisely has the structure of a hook, such that $\delta e_\mu{}^{a(s-1)} = -\eta_{\mu b}\lambda^{a(s-1),b}$. We can then algebraically fix the hook component in $e_\mu{}^{a(s-1)}$ to zero, similarly to the metric gauge in linearised Cartan gravity. Since each new gauge parameter are associated to a symmetry, we have a new generator $Z^{a(s-1),b}$, and a new field $\omega_\mu{}^{a(s-1),b}$ whose components are given by the right-hand-side of (B.16) for $t = 1$. The only gauge-invariant equation that we can impose at this stage is

$$de^{a(s-1)} - h_b \wedge \omega^{a(s-1),b} = 0 \qquad \text{(by hand)}\,, \tag{4.33}$$

with the gauge transformation

$$\delta\omega^{a(s-1),b} = d\lambda^{a(s-1),b}\,. \tag{4.34}$$

This is precisely the equation of motion (4.26) that we derived from the action (4.21). However, the field $\omega^{a(s-1),b}$ itself has more components than the ones we can express in terms of $de^{a(s-1)}$ (this can be observed by using rules for the tensor product given in appendix B.2). Upon imposing eq. (4.33) (which we will refer to as the first torsion constraint), one can find that all components of $\omega^{a(s-1),b}$ are expressed in terms of $de^{a(s-1)}$, except for the component[9]



. (4.35)

[8]The gauge variation of $\omega^{a(s-1),b}$ also picks up a term proportional to ξ and linear in the cosmological constant, as is the case of (linearised) AdS gravity.

[9]For $s = 2$, this component is absent and there is a one-to-one correspondence between the components of the spin connection $\omega_\mu{}^{[ab]}$ and the exterior derivative of the vielbein $\partial_{[\mu}e_{\nu]}{}^a$.

This can be inferred by comparing the components of $\omega^{a(s-1),b}$, given by the right-hand-side of (B.16) for $t = 1$ and those of $de^{a(s-1)}$ given by eq. (B.18).

In order not to introduce additional degrees of freedom (we already have enough to describe a Fronsdal field), this component must be gauged away. The idea is to introduce yet another set of gauge parameters and fields to algebraically gauge away this new component, etc. and repeat the process until all components of all fields are accounted for (either zero or fixed to zero by torsion constraint). This procedure yields a chain of torsion equations

$$T^{a(s-1),b(t)} \equiv d\omega^{a(s-1),b(t)} - h_b \wedge \omega^{a(s-1),b(t+1)} = 0 \qquad \text{(e.o.m.)}\,, \tag{4.36}$$

for all $t \in \{0, \ldots, s-2\}$, with

$$\delta\omega^{a(s-1),b(t)} = d\lambda^{a(s-1),b(t)} - h_b \lambda^{a(s-1),b(t+1)}\,, \tag{4.37}$$

and does indeed stop, when the last field and gauge parameter are introduced, with two rows of frame indices

$$d\omega^{a(s-1),b(s-2)} - h_b \wedge \omega^{a(s-1),b(s-1)} = 0 \qquad \text{(e.o.m.)}\,, \tag{4.38}$$

with

$$\delta\omega^{a(s-1),b(s-1)} = d\lambda^{a(s-1),b(s-1)}\,. \tag{4.39}$$

The last field strength tensor, that we will call the curvature $R^{a(s-1),b(s-1)}$ verifies

$$h_b \wedge R^{a(s-1),b(s-1)} = h_b \wedge d\omega^{a(s-1),b(s-1)} = 0\,. \tag{4.40}$$

All in all, we find that the complete set of auxiliary fields is given by

$$e_\mu{}^{a(s-1)}\,,\ \omega_\mu{}^{a(s-1),b}\,,\ \ldots\,,\ \omega_\mu{}^{a(s-1),b(s-1)}\,, \tag{4.41}$$

together with the gauge parameters

$$\xi^{a(s-1)}\,,\ \lambda^{a(s-1),b}\,,\ \ldots\,,\ \lambda^{a(s-1),b(s-1)}\,. \tag{4.42}$$

Chain of structure equations We have the set of torsion equations (4.36) from $t = 0$ to $t = s-2$, as well as $h_b \wedge R^{a(s-1),b(s-1)} = 0$. We already know how to move from one torsion equation to the next (by studying its integrability, yielding a Bianchi identity). For example, one can ensure that the consequences of eq. (4.33) are properly taken care of. By taking the exterior derivative and using $d^2 = 0$, one obtains

$$dT^{a(s-1)} = 0 = d\left(-h_b \wedge \omega^{a(s-1),b}\right) = h_b \wedge d\omega^{a(s-1),b} \qquad \text{(integrability)}\,, \tag{4.43}$$

whose general solution is[10]

$$d\omega^{a(s-1),b} = h_b \wedge \hat{\omega}^{a(s-1),b(2)} \qquad (\sigma_- \text{ cohomology})\,. \tag{4.44}$$

[10]Proving this requires to study the cohomology of the nilpotent operator $h_b \wedge \ldots$, known as σ_-. This goes beyond the scope of these lecture notes and we refer the interested reader to [56] for more details.

This is indeed the second torsion equation for $\hat{\omega}^{a(s-1),b(2)} = \omega^{a(s-1),b(2)}$, and so on. Alternatively, one can take the second torsion equation and take its wedge product with a background vielbein, contracting one index

$$h_b \wedge T^{a(s-1),b} = 0 = h_b \wedge d\omega^{a(s-1),b} = -d\left(h_b \wedge \omega^{a(s-1),b}\right), \tag{4.45}$$

which can be written, using the generalised Poincaré lemma [107], as

$$h_b \wedge \omega^{a(s-1),b} = d\hat{e}^{a(s-1)}, \tag{4.46}$$

which is the first torsion equation for $\hat{e}^{a(s-1)} = e^{a(s-1)}$. All in all, we have the hierarchy

$$T^{a(s-1)} = 0 \underset{PL}{\overset{int}{\rightleftarrows}} T^{a(s-1),b} = 0 \underset{PL}{\overset{int}{\rightleftarrows}} \dots \underset{PL}{\overset{int}{\rightleftarrows}} T^{a(s-1),b(s-2)} = 0 \underset{PL}{\overset{int}{\rightleftarrows}} h_b \wedge R^{a(s-1),b(s-1)} = 0, \tag{4.47}$$

where one moves from left to right by using integrability

$$T^{a(s-1),b(t)} = 0 \;\Rightarrow\; dT^{a(s-1),b(t)} = 0 \;\Rightarrow\; T^{a(s-1),b(t+1)} = 0, \tag{4.48}$$

and one moves from right to left by using the generalised Poincaré lemma

$$T^{a(s-1),b(t)} = 0 \;\Rightarrow\; h_b \wedge T^{a(s-1),b(t)} = 0 \;\Rightarrow\; T^{a(s-1),b(t-1)} = 0. \tag{4.49}$$

As in the case of Cartan gravity, the last thing to do is to impose an equation of motion on the curvature $R^{a(s-1),b(s-1)}$, which fixes the dynamical content. A natural choice is to impose

$$R^{a(s-1),b(s-1)} = h_a \wedge h_b \, C^{a(s),b(s)}, \tag{4.50}$$

where $C^{a(s),b(s)}$ is a traceless zero-form, to be interpreted as the Weyl[11] component of the generalised curvature of the Fronsdal field (it contains s derivatives of ϕ). One can verify, using techniques from unfolding [108, 109], that this choice indeed describes the propagation of a single massless spin-s field.

4.3.2 (A)dS space

Similarly to the case of linearised gravity, all derivatives d are replaced with ∇ and, starting from $t = 1$, there is an extra piece proportional to the cosmological constant in $T^{a(s-1),b(t)}$, as well as in $R^{a(s-1),b(s-1)}$. It can be proven that the torsions take the form

$$T^{a(s-1)} = \nabla e^{a(s-1)} - h_b \wedge \omega^{a(s-1),b}, \tag{4.51}$$

and

$$T^{a(s-1),b(t)} = \nabla\omega^{a(s-1),b(t)} - h_b \wedge \omega^{a(s-1),b(t+1)} - \frac{1}{\ell^2}\sigma_+\left(\omega^{a(s-1),b(t-1)}\right), \tag{4.52}$$

for $t \geq 1$, and the curvature is

$$R^{a(s-1),b(s-1)} = \nabla\omega^{a(s-1),b(s-1)} - \frac{1}{\ell^2}\sigma_+\left(\omega^{a(s-1),b(s-2)}\right), \tag{4.53}$$

where σ_+ is an operator enforcing the projection in the indices. This is similar to what happens with the cosmological constant term appearing in General Relativity *à la* Cartan. Even though calculations become more involved, the above argument does not change.

[11] For the case $s = 2$, it is precisely the linearised Weyl tensor, i.e. the traceless part of the linearised Riemann tensor.

4.4 Initial data for a gauge algebra

Having seen the complete set of fields and gauge transformations in the frame-like formulation of higher-spin dynamics, we now wish to answer the following question: *is there an algebra underlying the dynamics?* In other words, can we find a Lie algebra containing generators associated the complete set of reducibility parameters presented in section 3.3, and whose Cartan equations (at the linearised level) reproduce the system of torsions and curvature presented in section 4.3?

4.4.1 Looking for a non-Abelian algebra

Let us first reformulate the initial data of this problem algebraically. A putative higher-spin symmetry algebra should contain generators whose spectrum is described in 3.3, denoted by

$$\bigoplus_{\substack{s\geq 1\\ 0\leq t\leq s-1}} Z_{a(s-1),b(t)}\,, \tag{4.54}$$

with the following commutation relations with Lorentz generators (fixed by Lorentz covariance)

$$[J_{ab}, Z_{c(s-1),d(t)}] = 2\,(s-1)\,\eta_{c[b}\,Z_{a]c(s-2),d(t)} + 2\,t\,Z_{c(s-1),d(t-1)[a}\,\eta_{b]d}\,. \tag{4.55}$$

By looking at the form of the torsions and curvatures in flat and (A)dS space, we can read off the structure constants

$$[P_a, Z_{b(s-1),c(t)}] = \alpha\,\big(\eta_{ac}\,Z_{b(s-1),c(t-1)} + \text{permutations} - \text{traces}\big) + \beta\frac{1}{\ell^2}Z_{b(s-1),ac(t)}\,, \tag{4.56}$$

where α and β are non-vanishing, spin- and dimension-dependent structure constants, and the terms referred to as "permutations" and "traces" within brackets can be determined in such a way that the symmetries of the fibre indices on the right-hand side of eq. (4.56) matches those on the left-hand side. The convention that $Z_{b(s-1),c(t)} = 0$ whenever $t < 0$ or $t > s-1$ is also assumed. At this stage, we can rescale the generators such that $\alpha = t$ without loss of generality. This fixes all the commutators of the linearised algebra between a spin-2 field and a higher-spin field (also spin 2, closes on Poincaré or (A)dS) and constitutes the initial data for a candidate non-Abelian higher-spin symmetry algebra, where $[Z_{a(s-1),b(t)}, Z_{c(s'-1),d(t')}] \neq 0$ in general.

4.4.2 Structure constants from global symmetries

As a side note, although there is not a unique candidate to generalise the Lie bracket, one can compute the Lie derivative of a Killing tensor of eq. (3.31) along a Killing vector corresponding to translations in flat space

$$\mathcal{L}_{\kappa^c\partial_c}\left[M_{a(s-1),b(t)}\,\underbrace{x^b\cdots x^b}_{t}\right] = t\,\kappa^c\,M_{a(s-1),cb(t-1)}\,\underbrace{x^b\cdots x^b}_{t-1}\,, \tag{4.57}$$

and remark that they agree with the structure constants of the linearised unfolding strategy for $\ell\to\infty$ of eq. (4.56). Below, we present some attempts to define a Lie algebra fulfilling this rule.

Schouten bracket One generalisation of the Lie bracket to higher-rank tensors is the so-called Schouten(-Nijenhuis) bracket [110–113]. Given two symmetric tensors $v^{a(p)}$ and $w^{a(q)}$, their Schouten bracket is given by

$$[v, w]_{\mathrm{S}}^{a(p+q-1)} \equiv \frac{(p+q-1)!}{p!q!} \left(p\, v^{ba(p-1)} \partial_b w^{a(q)} - q\, w^{ba(q-1)} \partial_b v^{a(p)} \right) . \tag{4.58}$$

It is easy to prove that the Schouten bracket of Killing tensors remains a Killing tensor, however the Schouten bracket of two traceless tensors is not necessarily a traceless tensor (nor can it be decomposed into a sum of traceless tensors in general), thus making it a poor candidate for our needs.[12]

Weyl algebra A particularly useful representation of Killing symmetries of the type defined in eq. (3.30) is obtained using (Weyl-ordered) polynomials in the variables X_a and P_a with $[X_a, P_b] = i\eta_{ab}$ [20]. In this basis, Minkowski translations are realised by P_a and Lorentz transformations by

$$J_{ab} = X_{[a} P_{b]} . \tag{4.59}$$

One can therefore try to construct higher-spin isometries using higher products of these oscillators, and an algebra will be given by using the Leibniz rule.[13] This approach gives a meaningful answer in (A)dS space but fails in Minkowski, again due to the problem of trace constraints pointed out earlier.

(In)existence of the algebra Without specifying any particular representation, one can try to see if, within the Fronsdal formulation, there exist a non-Abelian algebra with the initial data (4.55) and (4.56) that satisfies the Jacobi identity and some other assumptions on the form of the commutators (see appendix A.2). It was found that the answer is no in the case of Poincaré [73, 74] (see also the comments [73, 115–120]), and that there is only one answer [121] in (A)dS (was first constructed in $d = 4$ in [3], then generalised to $d \geq 4$ in [16, 20, 122–125]): the Fradkin-Vasiliev-Eastwood (FVE) algebra (the precise set of assumptions is spelled out in appendix A.2).

Towards the gauging of the FVE algebra The gauging of this algebra was performed in [10, 11, 126] and non-linear equations of motion were obtained. Eventually, it was proven in [121] that the FVE algebra reproduces all known higher-spin cubic vertices in $d = 4$ and $d \geq 7$, making it the best (and only) candidate algebra for higher-spin symmetry. In particular, the non-Abelian cubic vertices for the $2 - s - s$ case in flat-space are precisely the seeds (highest-derivative term, non-Abelian gauge transformations) for the cubic vertices found by Fradkin and Vasiliev through their gauge-restoration procedure in (A)dS. The idea is to start from the non-Abelian vertex and add a 'tail' of lower-derivative couplings that render the action gauge-invariant. The fact that this algebra only exists in the case of non-vanishing cosmological constant was interpreted as yet another no-go for higher-spin theories in flat space, see appendix A.2.

[12]In some sense, the non-closure of the algebra mimics the non-closure of non-Abelian gauge transformations at the cubic order in the metric-like formulation. It looks like, in flat space, the trace constraint is at odds with the problem of finding a higher-spin symmetry algebra, as was pointed in [20].

[13]Another option is to start instead with commuting variables x_a and p_a and to deform the pointwise product into a star product *à la* deformation quantisation [114].

5. Higher-spin algebras

Having stressed the importance of a non-Abelian, infinite-dimensional symmetry algebra underlying any candidate gauge theory of interacting higher-spin gauge theory (in the frame-like formulation of the dynamics), we present a way to build such an algebra using the tensor algebra of space-time isometries, which was shown to be the unique algebra reproducing known cubic vertices in $d = 4$. We make a number of remarks in the direction of higher-spin holography.

We also spend some time on the $d = 3$ version of these algebras, as well as other candidates, describing the propagation and interaction of a finite number fields. Although the associated theories are topological (they can be built as Chern-Simons theories) the construction admits some contractions and deformations which makes it interesting for massive and flat higher-spin gravity.

5.1 Fradkin-Vasiliev-Eastwood algebra in general $d \geq 4$

We present first the general construction [16, 127, 128] in any $d \geq 4$. The case $d = 3$ also works, but due to the low dimensionality (and the fact that there is a more refined construction that only works in this dimension) we will present it later.

5.1.1 Universal enveloping construction

Let us present first the construction in general $d \geq 4$ through a Universal Enveloping Algebra (UEA). In $d = 3$ and $d = 4$ there are some specific realisations that we will briefly address. We are now in AdS space, whose isometry generators are embedded in ambient space in the following way:

$$[J_{AB}, J_{CD}] = \eta_{BC}\, J_{AD} - \eta_{AC}\, J_{BD} - \eta_{BD}\, J_{AC} + \eta_{AD}\, J_{BC}\,, \tag{5.1}$$

with, e.g. $J_{0a} = \ell^{-1}\, P_a$. One can then construct the algebra as a quotient of the AdS UEA

$$\mathfrak{hs}_d \equiv \frac{\mathcal{U}(\mathfrak{so}(d-1,2))}{\langle \mathcal{I}_{ABCD} \oplus \mathcal{I}_{AB} \rangle}\,, \tag{5.2}$$

where $\mathcal{U}(\mathfrak{so}(d-1,2))$ denotes the UEA of $\mathfrak{so}(d-1,2)$, and $\langle \mathcal{I}_{ABCD} \oplus \mathcal{I}_{AB} \rangle$ is generated by the contraction of $\mathcal{U}(\mathfrak{so}(d-1,2))$ on the left or the right of the elements $\mathcal{I}_{ABCD}$ and $\mathcal{I}_{AB}$ given below.

This construction deserves more explanations. To construct the UEA of a (real) Lie algebra $\mathfrak{g}$, one starts with an associative product (denoted here by $\star$, omitted in the following) such that

$$[a, b] = a \star b - b \star a\,, \tag{5.3}$$

for any $a, b \in \mathfrak{g}$. The tensor algebra of $\mathfrak{g}$ is given (as a vector space) by

$$\mathcal{T}(\mathfrak{g}) = \mathbb{R} \oplus \mathfrak{g} \oplus (\mathfrak{g} \star \mathfrak{g}) \oplus \cdots\,, \tag{5.4}$$

and the UEA is given by

$$\mathcal{U}(\mathfrak{g}) = \frac{\mathcal{T}(\mathfrak{g})}{\langle [a, b] - a \star b + b \star a \rangle}, \tag{5.5}$$

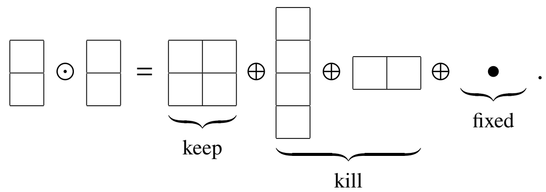
where $\mathcal{T}(\mathfrak{g})$ is quotiented by the equivalence relation defined by (5.3). A two-sided ideal generated by an element $\mathcal{I} \in \mathcal{U}(\mathfrak{g})$ is given by

$$\langle \mathcal{I} \rangle = \mathcal{U}(\mathfrak{g}) \star \mathcal{I} \star \mathcal{U}(\mathfrak{g})\,. \tag{5.6}$$

Let us take higher symmetric products of generators of AdS_d. At the quadratic order, we have

$$\begin{aligned}
\mathcal{K}_{AB,CD} &\equiv J_{C(A} \odot J_{B)D} - \frac{2}{d-1}\left(\eta_{AB}\mathcal{I}_{CD} + \eta_{CD}\mathcal{I}_{AB} - \eta_{C(A}\mathcal{I}_{B)D} - \eta_{D(A}\mathcal{I}_{B)C}\right) \\
&\quad + \frac{1}{d(d+2)}\left(\eta_{AB}\eta_{CD} - \eta_{C(A}\eta_{B)D}\right) C_2\,, \qquad (5.7a)\\
\mathcal{I}_{ABCD} &\equiv J_{[AB} \odot J_{CD]}\,, \qquad (5.7b)\\
\mathcal{I}_{AB} &\equiv J^{C}{}_{(A} \odot J_{B)C} + \frac{2}{d+1}\eta_{AB}\, C_2\,, \qquad (5.7c)\\
C_2 &\equiv \frac{1}{4} J_{AB} \odot J^{BA}\,, \qquad (5.7d)
\end{aligned}$$

where $a \odot b = a \star b + b \star a$, and in terms of Young diagrams



(5.8)

It can be shown that the generator with the symmetries of the second tableau spans an ideal and therefore can be factorised and so does the third one. Using the relation

$$\frac{3}{4}\mathcal{I}_{ABCD}\, J^{CD} - \mathcal{I}_{C[A}\, J_{B]}{}^{C} + \frac{d-1}{d+1}\left(C_2 + \frac{(d+1)(d-3)}{4}\right) J_{AB} = 0\,, \tag{5.9}$$

and requiring that the algebra be non-trivial (i.e. J_{AB} is not factorised), the scalar generator C_2 (which is the quadratic Casimir) must take a precise value which is determined to be a multiple of the identity by the previous relation, thereby avoiding multiplicity in the spectrum.

Exercise (harder): prove the relation (5.9).

Among quadratic generators, we are only left with the 'window' combination (5.7a) as an independent quadratic generator, which is what we wanted. Upon quotienting by just these two elements, all higher products in the UEA reduces to reduce to only the two-row traceless diagrams[14]

$$\bigoplus_{s\geq 1} \begin{array}{|c|}\hline \quad s-1 \quad \\ \hline \quad s-1 \quad \\ \hline \end{array}\,. \tag{5.10}$$

This is precisely what we want, since the branching of the Young diagrams in (5.10) from $d+1$-dimensional ambient space to d-dimensional physical space-time reproduces the spectrum of generators (4.54) (see appendix B.2).

Note that in $d = 4$, in the case of the Fradkin-Vasiliev construction (which relied on a particular oscillator representation of the AdS_4 algebra making use of a vector-spinor dictionary, see [3]), the elements $\mathcal{I}_{ABCD}$ and $\mathcal{I}_{AB}$ were automatically factorised. See also [17, 130–132] for representations of this algebra in various dimensions.

[14] This is not at all obvious and requires a careful study, see e.g. [129]. For instance, factorising either $\mathcal{I}_{AB}$ or $\mathcal{I}_{ABCD}$ fixes the higher-order Casimirs as polynomial functions of the quadratic one C_2, and it is remarkable that the value $-\frac{(d+1)(d-3)}{4}$ is the unique one for which all polynomials agree.

5.1.2 The Eastwood algebra as the higher symmetries of the d'Alembertian

The FVE algebra has an interpretation in the dual CFT_{d-1}: it corresponds to the algebra of higher differential symmetries of the d'Alembertian operators. The higher symmetries of the on-shell equation

$$\partial^2 \varphi \approx 0\,, \tag{5.11}$$

for a scalar field φ with fixed conformal dimension Δ, are differential operators $\hat{D}$ that weakly commute with the Laplacian [16]

$$\partial^2 \circ \hat{D} = \hat{D}' \circ \partial^2\,, \tag{5.12}$$

in the sense that it maps solutions of eq. (5.11) to themselves

$$\delta\left(\partial^2 \varphi\right) = \partial^2\,\delta\varphi = \partial^2 \circ \hat{D}\varphi = \hat{D}' \circ \partial^2 \varphi \approx 0\,. \tag{5.13}$$

The space of differential operators is naturally graded by the order, and a differential operator $\hat{D}$ of order $s-1$ can be decomposed as follows

$$\hat{D} = V^{\mu(s-1)}\partial_\mu \ldots \partial_\mu + \text{lower derivative}\,, \tag{5.14}$$

where $V^{\mu(s-1)}$ is called the symbol of the operator $\hat{D}$. Eastwood showed that there is a one-to-one correspondence between the symbols of the $\hat{D}$'s that verify the condition (5.12) (quotiented by trivial solutions of the form $\hat{D} = \hat{E} \circ \partial^2$) and the Young tableaux having the symmetry of (5.10), this time viewed as conformal Killing tensors of $\mathbb{R}^{d-2,1}$, which shows that the algebra of higher symmetries of the d'Alembertian in $d-1$ dimensions has the same spectrum as the higher-spin algebra in AdS_d.

In fact, they are the same algebra. To view this, we now take a closer look at the field φ in eq. (5.11), known as a *singleton* and which plays a central role in higher-spin holography.

5.1.3 Defining module: the singleton

The singleton is a scalar AdS_d field whose degrees of freedom are localised on the conformal boundary $\mathbb{M}_{d-1}$. It carries a conformal weight $\Delta = \frac{d-3}{2}$, sitting right at the unitarity bound, and is 'shortened' as compared to an ordinary massless scalar.

As a Verma module From the point of view of representations of $SO(d-1,2)$, the singleton is an ultrashort representation. It is the conformal (quasi-)primary $|\varphi\rangle$ corresponding to the unitary scalar field in conformally flat space-time verifying

$$K_\mu|\varphi\rangle = 0\,, \quad J_{\mu\nu}|\varphi\rangle = 0\,, \quad D|\varphi\rangle = \Delta|\varphi\rangle\,, \tag{5.15}$$

where the conformal algebra is spanned by $\{J_{\mu\nu}, P_\mu, D, K_\mu\}$ representing Lorentz transformations, translations, dilations and special conformal transformations and $\Delta = \frac{d-3}{2}$ (see appendix C for the definition of the generators of conformal isometries). It gives rise to the Verma module

$$\mathcal{D} \simeq \{P_{\mu_1} \ldots P_{\mu_n}|\varphi\rangle, n \geq 0\}\big/\{\eta^{\mu_1\mu_2}P_{\mu_1} \ldots P_{\mu_n}|\varphi\rangle, n \geq 2\}\,, \tag{5.16}$$

where the quotient is understood as the factorisation of the maximal sub-module (see, e.g., [133]) corresponding the equation of motion $P^2|\varphi\rangle = 0$.

As an AdS representation In AdS_d/CFT_{d-1} representation theory, the singleton is called "ultra-short" because it is a massless scalar field in AdS that is described by a massles field in the *dual CFT*.[15] It is often denoted $|Rac\rangle$ since its $d = 4$ avatar was first found by Dirac [134].

It has an energy (or conformal dimension in CFT language) $E = \frac{d-3}{2}$ (right at unitarity bound) and the conjugated one is $d - 1 - E = \frac{d+1}{2}$, therefore the Casimir takes value

$$C_2 = E(E - d + 1) = -\frac{(d-3)(d+1)}{4}\,. \tag{5.17}$$

Note that this is the same value appearing in eq. (5.9). This is no coincidence! The higher-spin algebra (5.2) is precisely the UEA of AdS_d, evaluated on the singleton representation. The two elements of the ideal $\mathcal{I}_{ABCD}$ and $\mathcal{I}_{AB}$ can be understood as the manifestation of the field being scalar, and verifying a wave equation respectively.[16]

As an AdS field The Fefferman-Graham expansion of a massless scalar field with energy $E = \frac{d-3}{2}$ in AdS (using in Poincaré coordinates) close to the boundary $z \to 0$ reads

$$\varphi = z^{\frac{d-3}{2}} \left(\bar{\varphi} + O(z^2)\right). \tag{5.18}$$

The singleton is captured by quotienting by the subleading contributions in z, so it is an AdS scalar field whose degrees of freedom are localised on the boundary. If one performs this quotient, the fact that φ verifies the d'Alembert equation in AdS_d implies that $\bar{\varphi}$ also verifies the d'Alembert equation in the CFT_{d-1} (see, e.g., [133, 135, 136]), so we are back in the situation of eq. (5.11).

5.1.4 Flato-Fronsdal theorem

Not only does the singleton provide a natural realisation of the higher-spin algebra, but it is also a central object in higher-spin holography. Indeed, from the product of two singletons, it is possible to create conserved higher-spin currents, whose spectrum is in one-to-one correspondence with the spectrum of Vasiliev theory. This is known a the Flato-Fronsdal theorem [14]

$$|Rac\rangle \otimes |Rac\rangle = \sum_{s\geq 0} D(s + d - 3, s)\,, \tag{5.19}$$

where the notation $D(E, s)$ refers to an irreducible representation of the isometry algebra of AdS_d with energy E and spin s. The value of the energy $E = s + d - 3$ corresponds to a Fronsdal (massless) field.

This theorem can be understood as follows: bulk massless higher-spin fields couple to a boundary theory through higher-spin conserved currents, built as bilinears in the singleton field. These bilinear currents actually have a simple expression

$$J_{\mu(s)}(\mathbf{x}) \sim i^s\, \bar{\varphi}^*(\mathbf{x}) \underbrace{\overleftrightarrow{\partial}_\mu \cdots \overleftrightarrow{\partial}_\mu}_{s} \bar{\varphi}(\mathbf{x}) - \text{traces}\,. \tag{5.20}$$

[15] One way to make this statement more precise is to use the Gelfand-Kirillov (GK) dimension. A massless field in AdS_d has GK dimension $d - 1$ (the dimension of space minus the number of equations) whereas the singleton has GK dimension $d - 2$, i.e. it lives effectively on the CFT.

[16] This statement can be proven easily via ambient space techniques, a more pedestrian proof making use of the differential representation of the CFT_{d-1} algebra is presented in appendix C.

For spin $s = 0$, and $s = 1$, one recovers the usual scalar and electromagnetic current densities $\bar{\varphi}^*\bar{\varphi}$ and $i(\bar{\varphi}^*\partial_\mu\bar{\varphi} - \bar{\varphi}\partial_\mu\bar{\varphi}^*)$, while for $s = 2$ and greater, one obtains the stress-energy tensor and higher-spin analogues obtained via an appropriate transformation subtracting the traces. The $J_{\mu(s)}(\mathbf{x})$ can be generated thanks to a function of an auxiliary variable $\mathbf{p}^\mu$ [15]

$$J(\mathbf{x};\mathbf{p}) \equiv \bar{\varphi}^*(\mathbf{x}+\mathbf{p})\,\bar{\varphi}(\mathbf{x}-\mathbf{p}) = \sum_{s\geq 0}\frac{i^s}{s!}J_{\mu(s)}(\mathbf{x})\mathbf{p}^\mu\cdots\mathbf{p}^\mu\,. \tag{5.21}$$

Exercise: check that the generating function of currents verifies $\partial_\mathbf{x}\cdot\partial_\mathbf{p}\,J(\mathbf{x};\mathbf{p}) = 0$ and deduce that the currents $J_{\mu(s)}(\mathbf{x})$ are conserved. Verify that $\left(\partial_\mathbf{x}\cdot\partial_\mathbf{x} + \partial_\mathbf{p}\cdot\partial_\mathbf{p}\right)J(\mathbf{x};\mathbf{p}) = 0$ and explain how to obtain traceless currents from there.

In higher-spin holography and according to the AdS/CFT dictionary [35, 36, 137], bulk (higher-spin) gauge fields $\phi_{\mu(s)}$ are dual to boundary (higher-spin) conserved currents $J_{\mu(s)}$. They couple to each other through the canonical term

$$\int d^{d-1}\mathbf{x}\; J_{\mu(s)}(\mathbf{x})\,\bar{\phi}^{\mu(s)}(\mathbf{x})\,, \tag{5.22}$$

where $\bar{\phi}^{\mu(s)}(\mathbf{x})$ is the *boundary value* of the field $\phi^{\mu(s)}(x)$ that are defined as the leading part of a bulk field in a Fefferman-Graham expansion $x = (z, \mathbf{x})$, see e.g. [121].

5.2 The three-dimensional case

Although the previous construction of the FVE algebra also applies to the case of $d = 3$, higher-spin gravity in 3 dimensions has followed another route historically, associated to the higher-spin extensions of Chern-Simons theory. There are a number of key features that distinguishes $d = 3$ from the other dimensions: gravity and higher-spin theories are topological since they do not propagate any degrees of freedom; and it is possible to cook up higher-spin theories with a finite number of fields, both in (A)dS and flat space-time. The fact that these theories exist and admit a tractable Chern-Simons formulation makes them ideal candidates to study quantum gravity as well as holography in three dimensions [138–142]. For a review on three-dimensional higher-spin theories and their Chern-Simons formulation, see [143].

5.2.1 Pure gravity

Recall that the $d = 3$ Anti de Sitter algebra $\mathfrak{so}(2,2)$ is not semi-simple since it can be split as two orthogonal copies $\mathfrak{so}(2,2) \simeq \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2)$

$$[J_a, J_b] = \epsilon_{abc}J^c\,, \quad [J_a, P_b] = \epsilon_{abc}P^c\,, \quad [P_a, P_b] = \frac{1}{\ell^2}\,\epsilon_{abc}J^c\,, \tag{5.23}$$

where we used the $d = 3$ Levi-Civita tensor ϵ_{abc} to dualise the generator of Lorentz transformations into a vector J_a. Alternatively, using $\mathfrak{so}(1,2) \simeq \mathfrak{sl}(2,\mathbb{R})$, we can write

$$[J_m, J_n] = (m-n)J_{m+n}\,, \quad [J_m, P_n] = (m-n)P_{m+n}\,, \quad [P_m, P_n] = \frac{1}{\ell^2}\,(m-n)J_{m+n}\,. \tag{5.24}$$

The two orthogonal copies can be recovered by considering

$$\mathcal{L}_m = \frac{1}{2}\left(J_m + \ell P_m\right), \quad \bar{\mathcal{L}}_m = \frac{1}{2}\left(J_m - \ell P_m\right), \tag{5.25}$$

verifying

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}, \quad [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n}, \quad [\mathcal{L}_m, \bar{\mathcal{L}}_n] = 0. \tag{5.26}$$

We can rewrite the Einstein-Cartan action as a difference of two Chern-Simons terms [141]

$$S = I_{\text{CS}}[A] - I_{\text{CS}}[\bar{A}] \tag{5.27}$$

where

$$I_{\text{CS}}[A] = \int \text{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right), \tag{5.28}$$

and where A, $\bar{A}$ take values in the two copies of $\mathfrak{sl}(2,\mathbb{R})$, with Tr denoting the invariant bilinear form of the algebra, also known as the Killing-Cartan form.

> *Exercise:* prove this. By using the Killing metric on $\mathfrak{sl}(2,\mathbb{R})$ and the change of basis (5.25), you should find $\text{Tr}(P_a J_{bc}) = \epsilon_{abc}$.

5.2.2 Algebra for higher-spin gravity in AdS$_3$

First, let us remark that most of the extra fields and generators found in section 4.3 do not exist because of low dimensionality. Indeed, the tableaux $|\mathbb{Y}_3(s-1,t)| = 0$ for $s \geq 2$ for all $2 \leq t \leq s-1$. This means that the two-field frame formulation of section 4.2 is actually complete! Moreover, we can dualise a $\mathbb{Y}_3(s-1,1)$ to $\mathbb{Y}_3(s-1)$, so we will be working with two *completely symmetric* traceless one-forms (like in three-dimensional gravity)

$$e_\mu{}^{a(s-1)}, \quad \omega_\mu{}^{a(s-1)}. \tag{5.29}$$

Discrete family We can have a complete (perturbative) non-linear description of higher-spin gravity in three dimensions by writing a Chern-Simons theory for a higher-spin extension of the potentials A, $\bar{A}$ taking values in a higher-spin algebra. We can take the finite-dimensional algebras $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$ as the improvement of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$.

Taking the adjoint representation $\underline{N^2-1}_N$ of a single $\mathfrak{sl}(N,\mathbb{R})$ and decomposing it into irreducible representations $\underline{n}_2$ of $\mathfrak{sl}(2,\mathbb{R})$, we have

$$\underline{N^2-1}_N = \underline{2N-1}_2 \oplus \underline{2N-3}_2 \oplus \cdots \oplus \underline{5}_2 \oplus \underline{3}_2, \tag{5.30}$$

where the final $\underline{3}_2$ is the adjoint representation of $\mathfrak{sl}(2,\mathbb{R})$. Since the gauge fields $e^{a(s-1)}$ and $\omega^{a(s-1)}$ each have $|\mathbb{Y}_3(s-1)| = 2s-1$ components in their fibre indices, we can deduce that the algebra $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$ is a candidate to describe the propagation and interaction of massless fields of spin $s, \ldots, 2$ in AdS$_3$.[17]

[17]Note that in this construction, there is actually no generator of spin-1 isometries. The latter can be added as an extra Abelian $\mathfrak{u}(1)$ generator.

Change of basis We have already flashed the change of basis from $\mathfrak{so}(1,2)$ to $\mathfrak{sl}(2,\mathbb{R})$ in (5.26), making it possible to rewrite the spin-2 isometry generators as $\mathcal{L}_m$ and $\bar{\mathcal{L}}_m$ with $|m| \leq 1$. For higher spins s, the isometry generators will be denoted by $\mathcal{W}^s_m$ and $\bar{\mathcal{W}}^s_m$, with $|m| \geq s-1$.

In order to construct a Chern-Simons action, we take the Cartan-Killing[18] metric of $\mathfrak{sl}(N,\mathbb{R})$ in our $\mathcal{W}^s_m$ basis, which reads

$$\mathrm{Tr}(\mathcal{W}^s_m \mathcal{W}^t_n) = \mathrm{Tr}(\bar{\mathcal{W}}^s_m \bar{\mathcal{W}}^t_n) = \delta^{s,t} A_s \kappa^s_{m,n} , \tag{5.31}$$

where A_s is a spin-dependent normalisation factor and

$$\kappa^s_{m,n} = \sum_{k=0}^{2s-2} (-1)^k \binom{2s-2}{k} [s-1+m]_{2s-2-k} [s-1-m]_k [s-1+n]_k [s-1-n]_{2s-2-k} , \tag{5.32}$$

with $[a]_b = a(a-1)\cdots(a-b+1)$ the falling Pochhammer symbol.

UEA construction In the basis of eq. (5.25), it can be proven (see, e.g., [145]) that factorising the ideal of the construction (5.2) factorises all products that mix $\mathcal{L}_m$ and $\bar{\mathcal{L}}_m$

$$\mathcal{I}_{AB} \quad \leftrightarrow \quad \mathcal{L}_m \bar{\mathcal{L}}_n , \tag{5.33}$$

as well as set the value of the quadratic Casimir in each chiral sector

$$\epsilon^{ABCD} \mathcal{I}_{ABCD} \, \oplus \, C_2 \quad \leftrightarrow \quad \mathcal{L}^2 \, \oplus \, \bar{\mathcal{L}}^2 , \tag{5.34}$$

where $\mathcal{L}^2 = \gamma^{mn} \mathcal{L}_m \mathcal{L}_m$ and similarly for the barred sector. In the end, the algebra $\mathfrak{hs}_3$ is actually the direct sum of the universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{R})$ in the zero-Casimir representation. This algebra is infinite-dimensional (as is the case of the general d construction).

Continuous family It is possible to relax the latter conditions (5.34) by imposing a weaker set of conditions, leading to a one-parameter[19] family $\mathfrak{hs}_3[\lambda] \simeq \mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda]$ of bosonic algebras constructed as follows

$$1 \oplus \mathfrak{hs}[\lambda] = \frac{\mathcal{U}\left(\mathfrak{sl}(2,\mathbb{R})\right)}{\left\langle \mathcal{L}^2 - \frac{\lambda^2-1}{4} 1 \right\rangle} , \tag{5.35}$$

and similarly for the anti-holomorphic sector, with the identity element denoted by $\bar{1}$. This time, the Casimirs of AdS_3 are not zero and given by

$$C_2 = P^2 + J^2 = 2(\mathcal{L}^2 + \bar{\mathcal{L}}^2) = \frac{\lambda^2-1}{2}(1+\bar{1}) = \frac{\lambda^2-1}{2} id , \tag{5.36a}$$

$$W = J^a P_a = \mathcal{L}^2 - \bar{\mathcal{L}}^2 = \frac{\lambda^2-1}{4}(1-\bar{1}) = \frac{\lambda^2-1}{4}\kappa , \tag{5.36b}$$

where we defined

$$id \equiv 1 + \bar{1} , \quad \kappa \equiv 1 - \bar{1} , \tag{5.37}$$

[18]For this construction to work, it is important that the spin-2 subalgebra $\mathfrak{sl}(2,\mathbb{R})$ be principally embedded within $\mathfrak{sl}(N,\mathbb{R})$, see e.g. [144].

[19]One can also construct a two-parameter family [146] by considering $\mathfrak{hs}[\lambda_1] \oplus \mathfrak{hs}[\lambda_2]$, which are relevant for partially-massless higher-spin gravity in three dimensions, see [147].

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with the extra scalar denoted κ verifying $\kappa^2 \sim id$. Moreover,

$$\kappa \mathcal{L}_m = \mathcal{L}_m \,, \quad \kappa \bar{\mathcal{L}}_m = -\bar{\mathcal{L}}_m \,, \tag{5.38}$$

so that the product of κ with any element of $\mathcal{U}(\mathfrak{so}(2,2))$ can be reduced to an element of $\mathcal{U}(\mathfrak{so}(2,2))$.

The generators are P^s_m and J^s_m for $2 \leq s \leq N$ and with $|m| \leq s-1$. Schematically, the structure constants of $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$ look like

$$[J^s_m, J^t_n] = \sum_{\substack{u=|s-t|+2 \\ s+t+u \text{ even}}}^{s+t-2} g(s,t,u,m,n;\lambda) J^u_{m+n} \,, \tag{5.39a}$$

$$[J^s_m, P^t_n] = \sum_{\substack{u=|s-t|+2 \\ s+t+u \text{ even}}}^{s+t-2} g(s,t,u,m,n;\lambda) P^u_{m+n} \,, \tag{5.39b}$$

$$[P^s_m, P^t_n] = \frac{1}{\ell^2} \sum_{\substack{u=|s-t|+2 \\ s+t+u \text{ even}}}^{s+t-2} g(s,t,u,m,n;\lambda) J^u_{m+n}, \tag{5.39c}$$

with $g(s,t,u,m,n;\lambda)$ known functions (see [148–150]).

The previous algebras (finite and infinite-dimensional) can be unified in a single UEA construction, of which they are particular cases: it was noted in [148] that the algebra $\mathfrak{hs}[\lambda = N]$ develop an infinite dimensional ideal corresponding to generators of spin $s > N$ which, upon factorisation, reduces to $\mathfrak{sl}(N,\mathbb{R})$. The higher-spin theories with an infinite spectrum can formally be recovered as the $N \to \infty$ limit of the algebras $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$ studied in [151, 152]. Moreover, the algebra $\mathfrak{hs}_3$ can be recovered by setting λ to 1 in the one-parameter family.

Deformations and contractions of the $\mathfrak{hs}_3[\lambda]$ family From a mathematical perspective, it is tempting to see what happens if we relax the restriction on the range of the index m in the generators $\mathcal{W}^s_m$. Similarly to the Witt (resp. Virasoro) algebra being the centreless (resp. central) infinite-dimensional enhancement of the $\mathfrak{sl}(2,\mathbb{R})$ algebra, there exist infinite-dimensional enhancement of the $\mathfrak{sl}(N,\mathbb{R})$ algebra, called w_N algebras (centreless) or $\mathcal{W}_N$ (central, built as a non-linear deformations [148] of the former), as well as the one-parameter families $w_\infty[\lambda]$ and $\mathcal{W}_\infty[\lambda]$ that extend $\mathfrak{hs}[\lambda]$.

In full similarity with the BMS_3 algebra being the double copy of two Virasoro algebras with equal central charges, the central $\mathcal{W}_N$ and $\mathcal{W}_\infty[\lambda]$ algebras turn out to be related to asymptotic symmetries of $d = 3$ higher-spin gravity [144, 153]. Moreover, these algebras admit a straightforward $\ell \to \infty$ contraction, giving rise to higher-spin asymptotic symmetries in flat space [154–157].

Massive higher-spin theories in three dimensions can also be obtained through star-product deformations of the $\mathfrak{hs}[\lambda]$ algebras [158], while a quantum deformation of the $\mathcal{W}_N$ and $\mathcal{W}_\infty[\lambda]$ algebras play a role in the context of minimal model holography, as was pointed out in [38, 39]. Finally, the algebra $w_{1+\infty}$, which is a central extension of (the classical contraction of) the w_∞ algebras was recently shown to play a role in celestial holography [159] and in the higher-spin dynamics of a subsector of General Relativity [160].

6. Conclusion and outlook

In these notes, we reviewed the construction of higher-spin theories, guided by the principle of symmetry. After determining the free theory in the metric-like form, we moved on to the construction of higher-spin symmetry algebras, motivated by the Cartan approach to Einstein's General Relativity and made the link between the metric-like and the frame-like formulations. The existence of such algebras is established in AdS space-time with $d \geq 3$, and we showed the strong relation between the symmetries of the free theory and the construction of these algebras, in particular how the elimination of the extra gauge fields is performed thanks to its structure constants describing the coupling of higher-spin fields to gravity. Thanks to the frame-like formulation and the study of symmetries, we have paved the way to understanding the construction of the theory of higher-spin gravity with cubic interactions performed in [2–4, 126, 161–163].

The next logical step is to move on to a fully non-linear interacting theory including terms beyond the cubic order, for an infinite-dimensional multiplet transforming in the higher-spin algebra $\mathfrak{hs}_d$. The construction of such a theory is due to Vasiliev [11], and reviewed, e.g. in [56]. When it comes to constructing an interacting theory of higher-spin fields in flat space-time, there seems to be a general tension between unitarity, locality and a spin greater than two. Dropping one of these assumptions, for instance the first one, it is possible to write a local higher-spin theory in flat space – albeit with a different spectrum than the one we discussed in these notes – that generalises conformal General Relativity: conformal higher-spin gravity [164–166].

Stepping away from the manifestly Lorentz-covariant setup gives interesting results. While the free theory is generally construction-insensitive, working directly with the physical (i.e. non-gauge) degrees of freedom offers more possibilities for interaction terms. Theories in the light-cone gauge were constructed [95], based on the cubic vertices found in [81, 93, 94] and chiral higher-spin theories were studied in [167–169], as well as their connections with Yang-Mills-like theories [170–172], showing interesting quantum properties.

In these notes, we considered only bosonic higher-spin fields. The free theory of higher-spin fermions was determined in [173, 174], generalising the Dirac and Rarita-Schwinger actions, and the supersymmetric $\mathcal{N} = 1$ extension of the higher-spin algebras was built in [131, 162]. Cubic interactions between fermionic and bosonic fields were computed in [85]. The inclusion of gauge fermions does not fundamentally change the general picture of higher-spin gravity that we tried to paint in these notes.

Similarly, higher-spin theories with a different spectrum than the one described here were considered. Free and interacting theories for partially-massless (see section 3.2.2) and mixed-symmetry fields, as well as symmetry algebras for these theories were described in [129, 175–180]. The goal of considering more complicated spectrum is to progressively bridge the gap between the relatively simple spectrum of a gauge theory *à la* Vasiliev and String Theory. On the other hand, theories of massive higher-spin fields is an entirely different subject, since they do not rely on gauge symmetries. They are however better connected to String Theory in the finite-α' regime and interestingly do not suffer from the same no-go constraints that massless fields. Apart from the free theory [181–183] which is similar to Fronsdal's theory, we refer to [26, 27] for more details.

Higher-spin holography is a fascinating subject and the elements given in section 5.1 barely begin to scratch its surface. The original higher-spin AdS/CFT conjecture of [5, 6] has received numerous checks, as well as generalisations, for instance in dS holography [184] or Chern-Simons matter models [12, 13], related to 3-dimensional bosonisation [185, 186]. The results of [187–189] indicate that CFTs with an exact higher-spin symmetry, which is the case of the $O(N)$ vector models at large N, have a trivial holographic S-matrix, in perfect analogy with Weinberg's low-energy or Coleman-Mandula's no-go theorems. The check of the higher-spin holographic correspondance at finite N is a subject of active research. We refer to the excellent [190] for a more detailed review.

Finally, due to their relative simplicity, higher-spin theories in three dimensions have received a lot of developments in the direction of asymptotic symmetries, starting with the generalisation of the work of Brown and Henneaux for pure gravity [140] to higher spins [144, 153, 155, 191–193] as well as in more general contexts, involving asymptotically flat space-times [146, 157], cosmology [156], black holes [194] and generic horizons [195]. More recently, three-dimensional higher-spin theories formulated on non-Lorentzian backgrounds were considered in [196–201]. Asymptotic symmetries of (free) higher-spin fields in higher dimensional asymptotically flat space-time were also studied in [202, 203] and their link with asymptotic symmetry algebras studied in [204, 205].

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A. No-go and yes-go theorems in flat space

In addition to the set of frequently presented no-go theorems on massless higher-spin in flat space [1, 206–208] (see also [74, 209] for a review), we also recall another one based purely on algebraic arguments, as well as possible ways out.

A.1 Symmetries of the S-matrix

Most of the no-go theorems concern the symmetries of the perturbative S-matrix. As such, they are only valid in flat space in their original form.

The Weinberg low-energy theorem [206] Consider a scattering event with N particles with momenta $p_1, \ldots, p_N$ and one spin-s massless particle with momentum q, becoming soft ($q \to 0$). Under the assumptions of locality, Poincaré-invariance and analyticity of the poles of the S-matrix, Weinberg showed that in the limit $q \to 0$, the S-matrix element is dominated by the contribution

$$S(p_1, \ldots, p_N, q) \sim \sum_{i=1}^{N} g_i^s \frac{p_i{}^{\mu} \cdots p_i{}^{\mu} \varepsilon_{\mu(s)}(q)}{2q \cdot p_i} S_{\text{bare}}(p_1, \ldots, p_N)\,, \tag{A.1}$$

where g_i^s is the coupling constant of the spin-s particle to the particle i, $\varepsilon_{\mu(s)}$ is the polarisation tensor of the higher-spin particle, which is traceless and transverse and where $S_{\text{bare}}(p_1, \ldots, p_N)$

represents the amplitude of the N particles scattering together, computed in the absence of the higher-spin leg. Since the S-matrix element (A.1) must satisfy the Ward identity associated to invariance under (super)translations, it was found that

$$\sum_{i=1}^{N} g_{s,i} \underbrace{p_i{}^{\mu} \cdots p_i{}^{\mu}}_{s-1} = 0\,. \tag{A.2}$$

This can be understood by saying that the limit in which the external higher-spin particle acquires a zero momentum $q = 0$ should describe the same physics as having no external higher-spin particle at all, meaning that the pre-factor in eq. (A.1) encoding the pole $q \to 0$ must vanish, producing the conservation law (A.2).

For $s = 1$ or $s = 2$, this is the usual charge $\sum_{i=0}^{N} g_{1,i} = 0$ and momentum $\sum_{i=0}^{N} p_i{}^{\mu} = 0$ conservation, as well as the universal coupling of gravity to other fields, $g_{2,i} = g_2$. For $s > 2$ there is simply no solution to the conservation law, unless $g_{s,i} = 0$ (in this simple case, $g_{s,i} = g_s$ and elastic scattering, i.e. the momenta are shuffled, also satisfies the conservation equation).

Therefore, as a by-product of the low-energy theorem, Weinberg showed that massless higher-spin particles in flat space cannot produce non-trivial scattering amplitudes in the low-energy limit. In other words, they do not carry long-range interactions. For the moment, massive higher-spin particles are safe (although they suffer from their own type of no-go theorems [210]), and so are gauge theories of (massless) higher-spin particles, provided they undergo some sort of confinement *à la QCD* and are not visible as individual particles in the low-energy regime.

The Coleman-Mandula theorem [207] Coleman and Mandula classified all possible symmetries of the S-matrix, under the hypothesis of analyticity and that the spectrum is gapped. They found that, apart from the product of Poincaré symmetry and an internal symmetry, any other symmetry of the S-matrix makes it trivial. The generalisation [211] also includes supersymmetry to the list of admissible space-time symmetries, while dropping the hypothesis of a mass gap, ruling out massless theories as well.

Possible ways out It seems that the only way out of the Weinberg low-energy theorem is to accept the fact that the higher-spin sector is gauged and that fields with a spin strictly greater than two will never be seen in scattering experiments with finite energy, just like individual gluons in QCD. Recently however [212], it was pointed out that the Weinberg low-energy theorem has to be understood as a statement about the number of derivatives in the coupling to gravity, rather than a no-go on higher-spin interactions per se. It is expected that moving to a different formulation of the dynamics than the one of Fronsdal could also cure the problem of scattering in flat space.

A.2 (In)existence of a gauge algebra

The theorem Let $\mathfrak{g}$ be the Poincaré or (A)dS algebra in $d \geq 4$. Taking (4.55) and (4.56) as initial data for a non-Abelian higher-spin Lie algebra extending $\mathfrak{g}$ with the spectrum of (5.10) with the hypothesis that $\mathfrak{g}$ is contained as a subalgebra and that at least one 'higher-spin' bracket of the form $[Z^{s_1}, Z^{s_2}]$ is non-vanishing for s_1 and s_2 strictly greater than two[20] yields no solution if $\mathfrak{g}$ is the Poincaré algebra and exactly one if $\mathfrak{g}$ is the (A)dS algebra [3, 73, 121]. This solution is the Fradkin-Vasiliev-Eastwood algebra $\mathfrak{hs}_d$ described in (5.2) (there are also some dimension-dependent constructions, e.g. for $d = 5$, see [121, 125, 129, 213]).

Possible way out Surprisingly, there exists an Inonu-Wigner contraction of the higher-spin algebra that contains Poincaré as a subalgebra [145]. It is built as follows

$$\mathfrak{ihs}_d \equiv \frac{\mathcal{U}\left(\mathfrak{iso}(1, d-1)\right)}{\langle \mathcal{I}'_{abcd} \oplus \mathcal{I}'_{abc} \oplus \mathcal{I}'_{ab} \oplus \mathcal{I}'_a \rangle}, \tag{A.3}$$

where

$$\mathcal{I}'_{abcd} \equiv J_{[ab} \odot J_{cd]}, \quad \mathcal{I}'_{abc} \equiv J_{[ab} \odot P_{c]}, \quad \mathcal{I}'_{ab} \equiv P_a \odot P_b, \quad \mathcal{I}'_a \equiv J_{ab} \odot P^b. \tag{A.4}$$

Note that, as a result of factorising $\mathcal{I}'_a$ and $\mathcal{I}'_{abc}$, the value of the quadratic Casimir of the Lorentz algebra $\mathfrak{so}(d-1, 1)$ is fixed

$$\mathcal{I}'_{abc} J^{bc} - \frac{2}{3} J_a{}^b \mathcal{I}'_b - \frac{d-3}{3} \mathcal{I}'_a + \frac{4}{3}\left(J^2 + \frac{(d-1)(d-3)}{4}\right) P_a = 0, \tag{A.5}$$

where $J^2 = \frac{1}{2} J_{ab} J^{ba}$. The reason this algebra was not in the classification of [4] is because it does not reproduce the initial data (4.56). However, the unfolded dynamics of this algebra is equivalent to the usual one, which we will show in [214].

B. Reminders of tensor calculus

This section provides some basics of tensor calculus for higher representations of the Lorentz group, that is beyond the usual vector $\mathbb{Y}_d(1)$, rank-2 symmetric traceless $\mathbb{Y}_d(2)$ and rank-2 anti-symmetric $\mathbb{Y}_d(1,1)$ representations.

B.1 Conventions

As explained in the introduction, we work in the symmetric convention, i.e. indices within groups separated by commas are symmetrised. We also use the shorthand that repeated indices that are not summed are symmetrised as well. For example, for a generic tensor

$$T_{a(n_1), b(n_2), \cdots, d(n_k)} \sim T_{(a_1 \cdots a_{n_1}), (b_1 \cdots b_{n_2}), \cdots, (d_1 \cdots d_{n_k})}. \tag{B.1}$$

Symmetrisation and product As an example of symmetrisation between indices belonging to different tensors, consider the product of the mixed-symmetry tensor $T^{a(k-1),b}$ and a vector V^a, the quantity $V^a T^{a(k-1),b}$ represents the tensor

$$V^{(a_1} T^{a_2 \cdots a_k), b} = \frac{1}{k}\left(V^{a_1} T^{a_2 \cdots a_k, b} + V^{a_2} T^{a_1 a_3 \cdots a_k, b} + \cdots + V^{a_k} T^{a_1 \cdots a_{k-1}, b}\right), \tag{B.2}$$

where the $1/k$ factor 'normalises' the sum of k terms within brackets to one, and we used cyclic permutations to implement the symmetrisation since T is already symmetric in the indices of the first row.

[20] In [4], there is also the technical hypothesis that the bracket $[Z^s, Z^s]$ always gives a generator of the base algebra (i.e. Poincaré or (A)dS). Relaxing this hypothesis leads to a contraction of the higher-spin algebra $\mathfrak{hs}_d$ and the latter can always be recovered by deformation.

Compatibility condition The fact that all tensors have the symmetries of the Young tableau $\mathbb{Y}_d(n_1, n_2, \cdots, n_k)$ is encoded in the *compatibility condition*: symmetrisation of the indices in a row with (at least) one index from the following row gives zero, i.e.

$$T_{a(n_1),ab(n_2-1),\cdots,d(n_k)} \sim T_{(a_1\cdots a_{n_1},a_{n_1+1})b_2\cdots b_{n_2},\cdots,d_1\cdots d_{n_k}} = 0\,, \tag{B.3a}$$

$$T_{a(n_1),b(n_2),bc(n_3-1),\cdots,d(n_k)} \sim T_{a_1\cdots a_{n_1},(b_1\cdots b_{n_2},b_{n_2+1})c_2\cdots c_{n_3},\cdots,d_1\cdots d_{n_k}} = 0\,, \tag{B.3b}$$

and so on. For example, for a $\mathbb{Y}_d(1,1)$ tensor $J_{a,b}$, the compatibility condition reduces to

$$J_{(a,b)} = \frac{1}{2}\left(J_{a,b} + J_{b,a}\right) = 0\,, \tag{B.4}$$

which means that $J_{a,b}$ is antisymmetric. The projections $\mathbb{Y}_d(1,1,\cdots,1)$ are the only ones where antisymmetrisation between indices is manifest (this is because we are using the symmetric convention, in the antisymmetric convention the opposite statement will be true). For a $\mathbb{Y}_d(2,1)$ tensor $M_{ab,c}$, the compatibility condition requires

$$M_{(ab,c)} = \frac{1}{3}\left(M_{ab,c} + M_{bc,a} + M_{ca,b}\right) = 0\,, \tag{B.5}$$

where we used that the tensor $M_{ab,c}$ is symmetric in its first two indices to reduce the symmetrisation to a cyclic permutation. For completeness, let us also show the example of the $\mathbb{Y}_d(2,2)$ 'window' tensor

$$K_{(ab,c)d} = \frac{1}{3}\left(K_{ab,cd} + K_{bc,ad} + K_{ca,bd}\right) = 0\,. \tag{B.6}$$

Tracelessness The tensors we work with are furthermore in irreducible representations of the Lorentz group, which means that they are traceless on any pair of indices. As an example, in Minkowski space-time with metric η_{ab},

$$\eta^{a(2)}T_{a(n_1),b(n_2),\cdots,d(n_k)} = 0\,, \quad \eta^{ab}T_{a(n_1),b(n_2),\cdots,d(n_k)} = 0\,, \quad \ldots\,. \tag{B.7}$$

Thanks to the compatibility conditions, it is enough to require tracelessness in the first row. Indeed, taking again the example of $M_{ab,c}$, we have

$$0 = \eta^{ab}M_{(ab,c)} = \eta^{ab}\frac{1}{3}\left(M_{ab,c} + M_{bc,a} + M_{ca,b}\right) = \frac{1}{3}\eta^{ab}M_{ab,c} + \frac{2}{3}\eta^{ab}M_{ca,b}\,, \tag{B.8}$$

so tracelessness in the indices of the first row implies tracelessness in the indices between the first and the second row (there is no trace in the indices of the second row in this example). For the 'window'

$$0 = \eta^{ab}K_{(ab,c)d} = \eta^{ab}\frac{1}{3}\left(K_{ab,cd} + K_{bc,ad} + K_{ca,bd}\right) = \frac{1}{3}\eta^{ab}K_{ab,cd} + \frac{2}{3}\eta^{ab}K_{ca,bd}\,, \tag{B.9a}$$

$$0 = \eta^{cd}K_{(ab,c)d} = \eta^{cd}\frac{1}{3}\left(K_{ab,cd} + K_{bc,ad} + K_{ca,bd}\right) = \frac{1}{3}\eta^{cd}K_{ab,cd} + \frac{2}{3}\eta^{cd}K_{ca,bd}\,, \tag{B.9b}$$

so tracelessness in the indices of the first row implies tracelessness in all pairs of indices. Note that the Fronsdal field $\phi_{a(s)}$ is *not* in an irreducible representation of the Lorentz group, since for $s \geq 4$ it is *doubly* traceless (the double trace can be computed on any pairs of indices because of the symmetry between indices)

$$\phi''{}_{a(s-4)} \sim \eta^{a_1a_2}\eta^{a_3a_4}\phi_{a_1\cdots a_s}\,. \tag{B.10}$$

It can however be decomposed into a traceless components $\mathbb{Y}_d(s) \oplus \mathbb{Y}_d(s-2)$

$$\phi_{a_1\cdots a_s} = \left(\phi_{a_1\cdots a_s} - \frac{s(s-1)}{2d+(s-2)(s+1)}\eta_{(a_1a_2}\phi'_{a_3\cdots a_s)}\right) + \frac{s(s-1)}{2d+(s-2)(s+1)}\eta_{(a_1a_2}\phi'_{a_3\cdots a_s)}\,, \tag{B.11}$$

where $\phi'_{a_3\cdots a_s} = \eta^{bc}\phi_{bca_3\cdots a_s}$. Both the term within parenthesis and the second term are in the traceless projection of their indices.

> *Exercise:* check this.

B.2 Young tableaux manipulation

Young tableaux are used to depict symmetries pictorially. Each box represents an index, and indices within each rows are symmetrised. Except in section B.2.1, where we explain the 'branching rules' to relate Young tableaux in different space-time dimensions, the dimensionality of space-time is fixed. In section B.2.2, we display the decomposition in Lorentz-irreducible components of some tensor products.

B.2.1 Branching rules

The branching rule [215] states that the restriction of a $SO(d)$ diagram $\mathbb{Y}_d(a_1, a_2, \cdots, a_n)$ to $SO(d-1)$ yields several diagrams $\mathbb{Y}_{d-1}(b_1, b_2, \cdots, b_n)$, which we will denote by an arrow

$$\mathbb{Y}_d(a_1, a_2, \cdots, a_n) \longrightarrow \bigoplus_{b_i} \mathbb{Y}_{d-1}(b_1, b_2, \cdots, b_n) \tag{B.12}$$

such that

$$0 \le b_n \le a_n \le b_{n-1} \le \cdots \le b_2 \le a_2 \le b_1 \le a_1\,. \tag{B.13}$$

In practice, we will only use the branching for two-row Young diagrams, which greatly simplifies the branching rule. As an example, the branching of a rectangular two-row Young tableau of length $s-1$ from $d+1$ to d-dimensional space-time reads

$$\mathbb{Y}_{d+1}(s-1, s-1) \longrightarrow \bigoplus_{t=0}^{s-1} \mathbb{Y}_d(s-1, t)\,, \tag{B.14}$$

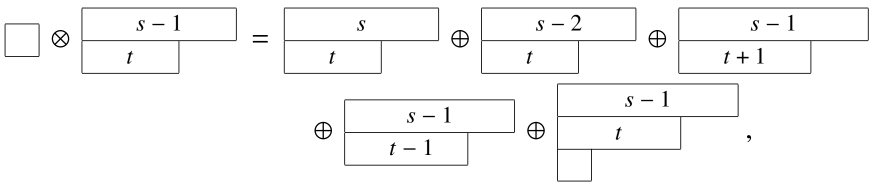
which shows that for each spin $s \ge 1$, the traceless Killing tensors of eq. (3.31) can be embedded inside of a two-row Young tableau as shown in eq. (5.10).

B.2.2 Tensor products

We work with trace-irreducible Young tableaux. For one-forms (gauge fields), we have the multiplication rules

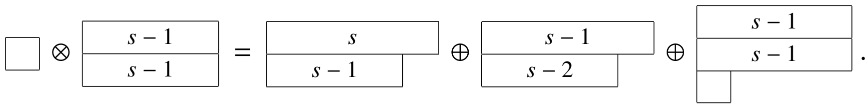
$$\square \otimes \boxed{\quad s-1 \quad} = \boxed{\quad s \quad} \oplus \boxed{\quad s-2 \quad} \oplus \begin{array}{l}\boxed{\quad s-1 \quad}\\ \square\end{array}\,, \tag{B.15}$$

and



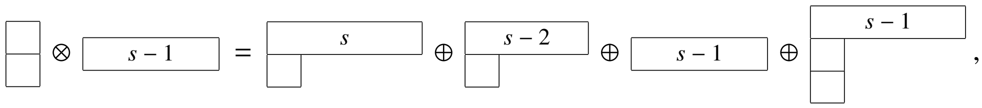
(B.16)

for $1 \leq t \leq s-2$, and



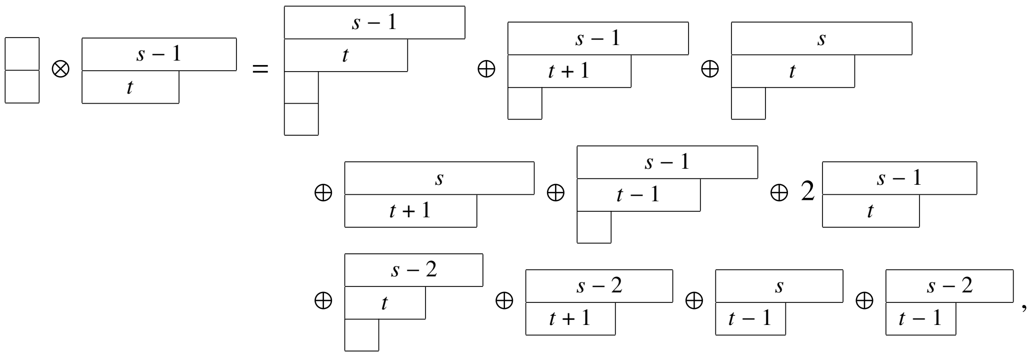
(B.17)

For two-forms (torsions and curvatures), we have



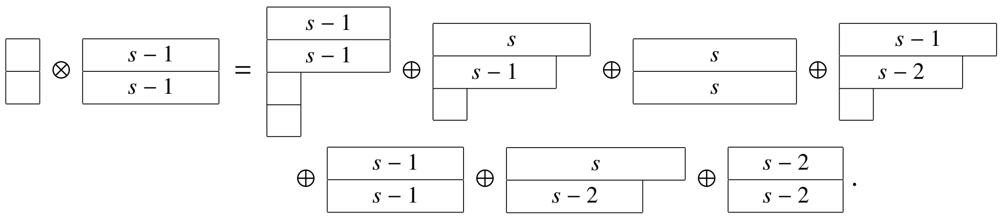
(B.18)

and



(B.19)

for $1 \leq t \leq s-2$, and



(B.20)

C. Enveloping algebra of conformal isometries

In this section, we show how the Eastwood algebra [16] can be realised as the Universal Enveloping Algebra of conformal isometries of $\mathbb{M}_{d-1}$, realised on the singleton module. We start by recalling the differential realisation of (a real form of) the conformal algebra in $d-1$ dimensions acting on a primary field φ with conformal dimension $\Delta = \frac{d-3}{2}$

$$J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \,, \tag{C.1a}$$

$$P_\mu = \partial_\mu \,, \tag{C.1b}$$

$$K_\mu = 2x_\mu \left(x^\nu \partial_\nu + \Delta\right) - x^\nu x_\nu \partial_\mu \,, \tag{C.1c}$$

$$D = x^\mu \partial_\mu + \Delta \,. \tag{C.1d}$$

The generators verify the $\mathfrak{so}(2, d-1)$ commutation relations

$$
\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= \eta_{\nu\rho}\, J_{\mu\sigma} - \eta_{\mu\rho}\, J_{\nu\sigma} - \eta_{\nu\sigma}\, J_{\mu\rho} + \eta_{\mu\sigma}\, J_{\nu\rho}\,, && \text{(C.2a)}\\
[J_{\mu\nu}, P_{\rho}] &= \eta_{\nu\rho}\, P_{\mu} - \eta_{\mu\rho}\, P_{\nu}\,, && \text{(C.2b)}\\
[J_{\mu\nu}, K_{\rho}] &= \eta_{\nu\rho}\, K_{\mu} - \eta_{\mu\rho}\, K_{\nu}\,, && \text{(C.2c)}\\
[D, P_{\mu}] &= -P_{\mu}\,, && \text{(C.2d)}\\
[D, K_{\mu}] &= +K_{\mu}\,, && \text{(C.2e)}\\
[K_{\mu}, P_{\nu}] &= -2\, J_{\mu\nu} - 2\, \eta_{\mu\nu}\, D\,. && \text{(C.2f)}
\end{aligned}
$$

Then, one may verify that the expressions

$$
J_{[\mu\nu} \odot J_{\rho\sigma]}\,, \quad J_{[\mu\nu} \odot P_{\rho]}\,, \quad J_{[\mu\nu} \odot K_{\rho]}\,, \quad P_{[\mu} \odot K_{\nu]} + D \odot J_{\mu\nu}\,, \tag{C.3}
$$

identically vanish in the differential realisation, and that

$$
\begin{aligned}
&\left[J^{\rho}{}_{(\mu} \odot J_{\nu)\rho} - P_{(\mu} \odot K_{\nu)} - \frac{1}{d-1}\,\eta_{\mu\nu}\left(4\,J^2 - P^{\rho} \odot K_{\rho}\right)\right]\varphi\,, && \text{(C.4a)}\\
&\left[J^{\rho}{}_{\mu} \odot P_{\rho} - P_{\mu} \odot D\right]\varphi\,, && \text{(C.4b)}\\
&\left[J^{\rho}{}_{\mu} \odot K_{\rho} + K_{\mu} \odot D\right]\varphi\,, && \text{(C.4c)}\\
&\left[P^{\mu} \odot P_{\mu}\right]\varphi\,, && \text{(C.4d)}\\
&\left[K^{\mu} \odot K_{\mu}\right]\varphi\,, && \text{(C.4e)}\\
&\left[\frac{1}{2} D \odot D - \frac{1}{4} P^{\mu} \odot K_{\mu} + \frac{d-3}{2}\right]\varphi\,, && \text{(C.4f)}
\end{aligned}
$$

where $J^2 = \frac{1}{4} J_{\mu\nu} \odot J^{\nu\mu}$, as well as

$$
\left[C_2 + \frac{(d+1)(d-3)}{4}\right]\varphi = \left[J^2 + \frac{1}{2} D \odot D - \frac{1}{2} P^{\mu} \odot K_{\mu} + \frac{(d+1)(d-3)}{4}\right]\varphi\,, \tag{C.5}
$$

vanish as a consequence of the equation of motion $\partial^2\varphi = 0$.

Using the branching rule (B.14) twice, one can show that the elements in (C.3) combine to form a rank-4 completely antisymmetric representation of $SO(2, d-1)$, while the elements in (C.4) combine to form a rank-2 symmetric traceless representation of $SO(2, d-1)$. Finally, the quadratic Casimir C_2 takes the value specified in (C.5). One can also check that there are no other independent relations involving elements quadratic in the generators of the conformal algebra.

This proves that the algebra of higher isometries of the singleton is isomorphic to $\mathfrak{hs}_d$ defined in (5.2), since the base algebra is the same (conformal isometries), its spectrum of generators matches with $\mathfrak{hs}_d$ as was proven in [16] and it automatically factorises exactly the ideal $\mathcal{I}_{ABCD} \oplus \mathcal{I}_{AB}$, with the right value of the quadratic Casimir C_2.

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