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Complete complemetarity relations and quantifiers of quantum correlations in neutrino oscillations

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We analyze complete complementarity relations, which characterize the interplay between different correlations encoded in a quantum system, for oscillating neutrinos. We also provide a short review of the main results obtained so far about quantum correlations in neutrino oscillations.

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1. Introduction

Quantum correlations provide a fertile testing ground for foundational aspects of quantum physics. In the last years, this very active research area has turned the attention towards subatomic physics [1]-[19], after being studied for a long time in a variety of physical context, such as quantum optics and condensed matter physics. In particular, the phenomenon of neutrino oscillations offers a rare example of quantum correlations on macroscopic scale. The quantum nature of neutrinos has been studied in terms of entanglement [2]-[5], Bell and Leggett- Garg inequalities [8]-[11] , and various aspects of quantum coherence, such as steering [13]-[14] and nonlocal advantage of quantum coherence (NAQC) [15]. Again, they have been considered in the context of Entropic Uncertainty Relations [16, 17] and so on. In recent years, interest has been focused on a quantitative characterization of these aspects. In fact, quantum correlations and effectiveness in detecting coherence and quantumness, are arguments which hide very delicate and subtle facets. It is worth pointing out that, while for a global pure state entanglement encompasses any possible form of correlations, for a mixed state several layers of non- classical correlations have been identified [20]. They show strict inclusion relations. In decreasing order, these can be classified as: NAQC $\subset$ Bell non-locality $\subset$ steering $\subset$ entanglement $\subset$ general quantum correlations (discord). The quantumness in neutrinos has also been studied by using complete complementarity relations (CCR) that fully characterize the interplay between different correlations encoded in a quantum system. In the following, we first provide a short review about quantum correlations quantifiers in the recent literature. Then we present the main outcomes related to CCR in neutrino oscillations.

2. A review of studies on quantum correlations in neutrino oscillations

In this section we review the principal results about quantum correlation quantifiers in the system of oscillating neutrinos, contained in Refs.[4, 13, 15–17, 21–25].

In [4] it is shown how, for oscillating neutrinos, some quantum correlations, as Bell non-locality, entanglement and discord, can be efficiently expressed in terms of oscillation probabilities. The authors consider the following decomposition for the density matrix ρ describing the system:

$$\rho = \frac{1}{4}[I_2 \otimes I_2 + \vec{r} \cdot \vec{\sigma} \otimes I_2 + I_2 \otimes \vec{s} \cdot \vec{\sigma} + \sum_{m,n} T_{mn}(\sigma_m \otimes \sigma_n)] \tag{1}$$

where $\vec{r} \equiv (r_x, r_y, r_z)$, $\vec{s} \equiv (s_x, s_y, s_z)$ and T_{mn} are the elements of the correlation matrix T. The decomposition coefficients can be found as: $r_m = \mathrm{Tr}[\rho(\sigma_m \otimes I_2)]$, $s_m = \mathrm{Tr}[\rho(I_2 \otimes \sigma_m)]$ and $T_{mn} = \mathrm{Tr}[\rho(\sigma_m \otimes \sigma_n)]$, $(m, n = x, y, z)$, where σ_m are the Pauli matrices. Let $u_i (i = 1, 2, 3)$ be the eigenvalues of the matrix $T^\dagger T$. The Bell-CHSH inequality can be written as $M(\rho) \leq 1$, where $M(\rho) = \max(u_i \otimes u_j), (i \neq j)$. For a neutrino state (Eq.(38) in the next section):

$$M(\rho) = 1 + 4P_{sur}P_{osc}. \tag{2}$$

where P_{sur} and P_{osc} are the survival and oscillation probabilites, respectively. It is rapid to conclude that a violation of the inequality occurs when $P_{sur} < 1$, with a maximal violation for $P_{osc} = P_{sur} = \frac{1}{2}$. The authors also consider the entanglement quantified by the concurrence measure:

$$C = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0), \tag{3}$$

where λ_i are the square roots of the eigenvalues of $\rho\tilde{\rho}$ in decrasing order, with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y$. For the neutrino state under consideration:

$$C = 2\sqrt{P_{sur}P_{osc}}. \tag{4}$$

Thus, entanglement is present when the oscillation probability is non-zero. A still weaker measure of quantum correlation is considered in [4], that is quantum discord, which is quantified by the geometric discord:

$$D_G(\rho) = \frac{1}{3}\left[||\vec{y}||^2 + ||T||^2 - \lambda_{max}\right], \tag{5}$$

in which $\vec{y}$ is the vector whose components are $y_m = Tr[\rho(\sigma_m \otimes I_2)]$ and λ_{max} is the maximum eigenvalues of the matrix $(\vec{y}\vec{y}^\dagger + TT^\dagger)$. For the neutrino state:

$$D_G(\rho) = \frac{8}{3}P_{sur}P_{osc}. \tag{6}$$

Quantum discord is non-zero when $P_{osc} \neq 0$.

Furthermore, the authors consider the teleportation fidelity, that defines the practical use of quantum correlations. The average fidelity F quantifies how well unkown input states can be transmitted to another location, showing the optimality of quantum teleportation. The maximum teleportation fidelity is:

$$F_{max} = \frac{1}{2}\left(1 + \frac{1}{3}N(\rho)\right). \tag{7}$$

$N(\rho) = (\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3})$, where u_i are the eigenvalues of the matrix $T^\dagger T$. The teleportation is possible whenever $F_{max} > \frac{2}{3}$, where $\frac{2}{3}$ is the classical value of teleportation fidelity [26].

In [13] it is analyzed the concept of coherence, which is fundamental in Quantum Mechanics. Coherence can be considered as a resource [20] and its quantitative characterization expresses the level of quantumness of a given system. The authors show a method for quantifying the quantumness of neutrino oscillation with the use of the l_1-norm coherence measure, defined as:

$$C_{l_1}(\rho) = \sum_{i\neq j} |\rho_{ij}|, \tag{8}$$

i.e., the sum of the absolute values of all off-diagonal elements ρ_{ij} of the density matrix ρ. The maximum value of $C(\rho)$ is $C_{max} = d - 1$, where d is the dimension of ρ. For a three flavor neutrino system, it can be expressed in terms of the transition probabilities as:

$$C_\alpha = 2\left(\sqrt{P_{\alpha e}P_{\alpha\mu}} + \sqrt{P_{\alpha e}P_{\alpha\tau}} + \sqrt{P_{\alpha\mu}P_{\alpha\tau}}\right). \tag{9}$$

In [13] it is shown a comparison between the coherence in experimentally observed neutrino oscillations from different sources, as Daya-Bay, KamLAND, MINOS and T2K, and the theoretical predictions. There is a good agreement in all the cases, especially for the KamLAND one, for which the authors found a value close to the theoretical maximum of coherence. A similar argument is presented in [22], in which is shown a comparison between the l_1-norm as a coherence measure with the concurrence as an entanglement measure by using a wave packet description for three flavor neutrino oscillations. The authors highlight that the origin of flavor entanglement is the same of the quantum coherence in neutrino. By quantifying the entanglement as in Eq.(3), for the initial flavor α, the concurrence between flavor β and γ is obtained as:

$$C^{\alpha}_{\beta\gamma} = 2\sqrt{P_{\alpha\beta}P_{\alpha\gamma}}, \tag{10}$$

in accordance with [4]. The authors conclude that, for a given flavor α, the sum of the three possible concurrences is equal to the l_1-norm measure of coherence.

In [15] the quantumness in experimentally observed neutrino oscillations has been investigated via the NAQC, quantum steering and Bell non-locality. It has been also shown that exists a strict hierarchical relationship among these correlations and that the NAQC is the strongest one. This highlights that NAQC is a reliable tool for the quantification of quantumness in two-flavor neutrino systems. The authors use the NAQC based on l_1-norm coherence measure [27]. For a neutrino bipartite state, it is possible to express it in terms of the transition probability as:

$$N_{l_1}(\rho^{\alpha}) = 2 + 2\sqrt{P_{\alpha\alpha}P_{\alpha\beta}}, \tag{11}$$

with $\alpha, \beta = e, \mu$. For the Bell non-locality the authors find the same outcome as in Eq.(2), according to Ref.[4]. The criterion for quantifying quantum steering for a bipartite system is given by [28]:

$$F_n(\rho, \varsigma) = \frac{1}{\sqrt{n}}\left|\sum_{i=1}^{n} \mathrm{Tr}(\rho A_i \otimes B_i)\right| \leq 1, \tag{12}$$

where $A_i = \zeta_i \cdot \sigma$ and $B_i = \xi_i \cdot \sigma$, $\zeta_i \in \mathcal{R}^3$ are unit vectors, $\xi_i \in \mathcal{R}^3$ are othonormal vectors, and $\varsigma = \{\zeta_1, ..., \zeta_n, \xi_1, ..., \xi_n\}$ denotes the set of measurement directions. For a neutrino bipartite state:

$$F_3(\rho, \varsigma) = \sqrt{\frac{1 + 8P_{\alpha\alpha}P_{\alpha\beta}}{3}}. \tag{13}$$

In [21] these considerations have been extended by using a wave packet approach for neutrino oscillations resorting to parameters from Daya-Bay and MINOS experiments. It has been found that, in the case of Daya-Bay experiment, corrections provided by the wave packet approach with respect the plane-wave one are practically irrelevant. At variance, when MINOS parameters are considered, the corrections are noticeable and lead to a better description of experimental data. From this analysis also comes out an interesting behaviour of NAQC. It is worth specifying that the local coherence bound of the single subsystem, beyond which we can reach a NAQC, is equal to $\sqrt{6}$ in the case of l_1-based NAQC. At large distances, when the oscillations are washed away, exceeding this limit depends solely on the mixing angle, leading to a violation in the case of MINOS at variance with the Daya- Bay case.

In [23] the hierarchy among three different definitions of NAQC, based on l_1-norm, relative entropy and skew information coherence measures [29, 30], has been investigated in neutrino systems. It has been found that the coherence content detected by the l_1-norm based NAQC overcomes the other two and represents an upper limit for them. Thus, l_1-norm based NAQC results to be more able to capture quantum resources with respect the other definitions. The expression of the l_1-norm based NAQC is given in Eq.(11). The expressions of the relative entropy and skew information based NAQCs are, respectively:

$$N_{re}(\rho) = 2 - P_{\alpha\alpha}\log_2 P_{\alpha\alpha} - P_{\alpha\beta}\log_2 P_{\alpha\beta}, \tag{14}$$

and

$$N_{sk}(\rho) = 2 + 4P_{\alpha\alpha}P_{\alpha\beta}. \tag{15}$$

In [16, 23] the quantumness in neutrino oscillations has been analyzed by exploiting the Entropic Uncertainty Relations (EUR) as a criterion to detect quantum correlations. The uncertainty principle provides a limit to our ability to predict the measurement results for a pair of incompatible observables of a quantum system. Uncertainty relations can be expressed in terms of the entropy [31, 32]. Recently, the EUR have been generalized to the case in which the parts of the considered system can be correlated in a non-classical way [33, 34]. The correlation between subsystems can be exploited to reduce the uncertainty below the usual limits: we then talk about Quantum Memory Assisted - Entropic Uncertianty Relations (QMA-EUR). Let us suppose Bob prepares a bipartite state ρ_{AB}, with correlations between A and B. He sends part A to Alice and keeps part B as a quantum memory. Alice can decide to measure one of two observables P and R and tells Bob her choice. Based on Alice's choice, Bob is able to guess her outcomes with minimal deviation limited by the uncertainty's lower bound by means of the part B which is correlated with A. The QMA-EUR, in terms of von Neumann entropy, can be expressed as:

$$S(P|B) + S(R|B) \geq -\log_2 c(P|R) + S(A|B). \tag{16}$$

where $S(A|B) = S(\rho_{AB}) - S(\rho_B)$ is the conditional von Neumann entropy of ρ_{AB} with $S(\rho_{AB}) = -tr(\rho_{AB}\log_2\rho_{AB})$, $\rho_B = tr_A(\rho_{AB})$, $S(X|B) = S(\rho_{XB}) - S(\rho_B)$ is the conditional von Neumann entropy of $\rho_{XB} = \sum_i(|\psi_i^X\rangle_A\langle\psi_i^X|\mathcal{I}_B)\rho_{AB}(|\psi_i^X\rangle_A\langle\psi_i^X|\mathcal{I}_B)$ (that is the state of B after performing a measurement on A of the observable X with eigenstates $|\psi_i^X\rangle$), $c(P|R) = \max_{j,k}|\langle\psi_j^P|\phi_k^R\rangle|^2$ represents the maximal overlap between the eigenstates $|\psi_j^P\rangle$ and $|\phi_k^R\rangle$ of the observables P and R. Since the correlations can reduce the uncertainty, in [16] it has been analyzed the relation between QMA-EUR and the more general quantum correlation (Quantum Discord) by using a plane-wave approximation for neutrino oscillations. For a neutrino bipartite state, it has been found that the entropic uncertainty U^α, $\alpha = e, \mu$, and the uncertainty lower bound U_b^α, are given by:

$$U^\alpha = 2(P_{\alpha\alpha}\log_2 P_{\alpha\alpha} + P_{\alpha\beta}\log_2 P_{\alpha\beta} + 1), \tag{17}$$

$$U_b^\alpha = P_{\alpha\alpha}\log_2 P_{\alpha\alpha} + P_{\alpha\alpha}\log_2 P_{\alpha\beta} + 1. \tag{18}$$

These are related to QD by:

$$U^\alpha = 2U_b^\alpha = 2 - 2QD(\rho_{AB}^\alpha). \tag{19}$$

In [23] it has been studied the relation between QMA-EUR and the strongest quantifier of quantum correlations (NAQC) by using a wave packet approach to NOs. For a bipartite neutrino state, by using the entropy-based NAQC, Eq.(15), it has been found:

$$U^\alpha = 2U_b^\alpha = 2[3 - N(\rho_{AB}^\alpha].\tag{20}$$

In both [16] and [23] it has been concluded that the uncertainty is anti-correlated to the quantum correlation considered: the stronger the quantum correlation, the smaller the uncertainty. It is worth discussing a further aspect that emerges from the analysis carried out in [23]. As shown in [21], in the wave packet approach the asymptotic trend of the NAQC depends on the mixing angle, with the

NAQC attaining its maximum value at great distances. In [23] it is shown that this happens when we consider parameters from KamLAND and MINOS neutrino experiments. Consequently, due to the anti-correlation between uncertainty and NAQC, this suggests that entropic uncertainty and its lower bound can go to zero asymptotically at large distance for sufficiently high values of the mixing angle.

In [17] it has been investigated the relation between the entanglement and the EUR in the context of three-flavor neutrino oscillations. Three different measure of entanglement have been considered: Entanglement of Formation (EOF) [35], Concurrence (C) [36] and Negativity ($\mathcal{N}$) [37]. The hierarchical relationship among them has been explored.

For a tripartite pure state ρ_{ABC}, the EOF can be expressed in terms of von Neumann entropy as [38]:

$$EOF(\rho_{ABC}) = \frac{1}{2}\left[S(\rho_A) + S(\rho_B) + S(\rho_C)\right]. \tag{21}$$

The Negativity is defined as:

$$\mathcal{N}(\rho_{ABC}) = (\mathcal{N}_{A-BC}\mathcal{N}_{B-CA}\mathcal{N}_{C-AB})^{\frac{1}{3}}, \tag{22}$$

where $\mathcal{N}_{A-BC} = -\sum_i \lambda_i^A$, $\mathcal{N}_{B-CA} = -\sum_j \lambda_j^B$, $\mathcal{N}_{C-AB} = -\sum_k \lambda_k^C$, where λ_ϵ^α ($\alpha = A, B, C$ and $\epsilon = i, j, k$) are the negative eigenvalues of the partial transpose of ρ_{ABC}.

The Concurrence measure of entanglement for a tri-qubit state is given by [39]:

$$C(\rho_{ABC}) = \left[3 - \mathrm{Tr}(\rho_A)^2 - \mathrm{Tr}(\rho_B)^2 - \mathrm{Tr}(\rho_C)^2\right]^{\frac{1}{2}}. \tag{23}$$

The tripartite generalization of the EUR is obtain by [40]:

$$S(\widehat{R}|B) + S(\widehat{S}|C) \geq q_{MU}, \tag{24}$$

where $S(\widehat{R}|B) = S(\rho_{\widehat{R}B}) - S(\rho_B)$ and analogously $S(\widehat{S}|C)$ are the conditional von Neumann entropy and $q_{MU} = -\log_2 c(\widehat{R}|\widehat{S})$, with $c = \max_{ij}\{|\langle\phi_i^{\widehat{R}}|\psi_j^{\widehat{S}}\rangle|^2\}$ representing the maximal overlap between the observable $\widehat{R}$ and $\widehat{S}$ with $|\phi_i^{\widehat{R}}\rangle$ and $|\psi_j^{\widehat{S}}\rangle$ denoting the corresponding eigenstates.

To explore EUR in three-flavor NOs, for a given flavor A, we have to consider the sum of Eq.(24) for three arbitrary non-commuting operators $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$:

$$\mathcal{U} = S(\mathbb{X}|B) + S(\mathbb{Z}|C) + S(\mathbb{Y}|B) + S(\mathbb{X}|C) + S(\mathbb{Z}|B) + S(\mathbb{Y}|C) \geq 3q_{MU}. \tag{25}$$

For a tri-partite neutrino state:

$$|\nu_\alpha(t)\rangle = a_{\alpha e}(t)\,|100\rangle + a_{\alpha\mu}(t)\,|010\rangle + a_{\alpha\tau}(t)\,|001\rangle\,, \tag{26}$$

the expression of Entanglement of Formation, Negativity and Concurrence in terms of transition probabilities are, respectively:

$$\begin{aligned} EOF = -\frac{1}{2}\big[&P_{\alpha e}\log_2 P_{\alpha e} + P_{\alpha\mu}\log_2 P_{\alpha\mu} + P_{\alpha\tau}\log_2 P_{\alpha\tau} \\ &+ (P_{\alpha\mu} + P_{\alpha\tau})\log_2(P_{\alpha\mu} + P_{\alpha\tau}) \\ &+ (P_{\alpha e} + P_{\alpha\tau})\log_2(P_{\alpha e} + P_{\alpha\tau}) \\ &+ (P_{\alpha\mu} + P_{\alpha e})\log_2(P_{\alpha\mu} + P_{\alpha e})\big], \end{aligned} \tag{27}$$

$$\mathcal{N} = \left(\sqrt{P_{\alpha e}}\sqrt{P_{\alpha\mu}+P_{\alpha\tau}}\sqrt{P_{\alpha e}}\sqrt{P_{\alpha\mu}}\sqrt{P_{\alpha e}+P_{\alpha\mu}}\sqrt{P_{\alpha\tau}}\right)^{\frac{1}{3}}, \tag{28}$$

$$C = \sqrt{3 - 3(P_{\alpha e}^2 + P_{\alpha\mu}^2 + P_{\alpha\tau}^2) - 2P_{\alpha\mu}P_{\alpha\tau} - 2P_{\alpha e}(P_{\alpha\mu}+P_{\alpha\tau})}. \tag{29}$$

The total entropic uncertianty $\mathcal{U}$ can be also expressed in terms of transition probabilities, as:

$$\begin{aligned}\mathcal{U} =& 4[H_{bin}(\lambda_1) - 1] + P_{\alpha\mu}\log_2 P_{\alpha\mu} + P_{\alpha\tau}\log_2 P_{\alpha\tau} \\ & - 2P_{\alpha e}\log_2 P_{\alpha e} + 3\big[(P_{\alpha e}+P_{\alpha\mu})\log_2(P_{\alpha e}+P_{\alpha\mu}) \\ & + (P_{\alpha\tau}+P_{\alpha e})\log_2(P_{\alpha\tau}+P_{\alpha e})\big],\end{aligned} \tag{30}$$

where $H_{bin}(\lambda_1) = -\lambda_1 \log_e \lambda_1 - (1-\lambda_1)\log_2(1-\lambda_1)$ is the binary entropy, with
$\lambda_1 = \frac{1}{2}\big(1 - \sqrt{(P_{\alpha e}+P_{\alpha\mu})^2 + 2(P_{\alpha e}-P_{\alpha\mu})P_{\alpha\tau} + P_{\alpha\tau}^2}\big)$.

It has been seen that for an initial electron neutrino, the concurrence is able to capture more quantumness compared with the other two measures, and the negativity is smaller than concurrence and EOF. Instead, for an initial muon neutrino, the amount of concurrence and EOF is greater than negativity. Furthermore, in accordance to [16, 23] it has been found an anti-correlation between uncertainty and entanglement.

In [24] various measures of bipartite and tripartite entanglement are explored. In particular, it is analyzed the genuine tripartite entanglement. In bipartite quantum system, all quantum correlations like tangle, concurrence, negativity coincides with the linear entropy. It reveals that the state $|\nu_e(t)\rangle$ is a bipartite entangled pure state. In the three flavor case, the neutrino oscillation satisfies the Coffman-Kundu-Wooters (CKW) monogamy inequality and exhibits the property of the class of W-states. Consequently, the residual entanglement inequalities $\pi_{e\mu\tau} > 0$ or $\pi_{\mu e\tau} > 0$ imply a generalized form of genuine tripartite entanglement in three flavor neutrino oscillations.

In [25] the authors investigated several trade-off relations in Quantum Resource Theory, based on Bell-CHSH violations, first-order coherence and intrinsic concurrence , and the relative entropy of coherence, for initial electron neutrino and muon neutrino oscillations. It is shown that the sum of the violation of CHSH tests on bipartite states $\langle CHSH\rangle^2_{\rho_{AB}}$, $\langle CHSH\rangle^2_{\rho_{BC}}$ and $\langle CHSH\rangle^2_{\rho_{AC}}$ is always ≤ 12 for the electron and the muon neutrinos. Therefore, it is impossible that all the three pairs of pairwise neutrino flavor systems violate the CHSH inequality simultaneously, and if one of the three pairs reaches the maximal violation of the CHSH inequality, the other two pairs cannot violate the CHSH inequality anymore. Thus, these relations give rise to strong restrictions on the distribution of nonlocality among the three reduced two-flavor neutrino systems.

In the next section, in the framework of neutrino oscillations we consider relations which share some conceptual similarities with the trade-off relations. In fact, these complete complementarity relations (CCR) are identities which, in correspondence to the variation of parameters, account for the balancing among fundamental characteristics of the system: quantum correlations, predictability and visibility. In the case of three flavors, CCR show how much the level of correlations in a two flavor subsystem constraints the amount of quantum correlations in the other flavor subsystems.

3. CCR in neutrino oscillations

CCR provide a way to characterize quantum correlations in multi-partite systems and they can be exploited to describe quantum correlations in neutrino systems. The concept of complementarity is summarized in the statement that a quantum system may possess properties which are equally real but mutually exclusive. It is often associated with wave-particle duality, the complementarity aspect between propagation and detection.

The first quantitative version of the wave-particle duality [41, 42] was summarized by a simple complementarity relation:

$$P^2 + V^2 \leq 1. \tag{31}$$

where P is the predictability, a measure of path information and V is the visibility of the interference pattern. Complementarity associated with wave-particle duality is thus related to competing properties of a quantum systems. Complementarity relations as in Eq.(31) are valid only for pure single-partite quantum states.

In [43] it is shown that for bipartite states we have to consider a *triality relation* formed by two quantities generating local, single-partite realities which can be related to wave-particle duality and a third entry representing the entanglement measure *concurrence*, which generates an exclusive bipartite non-local reality:

$$P_k^2 \; + \; V_k^2 \; + \; C^2 \; = \; 1, \qquad k = 1, 2 \tag{32}$$

where P_k and V_k are the predictability and visibility for the single-partite systems and C is the concurrence.

CCR can be efficiently expressed in terms of density matrix elements [44] . Let us consider a bipartite pure state in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, represented by the density matrix:

$$\rho_{A,B} = \sum_{i,k=0}^{d_A-1} \sum_{j,l=0}^{d_B-1} \rho_{ij,kl} \left| i, j \right\rangle \left\langle k, l \right| . \tag{33}$$

If the state of subsystem A is mixed, one has

$$P_{hs}(\rho_A) + C_{hs}(\rho_A) < \frac{d_A - 1}{d_A} \tag{34}$$

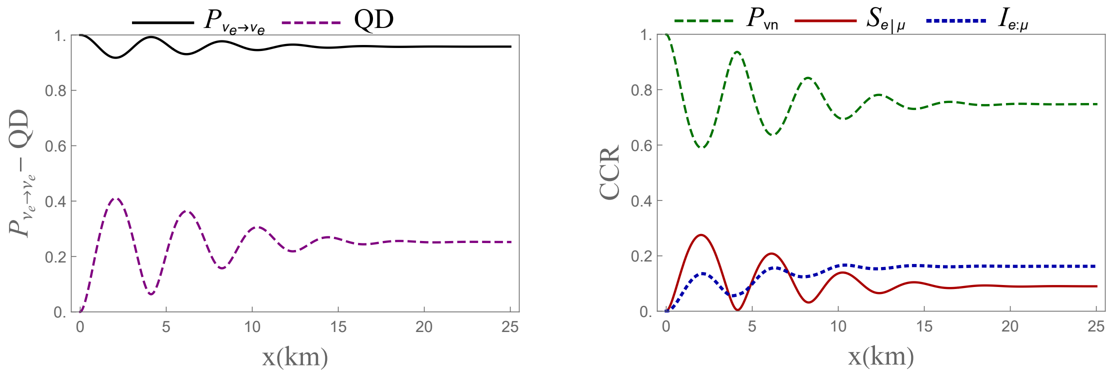
where $P_{hs}(\rho_A) \equiv \sum_{i=0}^{d_A-1} (\rho_{ii}^A)^2 - \frac{1}{d_A}$ and $C_{hs}(\rho_A) \equiv \sum_{i\neq k}^{d_A-1} |\rho_{ik}^A|^2$ are, respectively, the predictability measure and the Hilbert-Schmidt quantum coherence (a good generalization of the visibility measure [45]).

CCR is obtained by including the missing information about subsystem A, which is contained in the correlations with the subsystem B:

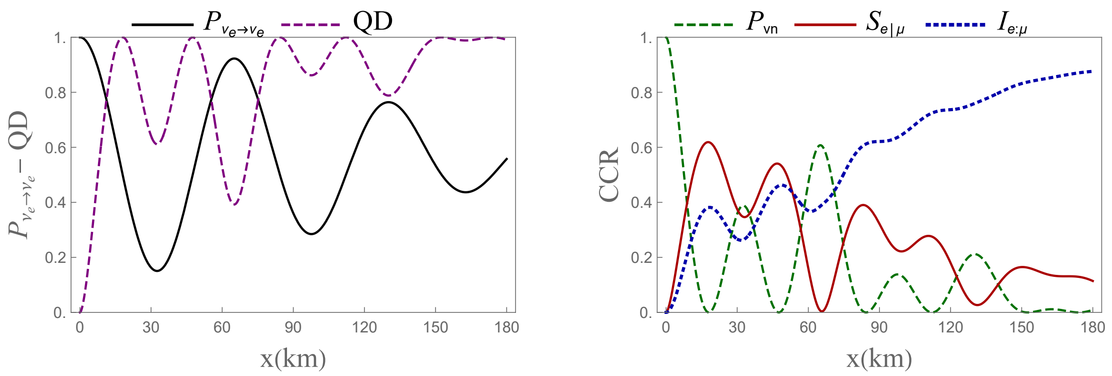
$$P_{hs}(\rho_A) + C_{hs}(\rho_A) + C_{hs}^{nl}(\rho_{A|B}) = \frac{d_A - 1}{d_A} \tag{35}$$

where $C_{hs}^{nl}(\rho_{A|B}) = \sum_{i\neq k, j\neq l} |\rho_{ij,kl}|^2 - 2 \sum_{i\neq k, j<l} \mathrm{Re}(\rho_{ij,kj}\rho_{il,kl}^*)$ is the non-local quantum coherence (entanglement), that, for a global pure state, is equivalent to the linear entropy of subsystem A. Another form of CCR can be obtained by defining the predictability and the coherence measures in terms of the von Neumann entropy:

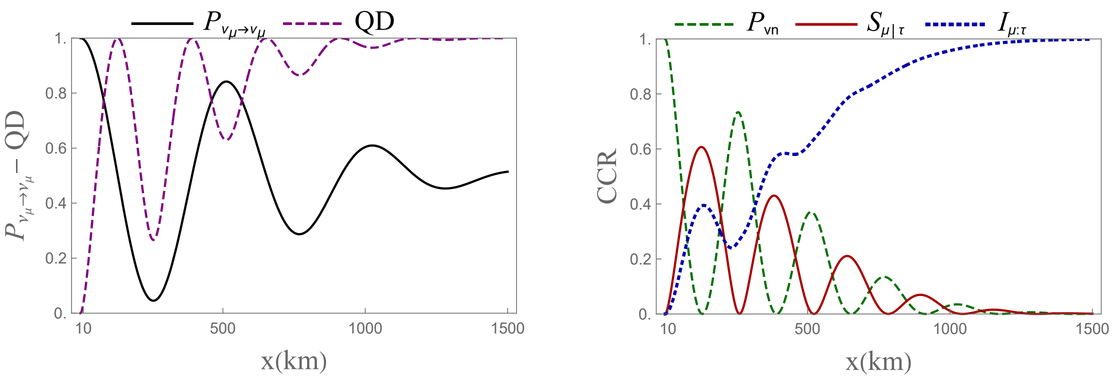
$$C_{\mathrm{re}}(\rho_A) + P_{\mathrm{vn}}(\rho_A) + S_{\mathrm{vn}}(\rho_A) = \log_2 d_A \tag{36}$$



(a) *DAYA BAY*



(b) *KamLAND*



(c) *MINOS*

Figure 1: On the left panels are shown the survival probability $P_{\nu_\alpha\to\nu_\alpha}$, $(\alpha = e, \mu)$, and the quantum discord QD as a function of the propagation distance x(km) for a neutrino state (44). On the right panels are shown the predictability $P_{vn}(\rho_\alpha)$, the conditional entropy $S_{\alpha|\beta}(\rho_{\alpha\beta})$ and the mutual information $I_{A:B}(\rho_{\alpha\beta})$, $(\alpha, \beta = e, \mu, \tau)$, as a function of distance x(km).

where $C_{\rm re}(\rho_A) = S_{\rm vn}(\rho_{A,\,\rm diag}) - S_{\rm vn}(\rho_A)$, $P_{\rm vn}(\rho_A) \equiv \log_2 d_A - S_{\rm vn}(\rho_{A,\,\rm diag})$. For pure states $S_{\rm vn}(\rho_A)$ is a measure of entanglement between A and B.

For bipartite mixed states, the CCR have to be suitably modified [46]. Indeed, $S_{\rm vn}(\rho_A)$ cannot be considered as a measure of entanglement, but only as a quantifier of the mixedness of A. The CCR for mixed states is

$$\log_2 d_A = I_{A:B}(\rho_{AB}) + S_{A|B}(\rho_{AB}) + P_{vn}(\rho_A) + C_{re}(\rho_A), \tag{37}$$

where: $C_{re}(\rho_A) = S_{vn}(\rho_{A\,diag}) - S_{vn}(\rho_A)$ is relative entropy of coherence, $P_{vn}(\rho_A) \equiv \ln d_A - S_{vn}(\rho_{A\,diag})$ is the predictability measure, $I_{A:B}(\rho_{AB}) = S_{vn}(\rho_A) + S_{vn}(\rho_B) - S_{vn}(\rho_{AB})$ is the mutual information of A and B and $S_{A|B}(\rho_{AB}) = S_{vn}(\rho_{AB}) - S_{vn}(\rho_B)$ indicates how much it is convenient knowing about the subsystem A with respect the whole system.

We now turn our attention to the case of a neutrino state [21]. For an initial electronic neutrino:

$$|\nu_e(t)\rangle = a_{ee}(t)\,|10\rangle + a_{e\mu}(t)\,|01\rangle \tag{38}$$

where we used the following correspondence [6]:

$$|\nu_e\rangle = |1\rangle_e \otimes |0\rangle_\mu = |10\rangle\,, \qquad |\nu_\mu\rangle = |0\rangle_e \otimes |1\rangle_\mu = |01\rangle\,, \tag{39}$$

that highlights the composite nature of neutrino flavor states. Starting from the density matrix of the system $\rho_{e\mu}$, it is simple obtaing the reduced density matrix of subsystem e (μ) by tracing over μ (e). We find that $P_{\rm hs}(\rho_e) = P_{ee}^2 + P_{e\mu}^2 - \frac{1}{2}$, $C_{\rm hs}(\rho_e) = 0$ and $C_{\rm hs}^{nl}(\rho_{e\mu}) = 2P_{ee}P_{e\mu}$, where we use $|a_{ee}(t)|^2 = P_{ee}$, $|a_{e\mu}(t)|^2 = P_{e\mu}$ and $P_{ee} + P_{e\mu} = 1$. Eq.(35) is verified. Furthermore, considering Eq.(36) is simple to see that $C_{\rm re}(\rho_e) = 0$, $P_{\rm vn}(\rho_e) = 1 + |a_{ee}|^2 \log_2 |a_{ee}|^2 + |a_{e\mu}|^2 \log_2 |a_{e\mu}|^2$ and $S_{\rm vn}(\rho_e) = -|a_{ee}|^2 \log_2 |a_{ee}|^2 - |a_{e\mu}|^2 \log_2 |a_{e\mu}|^2$. Since the dimension of subsystem e is $d_e = 2$ then $\log_2 d_e = 1$.

The above results are valid in the plane-wave approximation. In a more realistic wave-packet approach, one starts with a pure state $\rho_\alpha(x,t)$ ($\alpha = e, \mu$) which becomes mixed after time integration [6]:

$$\rho_\alpha(x) = \sum_{k,j} U_{\alpha k} U^*_{\alpha j} f_{jk}(x) \left|\nu_j\right\rangle \left\langle \nu_k\right|\,, \tag{40}$$

where $f_{jk}(x) = \exp\left[-i\frac{\Delta m^2_{jk} x}{2E} - \left(\frac{\Delta m^2_{jk} x}{4\sqrt{2}E^2\sigma_x}\right)^2\right]$. It is possible to express $\rho_\alpha(x)$ in terms of flavor eigenstates by establishing the identification $|\nu_\alpha\rangle = |\delta_{\alpha e}\rangle_e \left|\delta_{\alpha\mu}\right\rangle_\mu$. By using the relation $|\nu_i\rangle = \sum_\alpha U_{\alpha i}\,|\nu_\alpha\rangle$, we can write:

$$\rho_\alpha(x) = \sum_{\beta\gamma} F^\alpha_{\beta\gamma}(x) \left|\delta_{\beta e}\delta_{\beta\mu}\right\rangle \left\langle \delta_{\gamma e}\delta_{\gamma\mu}\right| \tag{41}$$

where

$$F^\alpha_{\beta\gamma}(x) = \sum_{kj} U^*_{\alpha j} U_{\alpha k} f_{jk}(x) U_{\beta j} U^*_{\gamma k} \tag{42}$$

The density matrix for an initial electron neutrino is:

$$\rho_{e\mu}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & F^e_{ee}(x) & F^e_{e\mu}(x) & 0 \\ 0 & F^e_{\mu e}(x) & F^e_{\mu\mu}(x) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{43}$$

By evaluating the terms of Eq.(37): $P_{vn}(\rho_e) = 1 + F^e_{ee}\log_2 F^e_{ee} + F^e_{\mu\mu}\log_2 F^e_{\mu\mu}$, $C_{re}(\rho_e) = 0$, $I_{e:\mu}(\rho_{e\mu}) + S_{e|\mu}(\rho_{e\mu}) = -F^e_{ee}\log_2 F^e_{ee} - F^e_{\mu\mu}\log_2 F^e_{\mu\mu}$, we find that the CCR for mixed states is satisfied [47].

We find that the sum of the first two terms of Eq.(37) is equal to the Quantum Discord, a measure of nonclassical correlations between two subsystems of a quantum system, defined as [48]:

$$QD(\rho_{AB}) = I(\rho_{AB}) - CC(\rho_{AB}) = S_{\text{vn}}(\rho_A) - S_{\text{vn}}(\rho_{AB}) + \min_{\{\Pi^b_i\}} S_{\text{vn},\{\Pi^b_i\}}(\rho_{A|B}) \tag{44}$$

where $I(\rho_{AB})$ is the total correlations between the subsystems A and B and $CC(\rho_{AB})$ quantifies the classical correlations. For the density matrix under consideration, we obtain:

$$QD(\rho_{e\mu}) = -F^e_{ee}\log_2 F^e_{ee} - F^e_{\mu\mu}\log_2 F^e_{\mu\mu}. \tag{45}$$

In [47] it is also shown the connection existing with the Non-local Advantage of Quantum Coherence (NAQC), a quantum correlation which occurs in a bipartite system when the average coherence of the conditional state of a subsystem B, after a local measurements on A, exceeds the coherence limit of the single subsystem. In the hierarchy of quantum correlations, NAQC has been classified as the strongest one, overtaking also the Bell non-locality. Mondal et al. [27] defined the NAQC of a bipartite state ρ_{AB} considering the average coherence of the post measurement state $\{p_{B|\Pi^a_i}, \rho_{B|\Pi^a_i}\}$ of B after a local measurement Π^a_i on A:

$$N(\rho_{AB}) = \frac{1}{2}\sum_{i\neq j, a=\pm} p_{B|\Pi^a_i} C^{\sigma_j}(\rho_{B|\Pi^a_i}), \tag{46}$$

where $\Pi^\pm_i = \frac{I\pm\sigma_i}{2}$, with I and $\sigma_i, (i = 1,2,3)$ being the identity and the three Pauli operators; $p_{B|\Pi^a_i} = \mathrm{Tr}(\Pi^a_i \rho_{AB})$, $\rho_{B|\Pi^a_i} = \mathrm{Tr}_A(\Pi^a_i\rho_{AB})/p_{B|\Pi^a_i}$. $C^{\sigma_j}(\rho_{B|\Pi^a_i})$ is the coherence of the conditional state of B with respect to the eigenbasis of σ_j.

For the neutrino state, by using the relative entropy as coherence measure, we find:

$$N(\rho_{e\mu}) = 2 - F^e_{ee}\log_2 F^e_{ee} - F^e_{\mu\mu}\log_2 F^e_{\mu\mu}, \tag{47}$$

and it is immediate to find the relation $N(\rho_{e\mu}) = 2 + I_{e:\mu}(\rho_{e\mu}) + S_{e|\mu}(\rho_{e\mu})$.

In Fig.1, the predictability, the conditional entropy and the mutual information are plotted for the Daya Bay, Kamland and MINOS parameters (see Table), along with the survival probability and the quantum discord.

Daya-Bay	KamLAND	MINOS
$\Delta m^2_{ee} = 2.42\times 10^{-3} eV^2$	$\Delta m^2_{12} = 7.49\times 10^{-5} eV^2$	$\Delta m^2_{32} = 2.32\times 10^{-3} eV^2$
$\sin^2 2\theta_{13} = 0.084$	$\tan^2 2\theta_{12} = 0.47$	$\sin^2 2\theta_{23} = 0.95$
$L \in [364, 1912] \quad m$	$L = 180 \quad Km$	$L = 735 \quad km$
$E = 4 \quad MeV$	$E = 2 \quad MeV$	$E = 0.5 \quad GeV$

The different values of the mixing angle associated to the three experiments lead to very different behaviors, especially in the asymptotic range. In the KamLand and Minos experiments, associated to higher values of the mixing angle, the mutual information grows almost monotonically keeping a high value even after oscillations are washed out. Due to the low value of the mixing angle, this aspect is not present for the Daya Bay parameters. We stress that, for KamLand and Minos experiments, it is very difficult to recognize in the mutual information a behaviour exclusively dependent on the oscillation probability.

4. Conclusions

We reviewed the main results about quantification of quantum correlations in neutrino oscillations. In literature several quantifiers have been considered, which can be organized in a hierarchical (decreasing) order: NAQC $\subset$ Bell non-locality $\subset$ steering $\subset$ entanglement $\subset$ general quantum correlations (discord).

In the framework of neutrino oscillations, these investigations have been made both for the plane wave approximation and in the wave packet approach, and usually the above quantifiers can be expressed in terms of oscillation probabilities. In the last section, we considered CCR which provide an exhaustive description of the quantumness in a generic quantum system.

In particular, we exploited CCR both in the plane wave approximation and in the wave packet approach to describe quantum correlations in neutrino oscillations. When a wave packet approach is exploited, the neutrino state becomes mixed. In this case, the NAQC can be expressed in terms of Quantum Discord, which describes non local terms in CCR. We have analyzed the behaviour of the CCR terms in connection with three neutrino experiments: Daya-Bay, KamLAND and MINOS. We found that different values of the mixing angle associated to the three experiments lead to very different behaviors, especially in the asymptotic range, where one can observe a persistence of quantum correlations. Finally we provided some preliminary results about the application of CCR in the case of three flavor neutrino oscillations.

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