PROCEEDINGS OF SCIENCE

Reduction of couplings and Finite Unified Theories

S. Heinemeyer,[a] J. Kalinowski,[b] W. Kotlarski,[c] M. Mondragon,[d,*] G. Patellis,[e] N. Tracas[f] and G. Zoupanos[f]

[a] *Instituto de Física Teórica (UAM/CSIC), Universidad Autónoma de Madrid Cantoblanco, 28049, Madrid, Spain*
[b] *Faculty of Physics, University of Warsaw Pasteura 5, 02-093 Warsaw, Poland*
[c] *National Centre for Nuclear Research Pasteura 7, 02-093 Warsaw, Poland*
[d] *Instituto de Física, Universidad Nacional Autónoma de México Apdo. Postal 20-364, CP 01000 CDMX, México*
[e] *CFTP, Departamento de Física, Instituto Superior Técnico Universidade de Lisboa Avenida Rovisco Pais 1, 1049-001, Lisboa, Portugal*
[f] *Physics Department, National Technical University 157 80 Zografou, Athens, Greece*
E-mail: Sven.Heinemeyer@cern.ch, kalino@fuw.edu.pl, wojciech.kotlarski@ncbj.gov.pl, myriam@fisica.unam.mx, patellis@central.ntua.gr, ntrac@central.ntua.gr, George.Zoupanos@cern.ch

We review the basic idea of the *reduction of couplings* method, both in the dimensionless and dimension 1 and 2 sectors. Then, we show the application of the method to $N = 1$ supersymmetric GUTs, and in particular to the construction of finite theories. We present the results for two phenomenologically viable finite models, an all-loop finite $SU(5)$ SUSY GUT, and a two-loop finite $SU(3)^3$ one. For each model we select three representative benchmark scenarios. In both models, the supersymmetric spectrum lies beyond the reach of the 14 TeV HL-LHC. For the $SU(5)$ model, the lower parts of the parameter space will be in reach of the FCC-hh, although the heavier part will be unobservable. For the two-loop finite $SU(3)^3$ model, larger parts of the spectrum would be accessible at the FCC-hh, although the highest possible masses would escape the searches.

Corfu Summer Institute 2022 "School and Workshops on Elementary Particle Physics and Gravity", 28 August - 1 October, 2022
Corfu, Greece

*Speaker

PoS(CORFU2022)055

1. Introduction

Although the Standard Model (SM) has proven to be extremely successful in describing the fundamental interactions, we know that it is most likely the low energy limit of a more fundamental theory. The many unanswered questions in the SM, coupled to the large number of free parameters, point in this direction. In the search for theories beyond the SM, the more common way to extend it is to add symmetries or fields, which usually imply more interactions and/or more particles, and more often than not leads to more, and not less, parameters than in the SM.

The *reduction of couplings* method [1–4] (see also [5–7]) relates seemingly independent parameters to a single, "primary" coupling, in the context of perturbative field theory. The method requires the original theory to which it is applied to be a renormalizable one, and the resulting relation among the parameters to be valid at all energy scales, i.e. Renormalization Group Invariant (RGI). It provides a complimentary approach to the addition of symmetries in the quest for a more fundamental theory. In order to be able to relate the "reduced" couplings to the primary one, they need to have the same ultraviolet behaviour. Thus, it is natural to attempt a reduction of couplings in the context of a supersymmetric Grand Unified Theory, therefore relating the gauge and Yukawa sector of the SUSY GUT (Gauge Yukawa Unification, GYU) [8–21]. In this approach, being in a GUT environment, RGI relations are set between the unification scale and the Planck one.

An additional advantage of this method is that one-loop uniqueness can guarantee the all-loop validity of these relations. Moreover, RGI relations can be found which guarantee all order finiteness of a theory. The method has predicted the top quark mass in the finite $N = 1$, $SU(5)$ model [8, 9] as well as in the minimal $N = 1$, $SU(5)$ one [10] before its experimental measurement [22].

Supersymmetry (SUSY) seems to be an essential ingredient for a successful *reduction of couplings*, thus we have to include a supersymmetry breaking sector (SSB), which involves dimension-1 and -2 couplings. The supergraph method and the spurion superfield technique played an important role for the progress in that sector, leading to complete all-loop finite models, i.e. including the SSB sector. The all-loop finite $N = 1$, $SU(5)$ model [25], and the two-loop finite $SU(3)^3$ [23, 24] models have given a prediction for the Higgs mass compatible with the experimental results [26–28] and a heavy SUSY mass spectrum, consistent with the experimental non-observation of these particles. The reduction of couplings method has been applied to several other cases. The full analysis of the most successful models, that includes predictions in agreement with the experimental measurements of the top and bottom quark masses for each model, can be found in a recent work [29, 30].

2. Theoretical Basis of Reduction of Couplings

The idea of reduction of couplings was introduced in [2] in order to explore the possibility to express seemingly unrelated parameters of a theory in terms of one basic parameter, denoted primary coupling. This is possible if there are RGI relations among couplings, which can be used to relate the seemingly free parameters. We will outline the method in this section, starting with the dimensionless parameters, and then extending it to dimension-1 and -2 parameters.

2.1 Reduction of Dimensionless Parameters

Any RGI relation among couplings $g_1, ..., g_A$ of a renormalizable theory can be written in the form $\Phi(g_1, \cdots, g_A)$ = const., which has to satisfy the partial differential equation

$$\mu \frac{d\Phi}{d\mu} = \vec{\nabla}\Phi \cdot \vec{\beta} = \sum_{a=1}^{A} \beta_a \frac{\partial \Phi}{\partial g_a} = 0 , \tag{1}$$

where β_a is the β-function of g_a. Solving this partial differential equation is equivalent to solving a set of ordinary differential equations, known as reduction equations (REs) [2–4],

$$\beta_g \frac{dg_a}{dg} = \beta_a , \; a = 1, \cdots, A , \tag{2}$$

where g and β_g are the primary coupling and its β-function, respectively, while the counting on a does not include g. The Φ_a's can impose a maximum of $(A-1)$ independent RGI "constraints" in the A-dimensional space of parameters, which could be expressed in terms of a single coupling g. Notice however, the general solutions of Eqs. (2) contain as many integration constants as the number of equations. This problem can be overcome by demanding power series solutions to the RE, which preserve perturbative renormalizability

$$g_a = \sum_n \rho_a^{(n)} \, g^{2n+1} , \tag{3}$$

This ansatz fixes the integration constant in each of the REs and chooses a special solution. The uniqueness of these power series solutions can be determined at one-loop level [2–4]. As an illustration, we assume β-functions of the form

$$\begin{aligned} \beta_a &= \frac{1}{16\pi^2}\left[\sum_{b,c,d\neq g} \beta_a^{(1)\,bcd} g_b g_c g_d + \sum_{b\neq g} \beta_a^{(1)\,b} g_b g^2\right] + \cdots , \\ \beta_g &= \frac{1}{16\pi^2} \beta_g^{(1)} g^3 + \cdots , \end{aligned} \tag{4}$$

where $\cdots$ stands for higher-order terms, and $\beta_a^{(1)\,bcd}$'s are symmetric in b, c, d. We will assume that the $\rho_a^{(n)}$'s with $n \leq r$ are uniquely determined. To obtain $\rho_a^{(r+1)}$'s we insert the power series (3) into the REs (2) and collect terms of $O(g^{2r+3})$:

$$\sum_{d\neq g} M(r)_a^d \, \rho_d^{(r+1)} = \text{lower order quantities} ,$$

where the right-hand side is known by assumption and

$$M(r)_a^d = 3 \sum_{b,c\neq g} \beta_a^{(1)\,bcd} \, \rho_b^{(1)} \, \rho_c^{(1)} + \beta_a^{(1)\,d} - (2r+1)\, \beta_g^{(1)} \, \delta_a^d , \tag{5}$$

$$0 = \sum_{b,c,d\neq g} \beta_a^{(1)\,bcd} \, \rho_b^{(1)} \, \rho_c^{(1)} \, \rho_d^{(1)} + \sum_{d\neq g} \beta_a^{(1)\,d} \, \rho_d^{(1)} - \beta_g^{(1)} \, \rho_a^{(1)} . \tag{6}$$

Therefore, the $\rho_a^{(n)}$'s for all $n > 1$ for a given set of $\rho_a^{(1)}$'s are uniquely determined if $\det M(n)_a^d \neq 0$ for all $n \geq 0$.

The couplings in SUSY theories have the same asymptotic behaviour, making them natural candidates in the search for unified reduced theories. Primary examples of the application of this method are finite Grand Unified Theories, as we will see later [8, 9, 19]. Having a complete reduction, i.e. all couplings determined in terms of only one, it is in general unrealistic, Thus, it is possible to impose fewer RGI constraints, achieving "partial reduction" [31, 32].

2.2 Reduction of Couplings in $N = 1$ SUSY Gauge Theories - Partial Reduction

Let us consider a chiral, $N = 1$ supersymmetric gauge theory with group G and gauge coupling g. The superpotential of the theory can be written:

$$W = \frac{1}{2} m_{ij}\, \phi_i\, \phi_j + \frac{1}{6} C_{ijk}\, \phi_i\, \phi_j\, \phi_k \,, \tag{7}$$

where m_{ij} and C_{ijk} are gauge invariant tensors and the chiral superfield ϕ_i belongs to the irreducible representation R_i of the gauge group. The renormalization constants associated with the superpotential, for preserved SUSY, are:

$$\phi_i^0 = \left(Z_i^j\right)^{(1/2)} \phi_j \,, \tag{8}$$

$$m_{ij}^0 = Z_{ij}^{i'j'}\, m_{i'j'} \,, \tag{9}$$

$$C_{ijk}^0 = Z_{ijk}^{i'j'k'}\, C_{i'j'k'} \,. \tag{10}$$

By virtue of the $N = 1$ non-renormalization theorem [33–36] there are no mass and cubic interaction term infinities, the only surviving infinities are the wave function renormalization constants Z_i^j, i.e. one infinity per field. The one-loop β-function of g is given by [37–41]

$$\beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2}\left[\sum_i T(R_i) - 3\, C_2(G)\right] , \tag{11}$$

where $C_2(G)$ is the quadratic Casimir operator of the adjoint representation of the gauge group G and $\mathrm{Tr}[T^a T^b] = T(R)\delta^{ab}$, where T^a are the group generators in the appropriate representation. Due to the non-renormalization theorem [33, 34, 36] the β-functions of C_{ijk} are related to the anomalous dimension matrices γ_j^i of the matter fields as:

$$\beta_{ijk} = \frac{dC_{ijk}}{dt} = C_{ijl}\,\gamma_k^l + C_{ikl}\,\gamma_j^l + C_{jkl}\,\gamma_i^l \,, \tag{12}$$

where the one-loop γ_j^i is given by [37]:

$$\gamma^{(1)}{}_j^i = \frac{1}{32\pi^2}\,[\, C^{ikl}\, C_{jkl} - 2\, g^2\, C_2(R_i)\delta_j^i \,], \tag{13}$$

and $C^{ijk} = C_{ijk}^*$.
We take C_{ijk} to be real so that C_{ijk}^2 are always positive. The squares of the couplings are convenient to work with, and the C_{ijk} can be covered by a single index i $(i = 1, \cdots, n)$:

$$\alpha = \frac{g^2}{4\pi} \,,\ \alpha_i = \frac{g_i^2}{4\pi} \,. \tag{14}$$

Then the evolution of α's in perturbation theory will take the form

$$\begin{aligned} \frac{d\alpha}{dt} = \beta &= -\beta^{(1)}\alpha^2 + \cdots , \\ \frac{d\alpha_i}{dt} = \beta_i &= -\beta_i^{(1)}\,\alpha_i\,\alpha + \sum_{j,k} \beta^{(1)}_{i,jk}\,\alpha_j\,\alpha_k + \cdots , \end{aligned} \tag{15}$$

Here, $\cdots$ denotes higher-order contributions and $\beta^{(1)}_{i,jk} = \beta^{(1)}_{i,kj}$. For the evolution equations (15), following ref [10] we investigate the asymptotic properties. First, we define [2, 4, 6, 42, 43]

$$\tilde{\alpha}_i \equiv \frac{\alpha_i}{\alpha} \ , \ i = 1, \cdots, n \ , \tag{16}$$

and derive from Eq. (15)

$$\begin{aligned} \alpha\frac{d\tilde{\alpha}_i}{d\alpha} = -\tilde{\alpha}_i + \frac{\beta_i}{\beta} &= \left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}}\right)\tilde{\alpha}_i \\ &- \sum_{j,k} \frac{\beta^{(1)}_{i,jk}}{\beta^{(1)}}\,\tilde{\alpha}_j\,\tilde{\alpha}_k + \sum_{r=2}\left(\frac{\alpha}{\pi}\right)^{r-1}\,\tilde{\beta}_i^{(r)}(\tilde{\alpha}) \ , \end{aligned} \tag{17}$$

where $\tilde{\beta}_i^{(r)}(\tilde{\alpha})$ $(r = 2, \cdots)$ are power series of $\tilde{\alpha}$'s and can be computed from the r^{th}-loop β-functions. We then search for fixed points ρ_i of Eq. (16) at $\alpha = 0$. We have to solve the equation

$$\left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}}\right)\rho_i - \sum_{j,k}\frac{\beta^{(1)}_{i,jk}}{\beta^{(1)}}\,\rho_j\,\rho_k = 0 \ , \tag{18}$$

assuming fixed points of the form

$$\rho_i = 0 \text{ for } i = 1, \cdots, n' \ ; \ \rho_i \ > 0 \text{ for } i = n' + 1, \cdots, n \ . \tag{19}$$

Next, we treat $\tilde{\alpha}_i$ with $i \leq n'$ as small perturbations to the undisturbed system (defined by setting $\tilde{\alpha}_i$ with $i \leq n'$ equal to zero). It is possible to verify the existence of the unique power series solution of the reduction equations (17) to all orders already at one-loop level [2–4, 42]:

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2}\rho_i^{(r)}\,\alpha^{r-1} \ , \ i = n' + 1, \cdots, n \ . \tag{20}$$

These are RGI relations among parameters, and preserve formally perturbative renormalizability. So, in the undisturbed system there is only one independent parameter, the primary coupling α.

The non-vanishing $\tilde{\alpha}_i$ with $i \leq n'$ cause small perturbations that enter in a way that the reduced couplings ($\tilde{\alpha}_i$ with $i > n'$) become functions both of α and $\tilde{\alpha}_i$ with $i \leq n'$. Investigating such systems with partial reduction is very convenient to work with the following PDEs:

$$\begin{aligned} \left\{\tilde{\beta}\,\frac{\partial}{\partial\alpha} + \sum_{a=1}^{n'}\tilde{\beta}_a\,\frac{\partial}{\partial\tilde{\alpha}_a}\right\}\,\tilde{\alpha}_i(\alpha,\tilde{\alpha}) = \tilde{\beta}_i(\alpha,\tilde{\alpha}) \ , \\ \tilde{\beta}_{i(a)} \ = \ \frac{\beta_{i(a)}}{\alpha^2} - \frac{\beta}{\alpha^2}\,\tilde{\alpha}_{i(a)}, \qquad \tilde{\beta} \ \equiv \ \frac{\beta}{\alpha} \ . \end{aligned} \tag{21}$$

These equations are equivalent to the REs (17), where, in order to avoid any confusion, we let a, b run from 1 to n' and i, j from $n' + 1$ to n. Then, we search for solutions of the form

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2} \left(\frac{\alpha}{\pi}\right)^{r-1} f_i^{(r)}(\tilde{\alpha}_a) \ , \ i = n' + 1, \cdots, n \ , \tag{22}$$

where $f_i^{(r)}(\tilde{\alpha}_a)$ are power series of $\tilde{\alpha}_a$. The requirement that in the limit of vanishing perturbations we obtain the undisturbed solutions (20) [32, 44] suggests this type of solutions. Once more, one can obtain the conditions for uniqueness of $f_i^{(r)}$ in terms of the lowest order coefficients.

2.3 Reduction of Dimension-1 and -2 Parameters

The extension of the reduction of couplings method to massive parameters is not straightforward, since the technique was originally aimed at massless theories on the basis of the Callan-Symanzik equation [2, 3]. Many requirements have to be met, such as the normalization conditions imposed on irreducible Green's functions [45], etc. Significant progress has been made towards this goal, starting from [46], where, as an assumption, a mass-independent renormalization scheme renders all RG functions only trivially dependent on dimensional parameters. Mass parameters can then be introduced similarly to couplings. This was justified later [47, 48], where it was demonstrated that, apart from dimensionless parameters, pole masses and gauge couplings, the model can also include couplings carrying a dimension and masses.

The reduction of dimensionless couplings was extended [46, 49] to the SSB dimensionful parameters of $N = 1$ supersymmetric theories. It was also found [19, 50] that soft scalar masses satisfy a universal sum rule. Consider the superpotential (7)

$$W = \frac{1}{2}\,\mu^{ij}\,\Phi_i\,\Phi_j + \frac{1}{6}\,C^{ijk}\,\Phi_i\,\Phi_j\,\Phi_k \ , \tag{23}$$

and the SSB Lagrangian

$$-\mathcal{L}_{\rm SSB} = \frac{1}{6}\,h^{ijk}\,\phi_i\phi_j\phi_k + \frac{1}{2}\,b^{ij}\,\phi_i\phi_j + \frac{1}{2}\,(m^2)^j_i\,\phi^{*\,i}\phi_j + \frac{1}{2}\,M\,\lambda_i\lambda_i + \text{h.c.} \tag{24}$$

The ϕ_i's are the scalar parts of chiral superfields Φ_i, λ are gauginos and M the unified gaugino mass.

The one-loop gauge and Yukawa beta-functions are given by (11) and (12), respectively, and the one-loop anomalous dimensions by (13). We make the assumption that the REs admit power series solutions:

$$C^{ijk} = g \sum_{n=0} \rho^{ijk}_{(n)} g^{2n} \ . \tag{25}$$

Since we want to obtain higher-loop results instead of knowledge of explicit β-functions, we require relations among β-functions. The spurion technique [36, 51–54] gives all-loop relations among

SSB β-functions [55–62]:

$$\beta_M = 2\mathcal{O}\left(\frac{\beta_g}{g}\right) , \tag{26}$$

$$\begin{aligned}\beta_h^{ijk} &= \gamma^i_l h^{ljk} + \gamma^j_l h^{ilk} + \gamma^k_l h^{ijl} \\ &\quad - 2\,(\gamma_1)^i_l\, C^{ljk} - 2\,(\gamma_1)^j_l\, C^{ilk} - 2\,(\gamma_1)^k_l\, C^{ijl} ,\end{aligned} \tag{27}$$

$$(\beta_{m^2})^i_j = \left[\Delta + X\frac{\partial}{\partial g}\right]\gamma^i_j , \tag{28}$$

where

$$\mathcal{O} = \left(Mg^2\frac{\partial}{\partial g^2} - h^{lmn}\frac{\partial}{\partial C^{lmn}}\right) , \tag{29}$$

$$\Delta = 2\mathcal{O}\mathcal{O}^* + 2|M|^2 g^2\frac{\partial}{\partial g^2} + \tilde{C}_{lmn}\frac{\partial}{\partial C_{lmn}} + \tilde{C}^{lmn}\frac{\partial}{\partial C^{lmn}} , \tag{30}$$

$$(\gamma_1)^i_j = \mathcal{O}\gamma^i_j, \tag{31}$$

$$\tilde{C}^{ijk} = (m^2)^i_l C^{ljk} + (m^2)^j_l C^{ilk} + (m^2)^k_l C^{ijl} . \tag{32}$$

Assuming (following [57]) that the relation among couplings

$$h^{ijk} = -M(C^{ijk})' \equiv -M\frac{dC^{ijk}(g)}{d\ln g} , \tag{33}$$

is RGI to all orders and the use of the all-loop gauge β-function of [63–65]

$$\beta_g^{\rm NSVZ} = \frac{g^3}{16\pi^2}\left[\frac{\sum_l T(R_l)(1-\gamma_l/2) - 3C_2(G)}{1-g^2C_2(G)/8\pi^2}\right] , \tag{34}$$

we are led to an all-loop RGI sum rule [66] (assuming $(m^2)^i_j = m^2_j\delta^i_j$),

$$\begin{aligned} m_i^2 + m_j^2 + m_k^2 = |M|^2 &\left\{\frac{1}{1-g^2C_2(G)/(8\pi^2)}\frac{d\ln C^{ijk}}{d\ln g} + \frac{1}{2}\frac{d^2\ln C^{ijk}}{d(\ln g)^2}\right\} \\ &+ \sum_l \frac{m_l^2 T(R_l)}{C_2(G) - 8\pi^2/g^2}\frac{d\ln C^{ijk}}{d\ln g} . \end{aligned} \tag{35}$$

It is worth noting that the all-loop result of Eq. (35) coincides with the superstring result for the finite case in a certain class of orbifold models [19, 67, 68] if $\frac{d\ln C^{ijk}}{d\ln g} = 1$ [9].

As mentioned above, the all-loop results on the SSB β-functions, Eqs.(26)-(32), lead to all-loop RGI relations. We assume:
(a) the existence of an RGI surface on which $C = C(g)$, or equivalently that the expression

$$\frac{dC^{ijk}}{dg} = \frac{\beta_C^{ijk}}{\beta_g} \tag{36}$$

holds (i.e. reduction of couplings is possible)
(b) the existence of a RGI surface on which

$$h^{ijk} = -M\frac{dC(g)^{ijk}}{d\ln g} \tag{37}$$

holds to all orders.

Then it can be proven [69–71] that the relations that follow are all-loop RGI (note that in both assumptions we do not rely on specific solutions of these equations)

$$M = M_0 \, \frac{\beta_g}{g}, \tag{38}$$

$$h^{ijk} = -M_0 \, \beta_C^{ijk}, \tag{39}$$

$$b^{ij} = -M_0 \, \beta_\mu^{ij}, \tag{40}$$

$$(m^2)^i_j = \frac{1}{2} \, |M_0|^2 \, \mu \frac{d\gamma^i{}_j}{d\mu}, \tag{41}$$

where M_0 is an arbitrary reference mass scale to be specified shortly. Assuming

$$C_a \frac{\partial}{\partial C_a} = C_a^* \frac{\partial}{\partial C_a^*} \tag{42}$$

for an RGI surface $F(g, C^{ijk}, C^{*ijk})$ we are led to

$$\frac{d}{dg} = \left(\frac{\partial}{\partial g} + 2 \frac{\partial}{\partial C} \, \frac{dC}{dg} \right) = \left(\frac{\partial}{\partial g} + 2 \frac{\beta_C}{\beta_g} \, \frac{\partial}{\partial C} \right) , \tag{43}$$

where Eq. (36) was used. Let us now consider the partial differential operator O in Eq. (29) which (assuming Eq. (33)), becomes

$$O = \frac{1}{2} M \frac{d}{d \ln g} \tag{44}$$

and β_M, given in Eq. (26), becomes

$$\beta_M = M \frac{d}{d \ln g} (\frac{\beta_g}{g}) , \tag{45}$$

which by integration provides us [62, 69] with the generalized, i.e. including Yukawa couplings, all-loop RGI Hisano - Shifman relation [58]

$$M = \frac{\beta_g}{g} M_0 \, . \tag{46}$$

M_0 is the integration constant and can be associated to the unified gaugino mass M (of an assumed covering GUT), or to the gravitino mass $m_{3/2}$ in a supergravity framework. Therefore, Eq. (38) becomes the all-loop RGI Eq. (46). β_M, using Eqs.(45) and (38) can be written as follows:

$$\beta_M = M_0 \frac{d}{dt} (\beta_g / g) \, . \tag{47}$$

Similarly

$$(\gamma_1)^i_j = O \gamma^i_j = \frac{1}{2} \, M_0 \, \frac{d\gamma^i_j}{dt} \, . \tag{48}$$

Next, from Eq.(33) and Eq.(38) we get

$$h^{ijk} = -M_0 \, \beta_C^{ijk} \, , \tag{49}$$

while β_h^{ijk}, using Eq.(48), becomes [69]

$$\beta_h^{ijk} = -M_0 \frac{d}{dt}\beta_C^{ijk}, \tag{50}$$

which shows that Eq. (49) is RGI to all loops. Eq. (40) can similarly be shown to be all-loop RGI as well.

It should be noted concerning the β-functions of the SBB parameters, as in Eqs. (47) and (50), that the vanishing of the dimensionless β-functions, even to all-orders, as will be discussed in the next section, is transferred to the dimensionful SSB sector of the theory.

3. Finiteness

An interesting consequence of the reduction of couplings programme, is that it led to the search of renormalizable field theories that are free of all logarithmic divergences, i.e. completely *Finite Theories*.

In SUSY there were already examples of completely finite theories, namely $N = 4$ supersymmetric unified gauge theories, since any ultraviolet (UV) divergences are absent in these theories. However, so far there are no phenomenologically viable models in the framework of $N = 4$ SUSY. The next possibility is to consider an $N = 2$ supersymmetric gauge theory, whose β-function receives corrections only at one loop. It is possible to select a spectrum to make the theory all-loop finite. A serious obstacle in order to make these kind of models phenomenologically viable is their mirror spectrum, which would need a particular mechanism to make it heavy. Therefore, one is naturally led to consider $N = 1$ supersymmetric gauge theories, which can be chiral and in principle realistic.

It should be noted that in the approach followed here (UV) finiteness means the vanishing of all the β-functions, i.e. the non-renormalization of the coupling constants, in contrast to a complete (UV) finiteness where even field amplitude renormalization is absent. Before the work of several members of our group, the studies on $N = 1$ finite theories were following two directions: (i) construction of finite theories up to two loops examining various possibilities to make them phenomenologically viable, (ii) construction of all-loop finite models without particular emphasis on the phenomenological consequences. The success of the work of our group started in refs [8, 9] with the construction of the first realistic all-loop finite model, based on the theorem presented below, realising in this way an old theoretical dream of field theorists.

Finiteness in N=1 Supersymmetric Gauge Theories

Let us, once more, consider a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g. The superpotential of the theory is given by (see Eq. (7))

$$W = \frac{1}{2} m_{ij}\, \phi_i\, \phi_j + \frac{1}{6} C_{ijk}\, \phi_i\, \phi_j\, \phi_k\ . \tag{51}$$

The $N = 1$ non-renormalization theorem, ensures the absence of mass and cubic-interaction-term infinities, and leads to wave-function infinities only; one for each superfield. As one can see from Eqs. (11) and (13), all the one-loop β-functions of the theory vanish if $\beta_g^{(1)}$ and $\gamma^{(1)i}_{\ j}$ vanish, i.e.

$$\sum_i T(R_i) = 3C_2(G)\,, \tag{52}$$

$$C^{ikl}C_{jkl} = 2\delta^i_j g^2 C_2(R_i)\,. \tag{53}$$

The conditions for finiteness for $N = 1$ field theories with $SU(N)$ gauge symmetry are discussed in [72], and the analysis of the anomaly-free and no-charge renormalization requirements for these theories can be found in [73]. A very interesting result is that the conditions (52) and (53) are necessary and sufficient for finiteness at the two-loop level [37–41].

In case SUSY is broken by soft terms, the requirement of finiteness in the one-loop soft breaking terms imposes further constraints among them [74]. In addition, the same set of conditions that are sufficient for one-loop finiteness of the soft breaking terms render the soft sector of the theory two-loop finite [75].

The one- and two-loop finiteness conditions of Eqs. (52) and (53) restrict considerably the possible choices of the irreducible representations (irreps) R_i for a given group G, as well as the Yukawa couplings in the superpotential (51). Note in particular that the finiteness conditions cannot be applied to the MSSM, since the presence of a $U(1)$ gauge group is incompatible with the condition (52), due to $C_2[U(1)] = 0$. This naturally leads to the expectation that finiteness should be attained at the grand unified level only, the MSSM being just the corresponding, low-energy, effective theory.

Another important consequence of one- and two-loop finiteness is that SUSY (most probably) can only be broken due to the soft breaking terms. Indeed, due to the unacceptability of gauge singlets, F-type spontaneous symmetry breaking [76] terms are incompatible with finiteness, as well as D-type [77] spontaneous breaking which requires the existence of a $U(1)$ gauge group.

A natural question to ask is what happens at higher loop orders. The answer is contained in a theorem [78, 79] which states the necessary and sufficient conditions to achieve finiteness at all orders. The finiteness conditions (52) and (53) impose relations between gauge and Yukawa couplings. To require such relations which render the couplings mutually dependent at a given renormalization point is trivial. What is not trivial is to guarantee that relations hold at any renormalization point. But we have seen from the previous sections (see Eq. (36)), that the necessary and also sufficient, condition to have RG invariance is to require that such relations are solutions to the REs

$$\beta_g \frac{dC_{ijk}}{dg} = \beta_{ijk} \tag{54}$$

and hold at all orders. Remarkably, the existence of all-order power series solutions to (54) can be decided at one-loop level, as already mentioned.

The all-order finiteness theorem [78, 79], states under which conditions an $N = 1$ supersymmetric gauge theory can become finite to all orders in perturbation theory, that is attain physical scale invariance. It is based on (a) the structure of the supercurrent in $N = 1$ supersymmetric gauge theory [80–82], and on (b) the non-renormalization properties of $N = 1$ chiral anomalies [78, 79, 83–85]. Details of the proof can be found in refs. [78, 79] and further discussion in Refs. [83–87]. One-loop finiteness, i.e. vanishing of the β-functions at one-loop, implies that the Yukawa couplings λ_{ijk} must be functions of the gauge coupling g. To find a similar condition to all orders it is necessary and sufficient for the Yukawa couplings to be a formal power series in g, which is solution of the REs (54). We state below the theorem for all-order vanishing β-functions [78].

Theorem:
Consider an $N = 1$ supersymmetric Yang-Mills theory, with simple gauge group. If the following conditions are satisfied

1. There is no gauge anomaly.

2. The gauge β-function vanishes at one-loop

$$\beta_g^{(1)} = 0 = \sum_i T(R_i) - 3\, C_2(G). \tag{55}$$

3. There exist solutions of the form

$$C_{ijk} = \rho_{ijk} g, \qquad \rho_{ijk} \in \mathbb{C} \tag{56}$$

to the conditions of vanishing one-loop matter fields anomalous dimensions

$$\gamma^{(1)}{}^i_j \; = \; 0 = \frac{1}{32\pi^2}\, [\, C^{ikl}\, C_{jkl} - 2\, g^2\, C_2(R)\delta^i_j]. \tag{57}$$

4. These solutions are isolated and non-degenerate when considered as solutions of vanishing one-loop Yukawa β-functions:

$$\beta_{ijk} = 0. \tag{58}$$

Then, each of the solutions (56) can be uniquely extended to a formal power series in g, and the associated super Yang-Mills models depend on the single coupling constant g with a β-function which vanishes at all-orders.

It is important to note a few things: The requirement of isolated and non-degenerate solutions guarantees the existence of a unique formal power series solution to the reduction equations. The vanishing of the gauge β-function at one-loop, $\beta_g^{(1)}$, is equivalent to the vanishing of the R current anomaly. The vanishing of the anomalous dimensions at one-loop implies the vanishing of the Yukawa couplings β-functions at that order. It also implies the vanishing of the chiral anomaly coefficients r^A. This last property is a necessary condition for having β-functions vanishing at all orders.[1]

Thus, it is clear that finiteness and reduction of couplings are intimately related. Since a relationship between the Noether current and the current belonging to the supercurrent multiplet J is absent in non-supersymmetric theories, one cannot extend the validity of a similar theorem to such theories.

A very interesting development was done in ref [56]. Based on the all-loop relations among the β-functions of the soft supersymmetry breaking terms and those of the rigid supersymmetric theory,

[1] There is an alternative way to find finite theories [88–91].

with the help of the differential operators discussed in the previous sections, it was shown that certain RGI surfaces can be chosen, so as to reach all-loop finiteness of the full theory. More specifically it was shown that on certain RGI surfaces the partial differential operators appearing in Eqs. (26,27) acting on the β- and γ- functions of the rigid theory can be transformed to total derivatives. Then the all-loop finiteness of the β- and γ-functions of the rigid theory can be transferred to the β-functions of the soft supersymmetry breaking terms. Therefore a totally all-loop finite $N = 1$ SUSY gauge theory can be constructed, including the soft supersymmetry breaking terms.

4. Successful Finite Unification Models

We now briefly review the basic properties of phenomenologically successful SUSY models with reduced couplings, which can be made finite either to all-loops or to two-loops in perturbation theory. Their predictions for the top and bottom quark masses, the SM Higgs boson mass, as well as the supersymmetric and the other Higgs spectra are discussed in 4.4, while experimental constraints considered are listed in 4.3. Other models with reduced couplings that were analyzed can be found in [29] and [92] (see also [93] and [94]) are the Reduced Minimal $N = 1$ $SU(5)$ [10], and the Reduced Minimal Supersymmetric Standard Model [95, 96].

4.1 An all-loop finite $N = 1$ supersymmetric $SU(5)$ model

The first model we will review is the finite to all-orders $SU(5)$, where we restrict the application of the reduction of couplings method to the third generation. In the latest version improved Higgs calculations predict a somewhat different interval that is still within current experimental limits.

The particle content of the model consists of three $(\overline{\mathbf{5}} + \mathbf{10})$ supermultiplets for the three generations of leptons and quarks, while the Higgs sector is accommodated in four supermultiplets $(\overline{\mathbf{5}} + \mathbf{5})$ and one $\mathbf{24}$. The one-loop anomalous dimensions are diagonal, fermions do not couple to $\mathbf{24}$ and the MSSM Higgs doublets are mostly composed from the 5 and $\bar{5}$ that couple to the third generation. The finite $SU(5)$ group is broken to the MSSM, which is no longer a finite theory, as expected [8–11, 15, 18]. When the GUT breaks to the MSSM, a suitable rotation in the Higgs sector [8, 9, 97–100], allows only two Higgs doublets (coupled mostly to the third family) to remain light and acquire vevs. Fast proton decay is avoided with the usual doublet-triplet splitting.

The superpotential (with an enhanced symmetry due to the reduction of couplings) is given by [19, 21]:

$$
\begin{aligned}
W = \sum_{i=1}^{3} \Big[\frac{1}{2} g_i^u \,\mathbf{10}_i \mathbf{10}_i H_i + g_i^d \,\mathbf{10}_i \overline{\mathbf{5}}_i \,\overline{H}_i \,\Big] + g_{23}^u \,\mathbf{10}_2 \mathbf{10}_3 H_4 \qquad (59)\\
+ g_{23}^d \,\mathbf{10}_2 \overline{\mathbf{5}}_3 \,\overline{H}_4 + g_{32}^d \,\mathbf{10}_3 \overline{\mathbf{5}}_2 \,\overline{H}_4 + g_2^f \,H_2 \,\mathbf{24}\, \overline{H}_2 + g_3^f \,H_3 \,\mathbf{24}\, \overline{H}_3 + \frac{g^\lambda}{3} \,(\mathbf{24})^3 \,.
\end{aligned}
$$

The *non-degenerate* and *isolated* solutions to the vanishing of $\gamma_i^{(1)}$ are:

$$
\begin{aligned}
&(g_1^u)^2 = \frac{8}{5}\, g^2 \,,\; (g_1^d)^2 = \frac{6}{5}\, g^2 \,,\; (g_2^u)^2 = (g_3^u)^2 = \frac{4}{5}\, g^2 \,,\\
&(g_2^d)^2 = (g_3^d)^2 = \frac{3}{5}\, g^2 \,,\; (g_{23}^u)^2 = \frac{4}{5}\, g^2 \,,\; (g_{23}^d)^2 = (g_{32}^d)^2 = \frac{3}{5}\, g^2 \,, \qquad (60)\\
&(g^\lambda)^2 = \frac{15}{7} g^2 \,,\; (g_2^f)^2 = (g_3^f)^2 = \frac{1}{2}\, g^2 \,,\; (g_1^f)^2 = \; (g_4^f)^2 = 0 \,.
\end{aligned}
$$

Regarding the parameters of non-zero dimension, we have the relation $h = -MC$, while the sum rules lead to:

$$m_{H_u}^2 + 2m_{\mathbf{10}}^2 = M^2 \ , \ m_{H_d}^2 - 2m_{\mathbf{10}}^2 = -\frac{M^2}{3} \ , \ m_{\overline{\mathbf{5}}}^2 + 3m_{\mathbf{10}}^2 = \frac{4M^2}{3} \ . \tag{61}$$

We therefore result in just two free dimensionful parameters, $m_{\mathbf{10}}$ and M. The model is discussed in more detail in [8–10].

4.2 A two-loop Finite $SU(N)^3$ Model

We will exemplify now how to construct a FUT based on a product gauge group. Consider an $N = 1$ SUSY theory with $SU(N)_1 \times SU(N)_2 \times \cdots \times SU(N)_k$ having n_f families transforming as $(N, N^*, 1, \ldots, 1) + (1, N, N^*, \ldots, 1) + \cdots + (N^*, 1, 1, \ldots, N)$. Then, the first order coefficient of the β-function, for each $SU(N)$ group is:

$$b = \left(-\frac{11}{3} + \frac{2}{3}\right) N + n_f \left(\frac{2}{3} + \frac{1}{3}\right)\left(\frac{1}{2}\right) 2N = -3N + n_f N \, . \tag{62}$$

Demanding the vanishing of the gauge one-loop β-function, i.e. $b = 0$, we are led to the choice $n_f = 3$. Phenomenological reasons lead to the choice of the $SU(3)_C \times SU(3)_L \times SU(3)_R$ model, discussed in Ref.[23], while a detailed discussion of the general well known example can be found in [101–104]. The leptons and quarks transform as:

$$q = \begin{pmatrix} d & u & D \\ d & u & D \\ d & u & D \end{pmatrix} \sim (3, 3^*, 1), \quad q^c = \begin{pmatrix} d^c & d^c & d^c \\ u^c & u^c & u^c \\ D^c & D^c & D^c \end{pmatrix} \sim (3^*, 1, 3), \quad \lambda = \begin{pmatrix} N & E^c & \nu \\ E & N^c & e \\ \nu^c & e^c & S \end{pmatrix} \sim (1, 3, 3^*) \tag{63}$$

where D are down-type quarks acquiring masses close to M_{GUT}. A cyclic Z_3 symmetry is imposed on the multiplets to achieve equal gauge couplings at the GUT scale and in that case the vanishing of the first-order β-function is satisfied. Continuing to the vanishing of the anomalous dimension of all the fields (see Eq. (53)), we note that there are two trilinear invariant terms in the superpotential, namely:

$$f\, Tr(\lambda q^c q) + \frac{1}{6} f' \ \epsilon_{ijk}\epsilon_{abc}(\lambda_{ia}\lambda_{jb}\lambda_{kc} + q^c_{ia} q^c_{jb} q^c_{kc} + q_{ia} q_{jb} q_{kc}), \tag{64}$$

with f and f' the corresponding Yukawa couplings. The superfields $(\tilde{N}, \tilde{N}^c)$ obtain vev's and provide masses to leptons and quarks

$$m_d = f\langle\tilde{N}\rangle, \ \ m_u = f\langle\tilde{N}^c\rangle, \ \ m_e = f'\langle\tilde{N}\rangle, \ \ m_\nu = f'\langle\tilde{N}^c\rangle. \tag{65}$$

Having three families, 11 f couplings and 10 f' couplings are present in the most general superpotential. Demanding the vanishing of all superfield anomalous dimensions, 9 conditions are imposed

$$\sum_{j,k} f_{ijk}(f_{ljk})^* + \frac{2}{3}\sum_{j,k} f'_{ijk}(f'_{ljk})^* = \frac{16}{9} g^2 \delta_{il} \, , \tag{66}$$

where

$$f_{ijk} = f_{jki} = f_{kij}, \qquad f'_{ijk} = f'_{jki} = f'_{kij} = f'_{ikj} = f'_{kji} = f'_{jik} \, . \tag{67}$$

The masses of leptons and quarks are acquired from the vev's of the scalar parts of the superfields $\tilde{N}_{1,2,3}$ and $\tilde{N}^c_{1,2,3}$.

At $M_{\rm GUT}$ the $SU(3)^3$ FUT breaks[2] to the MSSM, where as was already mentioned, both Higgs doublets couple mostly to the third generation. The FUT breaking leaves its mark in the form of Eq. (66), i.e. boundary conditions on the gauge and Yukawa couplings, the relation among the soft trilinear coupling, the corresponding Yukawa coupling and the unified gaugino mass and finally the soft scalar mass sum rule at $M_{\rm GUT}$. In this specific model the sum rule takes the form:

$$m^2_{H_u} + m^2_{\tilde{t}^c} + m^2_{\tilde{q}} = M^2 = m^2_{H_d} + m^2_{\tilde{b}^c} + m^2_{\tilde{q}} \ . \tag{68}$$

The model is finite to all-orders if the solution of Eq. (66) is both *isolated* and unique. But then $f' = 0$ and, at one-loop order, the lepton masses vanish. Since these masses cannot be generated, even radiatively, because of the finiteness conditions, we concentrate on finding a two-loop finite solution. If the solution of Eq. (66) is unique but not isolated (i.e. parametric), we can have non zero f' leading to non-vanishing lepton masses and at the same time achieving two-loop finiteness. In that case the set of conditions restricting the Yukawa couplings read:

$$f^2 = r\left(\frac{16}{9}\right)g^2 \, , \quad f'^2 = (1-r)\left(\frac{8}{3}\right)g^2 \, , \tag{69}$$

where r parametrises the different solutions and as such is a free parameter. It should be noted that we use the sum rule as boundary condition for the soft scalar masses.

4.3 Phenomenological Constraints

In this section we briefly review several experimental constraints that were applied in our phenomenological analysis. The used values do not correspond to the latest experimental results, which, however, has a negligible impact on our analysis.

We have evaluated the pole mass of the top quark while the bottom quark mass is evaluated at the M_Z scale (to avoid uncertainties to its pole mass). The experimental values, taken from ref.[107] are:[3]

$$m_t^{\rm exp} = 173.1 \pm 0.9 \text{ GeV} \quad , \quad m_b(M_Z) = 2.83 \pm 0.10 \text{ GeV} \ . \tag{70}$$

We interpret the Higgs-like particle discovered in July 2012 by ATLAS and CMS [26, 27] as the light CP-even Higgs boson of the MSSM [108–110]. The Higgs boson experimental average mass is [107] [4]

$$M_h^{\rm exp} = 125.10 \pm 0.14 \text{ GeV} \ . \tag{71}$$

The theoretical uncertainty [111, 112], however, for the prediction of M_h in the MSSM dominates the total uncertainty, since it is much larger than the experimental one. In our analysis we used the version 2.16.0 of the `FeynHiggs` code [111–118] to predict the Higgs mass.[5] This version gives a

[2][105, 106] and refs therein discuss in detail the spontaneous breaking of $SU(3)^3$.

[3]This is not the latest experimental value, but the changes are small and have no phenomenological consequences.

[4]This is the latest available LHC combination. More recent measurements confirm this value.

[5]An analysis of the impact of the improved M_h calculation in various SUSY models can be found in [119]. Further improvements that became available later do not have a relevant impact.

downward shift on the Higgs mass M_h of $O(2\text{ GeV})$ for large SUSY masses and in particular gives a reliable point-by-point evaluation of the Higgs-boson mass uncertainty [120]. The theoretical uncertainty calculated is added linearly to the experimental error in Eq. (71).

Furthermore, recent results from the ATLAS experiment [121] set limits to the mass of the pseudoscalar Higgs boson, M_A, in comparison with $\tan\beta$. For models with $\tan\beta \sim 45-55$, as the ones examined here, the lowest limit for the physical pseudoscalar Higgs mass is

$$M_A \gtrsim 1900\text{ GeV}. \tag{72}$$

For the production of the heavy Higgs sector and the full supersymmetric spectrum of each model a `SARAH` [122] generated, custom MSSM module for `SPheno` [123, 124] was used. The cross sections for their particle productions at the HL-LHC and FCC-hh, with $\sqrt{s} = 14$ TeV and 100 TeV respectively, were calculated with `MadGraph5_aMC@NLO` [125].

We also considered the following four flavor observables where SUSY has non-negligible impact. For the branching ratio $\text{BR}(b \to s\gamma)$ we take a value from [126, 152], while for the branching ratio $\text{BR}(B_s \to \mu^+\mu^-)$ we use a combination of [127, 153–155, 170]:[6]

$$\frac{\text{BR}(b \to s\gamma)^{\text{exp}}}{\text{BR}(b \to s\gamma)^{\text{SM}}} = 1.089 \pm 0.27 \quad , \quad \text{BR}(B_s \to \mu^+\mu^-) = (2.9 \pm 1.4) \times 10^{-9}\,. \tag{73}$$

For the B_u decay to $\tau\nu$ we use [156–158] and for ΔM_{B_s} we use [128, 129]:

$$\frac{\text{BR}(B_u \to \tau\nu)^{\text{exp}}}{\text{BR}(B_u \to \tau\nu)^{\text{SM}}} = 1.39 \pm 0.69 \quad , \quad \frac{\Delta M_{B_s}^{\text{exp}}}{\Delta M_{B_s}^{\text{SM}}} = 0.97 \pm 0.2\,. \tag{74}$$

Finally, we consider Cold Dark Matter (CDM) constraints. Since the Lightest SUSY Particle (LSP), which in our case is the lightest neutralino, is a promising CDM candidate [130, 131], we examine if the model is within the CDM relic density experimental limits. The current bound on the CDM relic density at $2\,\sigma$ level is given by [132]

$$\Omega_{\text{CDM}} h^2 = 0.1120 \pm 0.0112\,. \tag{75}$$

For the calculation of the CDM relic density the the `MicrOMEGAs 5.0` code [133–135] was used.

4.4 Numerical Analysis of the Finite $SU(5)$

We will briefly present now the analysis of the predicted spectrum of the all-loop finite $SU(5)$ model. Below the GUT scale we get the MSSM, where the third generation is given by the finiteness conditions (the first two remain unrestricted). However, these conditions do not restrict the low-energy renormalization properties, so the above relations between gauge, Yukawa and the various dimensionful parameters serve as boundary conditions at M_{GUT}. The third generation quark masses $m_b(M_Z)$ and m_t are predicted within 3σ and 2σ uncertainties, respectively, of their experimental values (see the complete analysis in [29]), as shown in Fig. 1. $\mu < 0$ is the only phenomenologically viable option, as shown in [29, 94, 136–142].

[6]As before, an update to the most recent values [171] would not change our resuls in a relevant way.

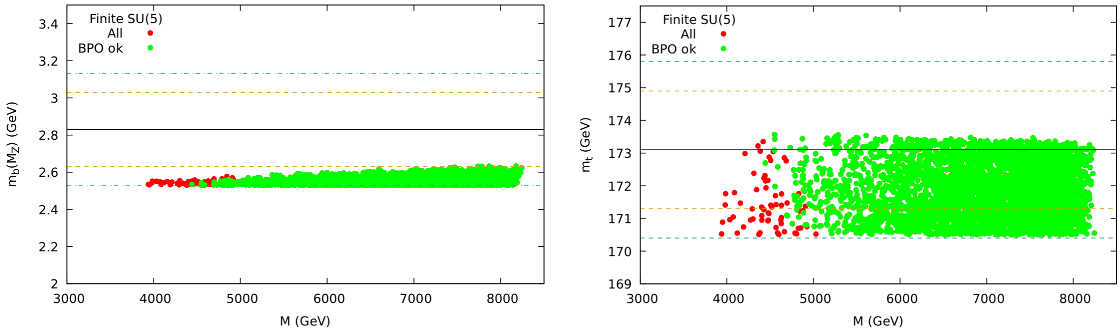

Figure 1: $m_b(M_Z)$ (left) and m_t (right) as a function of M for the Finite $N = 1$ $SU(5)$. The scattered points (red and green) come from varying the free parameter m_{10}, and the green points are the ones that satisfy the B-physics constraints. The orange (blue) dashed lines denote the 2σ (3σ) experimental uncertainties.

The plot of the light Higgs mass satisfies all experimental constraints considered in 4.3 (including B-physics constraints) for a unified gaugino mass $M \sim 4500 - 7500$ GeV, while its point-by-point theoretical uncertainty [120] drops significantly (w.r.t. the previous analysis) to $0.65-0.70$ GeV. This can be found in Fig. 2. The improved evaluation of M_h and its uncertainty prefer a heavier (Higgs) spectrum (compared to previous analyses [29, 94, 136–141, 143–147]), and thus allows only a heavy supersymmetric spectrum, which is in agreement with all existing experimental data. Very heavy colored supersymmetric particles are favored, in agreement with the non-observation of such particles at the LHC [162].

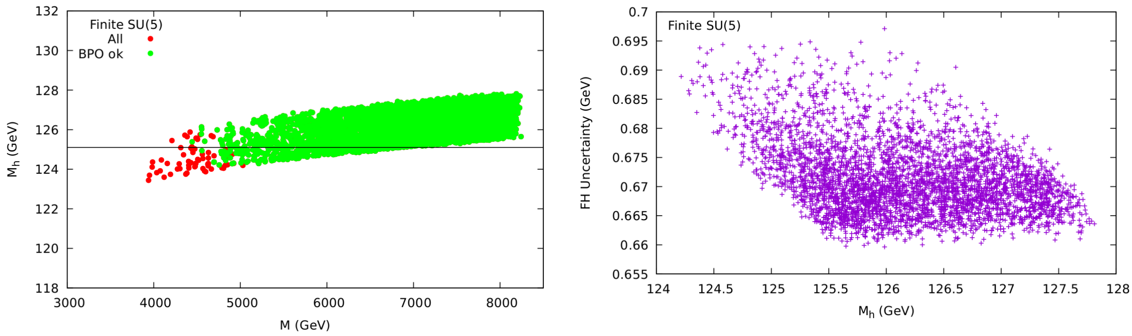

Figure 2: Left: M_h as a function of M. As in Fig. 1, the green points are the ones that comply with the B-physics constraints. Right: The lightest Higgs mass theoretical uncertainty calculated with `FeynHiggs` 2.16.0 [120].

At this point there is an important remark. No point fulfills the strict bound of Eq. (75), since we have overproduction of CDM in the early universe (for the original analysis see [93]). The LSP, which in our case is the lightest neutralino, is strongly Bino-like. Combined with the heavy mass it acquires (1-2 TeV), it cannot account for a relic density low enough to agree with experimental observation. Thus, we need a mechanism that reduces this CDM abundance. This could be related to the problem of neutrino masses, which cannot be generated naturally in this particular model. However, one could extend the model by considering bilinear R-parity violating terms (that preserve finiteness) and thus introduce neutrino masses [148, 160]. R-parity violation [161] would have a small impact on the masses and production cross sections, but remove the CDM bound of

Eq. (75) completely. Other mechanisms, not involving R-parity violation, that could be invoked if the amount of CDM appears to be too large, concern the cosmology of the early universe. For example, "thermal inflation" [163] or "late time entropy injection" [164] can bring the CDM density into agreement with Planck measurements. For the original discussion see [29].

As explained in more detail in [92], the three benchmarks chosen (for the purposes of collider phenomenology) feature the LSP above 2100 GeV, 2400 GeV and 2900 GeV, respectively. The resulting masses that are relevant to our analysis were generated by `SPheno` 4.0.4 [123, 124] and are listed in Table 1 for each benchmark (with the corresponding $\tan\beta$). The two first masses refer to the heavy Higgs bosons. The gluino mass is $M_{\tilde{g}}$, the neutralinos and the charginos are denoted as $M_{\tilde{\chi}^0_i}$ and $M_{\tilde{\chi}^\pm_i}$, while the slepton and sneutrino masses for all three generations are given as $M_{\tilde{e}_{1,2,3}}$, $M_{\tilde{\nu}_{1,2,3}}$. Similarly, the squarks are denoted as $M_{\tilde{d}_{1,2}}$ and $M_{\tilde{u}_{1,2}}$ for the first two generations. The third generation masses are given by $M_{\tilde{t}_{1,2}}$ for stops and $M_{\tilde{b}_{1,2}}$ for sbottoms.

	$\tan\beta$	$M_{A,H}$	$M_{H^\pm}$	$M_{\tilde{g}}$	$M_{\tilde{\chi}^0_1}$	$M_{\tilde{\chi}^0_2}$	$M_{\tilde{\chi}^0_3}$	$M_{\tilde{\chi}^0_4}$	$M_{\tilde{\chi}^\pm_1}$	$M_{\tilde{\chi}^\pm_2}$
FUTSU5-1	49.9	5.688	5.688	8.966	2.103	3.917	4.829	4.832	3.917	4.833
FUTSU5-2	50.1	7.039	7.086	10.380	2.476	4.592	5.515	5.518	4.592	5.519
FUTSU5-3	49.9	16.382	16.401	12.210	2.972	5.484	6.688	6.691	5.484	6.691
	$M_{\tilde{e}_{1,2}}$	$M_{\tilde{\nu}_{1,2}}$	$M_{\tilde{\tau}}$	$M_{\tilde{\nu}_\tau}$	$M_{\tilde{d}_{1,2}}$	$M_{\tilde{u}_{1,2}}$	$M_{\tilde{b}_1}$	$M_{\tilde{b}_2}$	$M_{\tilde{t}_1}$	$M_{\tilde{t}_2}$
FUTSU5-1	3.102	3.907	2.205	3.137	7.839	7.888	6.102	6.817	6.099	6.821
FUTSU5-2	3.623	4.566	2.517	3.768	9.059	9.119	7.113	7.877	7.032	7.881
FUTSU5-3	4.334	5.418	3.426	3.834	10.635	10.699	8.000	9.387	8.401	9.390

Table 1: Masses for each of the three benchmarks of the Finite $N = 1$ $SU(5)$ (in TeV) [92].

At 14 TeV HL-LHC none of the Finite $SU(5)$ scenarios listed above has a SUSY production cross section above 0.01 fb, and thus will most probably remain unobservable [165]. The discovery prospects for the heavy Higgs-boson spectrum is significantly better at the FCC-hh [166]. Theoretical analyses [167, 168] have shown that for large $\tan\beta$ heavy Higgs mass scales up to $\sim$ 8 TeV could be accessible. Since in this model we have $\tan\beta \sim 50$, the first two benchmark points are well within the reach of the FCC-hh (as explained in [92]). The third point, however, where $M_A \sim 16$ TeV, will be far outside the reach of the collider.

Since the production cross section of squark pairs and squark-gluino pairs are at the few fb level, their prospects for detection are very dim. The heavy LSP will keep charginos and neutralinos unobservable, and large parts of the possible mass spectra will not be observable at the FCC-hh.

4.5 Phenomenological Analysis of the $SU(3)^3$ Model

We will consider here the two-loop finite version of the $SU(3)^3$ model, where again below $M_{\rm GUT}$ we get the MSSM [23, 24]. We take into account two new thresholds for the masses of the new particles at $\sim 10^{13}$ GeV and $\sim 10^{14}$ GeV resulting in a wider phenomenologically viable parameter space [147].

It has to be noted that it is by no means a given fact that there will be a value of r which simultaneously fits both the top and bottom quark masses. Looking for the values of the parameter

r which comply with the experimental limits, we find that both the top and bottom masses are in the experimental range (within 2σ) for the same value of r between 0.65 and 0.80 (we singled out the $\mu < 0$ case as the most promising). The inclusion of the above-mentioned thresholds gives an important improvement on the top mass from past versions of the model [23, 24, 149, 150].

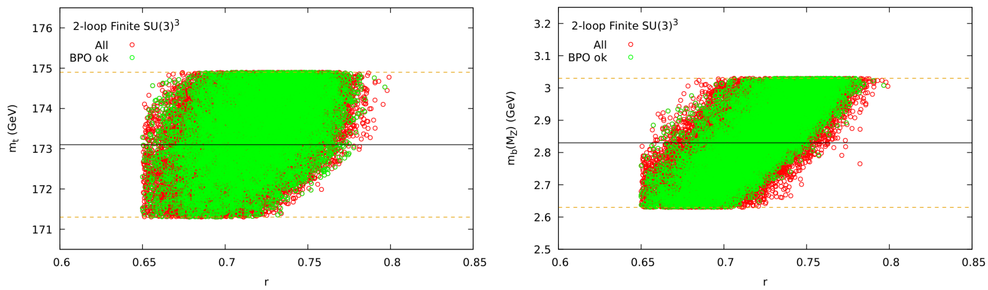

Figure 3: Bottom and top quark masses for the two-loop finite $SU(3)^3$ model, with $\mu < 0$, as functions of r. The scattered points are due to the fact that we vary five parameters, namely r and four of the parameters from the sum rule, the green points satisfy the B-physics constraints.

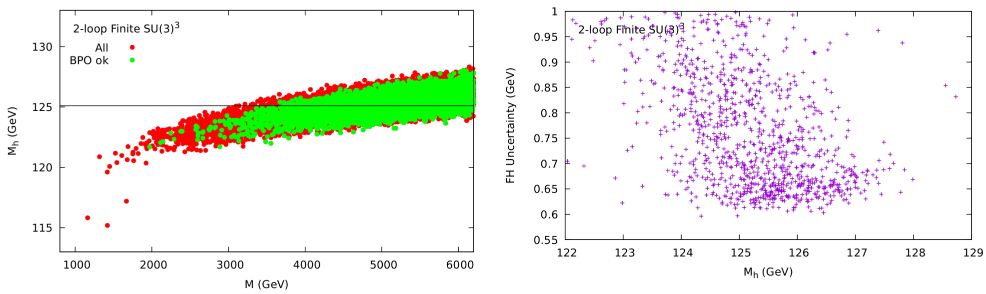

Figure 4: Left panel: Higgs boson mass M_h as a function of M (color code as in Fig. 3). Right panel: The Higgs mass theoretical uncertainty [120].

Concerning the SUSY spectra, we choose again three benchmarks, each featuring the LSP above 1500 GeV, 2000 GeV and 2400 GeV respectively (but the LSP can go as high as $\sim$ 4100 GeV, again with too small cross sections). For more details see ref. [92].

	M_H	M_A	$M_{H^\pm}$	$M_{\tilde{g}}$	$M_{\tilde{\chi}^0_1}$	$M_{\tilde{\chi}^0_2}$	$M_{\tilde{\chi}^0_3}$	$M_{\tilde{\chi}^0_4}$	$M_{\tilde{\chi}^\pm_1}$	$M_{\tilde{\chi}^\pm_2}$
FSU33-1	7.029	7.029	7.028	6.526	1.506	2.840	6.108	6.109	2.839	6.109
FSU33-2	6.484	6.484	6.431	8.561	2.041	3.817	7.092	7.093	3.817	7.093
FSU33-3	6.539	6.539	6.590	10.159	2.473	4.598	6.780	6.781	4.598	6.781
	$M_{\tilde{e}_{1,2}}$	$M_{\tilde{\nu}_{1,2}}$	$M_{\tilde{\tau}}$	$M_{\tilde{\nu}_\tau}$	$M_{\tilde{d}_{1,2}}$	$M_{\tilde{u}_{1,2}}$	$M_{\tilde{b}_1}$	$M_{\tilde{b}_2}$	$M_{\tilde{t}_1}$	$M_{\tilde{t}_2}$
FSU33-1	2.416	2.415	1.578	2.414	5.375	5.411	4.913	5.375	4.912	5.411
FSU33-2	3.188	3.187	2.269	3.186	7.026	7.029	6.006	7.026	6.005	7.029
FSU33-3	3.883	3.882	2.540	3.882	8.334	8.397	7.227	8.334	7.214	7.409

Table 2: Masses for each benchmark of the Finite $N = 1$ $SU(3)^3$ (in TeV).

It should be noted that in this model the scale of the heavy Higgs bosons does not vary monotonously with $M_{\tilde{\chi}_1^0}$, as in the previously considered models. This can be understood as follows. The Higgs bosons masses are determined by a combination of the sum rule at the unification scale, and the requirement of successful electroweak symmetry breaking at the low scale. Like in the finite scenario of the previous section, there are no direct relations between the soft scalar masses and the unified gaugino mass, but they are related through the corresponding sum rule and thus vary correlatedly, a fact that makes the dependence on the boundary values more restrictive. Furthermore (and even more importantly), the fact that we took into account the two thresholds at $\sim 10^{13}$ GeV and $\sim 10^{14}$ GeV [147], allows the new particles, mainly the Higgsinos of the two other families (that were considered decoupled at the unification scale in previous analyses) and the down-like exotic quarks (in a lower degree), to affect the running of the (soft) RGEs in a non-negligible way. Thus, since at low energies the heavy Higgs masses depend mainly on the values of $m^2_{H_u}$, $m^2_{H_d}$, $|\mu|$ and $\tan\beta$, they are substantially less connected to $M_{\tilde{\chi}_1^0}$ than in the other models, leading to a different exclusion potential, as will be discussed in the following.

Scenarios of Finite $SU(3)^3$ are beyond the reach of the HL-LHC. Not only superpartners are too heavy, but also heavy Higgs bosons with a mass scale of $\sim$ 7 TeV cannot be detected at the HL-LHC.

On the other hand, at the 100 TeV collider all three benchmark points are well within the reach of the $H/A \to \tau^+\tau^-$ as well as the $H^\pm \to \tau\nu_\tau, tb$ searches [167, 168], despite the slightly smaller values of $\tan\beta \sim 45$. This is a result of the different dependence of the heavy Higgs-boson mass scale on $M_{\tilde{\chi}_1^0}$, as discussed above. However, we have checked that M_A can go up to to $\sim$ 11 TeV, and thus the heaviest part of the possible spectrum would escape the heavy Higgs-boson searches at the FCC-hh.

Interesting are also the prospects for production of squark pairs and squark-gluino, which can reach $\sim$ 20 fb for the FSU33-1 case, going down to a few fb for FSU33-2 and FSU33-3 scenarios. The lightest squarks decay almost exclusively to the third generation quark and chargino/neutralino, while gluino enjoys many possible decay channels to quark-squark pairs each one with branching fraction of the order of a percent, with the biggest one $\sim 20\%$ to $t\tilde{t}_1 + h.c.$

We briefly discuss the SUSY discovery potential at the FCC-hh, referring again to [169]. Stops in FSU33-1 and FSU33-2 can be tested at the FCC-hh, while the masses turn out to be too heavy in FSU33-3. The situation is better for scalar quarks, where all three scenarios can be tested, but will not allow for a $5\,\sigma$ discovery. Even more favorable are the prospects for gluino. Possibly all three scenarios can be tested at the $5\,\sigma$ level. The charginos and neutralinos will not be accessible, due to the too heavy LSP. Taking into account that only the lower part of the possible mass spectrum is represented by the three benchmark points and that the LSP can go up to 4.1 TeV, we conclude that large parts of the parameter space will not be testable at the FCC-hh. The only partial exception here is the Higgs-boson sector, where only the part with the highest possible Higgs-boson mass spectra would escape the FCC-hh searches.

5. Conclusions

We reviewed the method of reduction of couplings, which is realized most naturally in SUSY theories, and with more phenomenological success in $N = 1$ theories, rendering them more predictive. We considered the, so far, most promising two models, an all-loop finite $SU(5)$ model, and a two-loop finite $SU(3)^3$ model. We presented the updated results for these models, using the Higgs-boson mass calculation of FeynHiggs, and then turning to the question of their discovering potential. In each case, low-mass region benchmark points have been chosen, for which the SPheno code was used to calculate the spectrum of supersymmetric particles and their respective decay modes. Finally, the MadGraph event generator was used for the computation of the production cross-sections of the relevant final states at the 14 TeV (HL-)LHC and 100 TeV FCC-hh colliders.

Both models predict relatively heavy spectra, which evade largely the detection of the SUSY particles at the HL-LHC. In the case of the all-loop finite $SU(5)$ model two of the three benchmarks points for the heavy Higgses are within the reach of the FCC-hh, although large parts of the rest of the spectrum will not be observable. Concerning the $SU(3)^3$ model, in the case of the heavy Higgs bosons, all three benchmark points analyzed are within the reach of the FCC-hh, but the heaviest part of the possible spectrum would be inaccessible. For the rest of the SUSY spectrum the lower part of the parameter space will be testable at 2σ level, with a small part, namely the gluinos, that could be tested at the 5σ level.

Acknowledgements

GZ thanks the ITP of Heidelberg, MPI Munich, CERN Department of Theoretical Physics for their hospitality and support and DFG Exzellenzcluster 2181:STRUCTURES of Heidelberg University for support. WK is supported by the National Science Centre (Poland) under the research grant 2020/38/E/ ST2/00126. The work of JK and WK has been supported by the National Science Centre, Poland, the HARMONIA project under Contract UMO-2015/18/M/ST2/00518 (2016-2020) and JK partially by the Norwegian Financial Mechanism 2014-2021, grant 2019/34/H/ST2/00707. S.H acknowledges support from the grant IFT Centro de Excelencia Severo Ochoa CEX2020-001007-S funded by MCIN/AEI/10.13039/501100011033. The work of S.H. was supported in part by the grant PID2019-110058GB-C21 funded by MCIN/AEI/10.13039/501100011033 and by “ERDF A way of making Europe”. The work of MM is partly supported by UNAM PAPIIT through Grant IN109321.

References

[1] J. Kubo, S. Heinemeyer, M. Mondragon, O. Piguet, K. Sibold, W. Zimmermann and G. Zoupanos, PoS (Higgs & top)001, Ed. Klaus Sibold, https://pos.sissa.it/cgi-bin/reader/conf.cgi?confid=222. A short version is published in arXiv:1411.7155 [hep-ph].

[2] W. Zimmermann, *Reduction in the Number of Coupling Parameters*, *Commun. Math. Phys.* **97** (1985) 211.

[3] R. Oehme and W. Zimmermann, *Relation Between Effective Couplings for Asymptotically Free Models*, *Commun. Math. Phys.* **97** (1985) 569.

[4] R. Oehme, *Reduction and reparametrization of quantum field theories*, *Prog. Theor. Phys. Suppl.* **86** (1986) 215.

[5] E. Ma, *Modified Quantum Chromodynamics. 1. Exact Global Color Symmetry and Asymptotic Freedom*, *Phys. Rev.* **D17** (1978) 623.

[6] N.-P. Chang, *Eigenvalue Conditions and Asymptotic Freedom for Higgs Scalar Gauge Theories*, *Phys. Rev.* **D10** (1974) 2706.

[7] S. Nandi and W.-C. Ng, *CAN COUPLING CONSTANTS BE RELATED?*, *Phys.Rev.* **D20** (1979) 972.

[8] D. Kapetanakis, M. Mondragon and G. Zoupanos, *Finite unified models*, *Z. Phys.* **C60** (1993) 181 [hep-ph/9210218].

[9] M. Mondragon and G. Zoupanos, *Finite unified theories and the top quark mass*, *Nucl. Phys. Proc. Suppl.* **37C** (1995) 98.

[10] J. Kubo, M. Mondragon and G. Zoupanos, *Reduction of couplings and heavy top quark in the minimal SUSY GUT*, *Nucl. Phys.* **B424** (1994) 291.

[11] J. Kubo, M. Mondragon, N.D. Tracas and G. Zoupanos, *Gauge Yukawa unification in asymptotically nonfree theories*, *Phys. Lett.* **B342** (1995) 155 [hep-th/9409003].

[12] J. Kubo, M. Mondragon and G. Zoupanos, *Finite unification and top quark mass*, hep-th/9409032.

[13] J. Kubo, M. Mondragon and G. Zoupanos, *Top quark mass predictions from gauge-Yukawa unification*, hep-ph/9512400.

[14] J. Kubo, M. Mondragon, M. Olechowski and G. Zoupanos, *Testing Gauge-Yukawa-Unified Models By M_t*, *Nucl. Phys.* **B479** (1996) 25 [hep-ph/9512435].

[15] J. Kubo, M. Mondragon, M. Olechowski and G. Zoupanos, *Gauge-Yukawa Unification and the Top-Bottom Hierarchy*, hep-ph/9510279.

[16] J. Kubo, M. Mondragon, S. Shoda and G. Zoupanos, *Gauge-Yukawa Unification in $SO(10)$ SUSY GUTs*, *Nucl. Phys.* **B469** (1996) 3 [hep-ph/9512258].

[17] J. Kubo, M. Mondragon and G. Zoupanos, *Gauge-Yukawa unified models*, hep-ph/9702391.

[18] J. Kubo, M. Mondragon and G. Zoupanos, *Unification beyond GUTs: Gauge Yukawa unification*, *Acta Phys. Polon.* **B27** (1997) 3911 [hep-ph/9703289].

[19] T. Kobayashi, J. Kubo, M. Mondragon and G. Zoupanos, *Constraints on finite soft supersymmetry-breaking terms*, *Nucl. Phys.* **B511** (1998) 45 [hep-ph/9707425].

[20] J. Kubo, M. Mondragon and G. Zoupanos, *Unification beyond GUTs top mass predictions*, hep-ph/9708225.

[21] M. Mondragon and G. Zoupanos, *Finite unified theories*, *J.Phys.Conf.Ser.* **171** (2009) 012095.

[22] TEVATRON ELECTROWEAK WORKING GROUP, CDF AND D0 COLLABORATIONS collaboration, *Combination of CDF and D0 results on the mass of the top quark using up to 5.8 fb-1 of data*, 1107.5255.

[23] E. Ma, M. Mondragon and G. Zoupanos, *Finite $SU(N)^k$ unification*, *JHEP* **12** (2004) 026 [hep-ph/0407236].

[24] S. Heinemeyer, E. Ma, M. Mondragon and G. Zoupanos, *Finiteness in SU(3)**3 models*, *Fortsch.Phys.* **58** (2010) 729.

[25] S. Heinemeyer, M. Mondragon and G. Zoupanos, *Confronting Finite Unified Theories with Low-Energy Phenomenology*, *JHEP* **07** (2008) 135 [0712.3630].

[26] ATLAS COLLABORATION collaboration, *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, *Phys.Lett.* **B716** (2012) 1 [1207.7214].

[27] CMS COLLABORATION collaboration, *Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC*, *Phys.Lett.* **B716** (2012) 30 [1207.7235].

[28] ATLAS COLLABORATION collaboration, *Combined measurements of the mass and signal strength of the Higgs-like boson with the ATLAS detector using up to 25 fb^{-1} of proton-proton collision data*, .

[29] S. Heinemeyer, M. Mondragón, G. Patellis, N. Tracas and G. Zoupanos, *Updates and New Results in Models with Reduced Couplings*, *Fortsch. Phys.* **68** (2020) 2000028 [2002.10983].

[30] S. Heinemeyer, J. Kalinowski, W. Kotlarski, M. Mondragón, G. Patellis, N. Tracas and G. Zoupanos, Int. J. Mod. Phys. A **37** (2022) no.07, 2240010 doi:10.1142/S0217751X22400103

[31] J. Kubo, K. Sibold and W. Zimmermann, *Higgs and Top Mass from Reduction of Couplings*, *Nucl.Phys.* **B259** (1985) 331.

[32] J. Kubo, K. Sibold and W. Zimmermann, *New results in the reduction of the Standard Model*, *Phys. Lett.* **B220** (1989) 185.

[33] J. Wess and B. Zumino, *A Lagrangian Model Invariant Under Supergauge Transformations*, *Phys. Lett.* **B49** (1974) 52.

[34] J. Iliopoulos and B. Zumino, *Broken Supergauge Symmetry and Renormalization*, *Nucl. Phys.* **B76** (1974) 310.

[35] S. Ferrara, J. Iliopoulos and B. Zumino, *Supergauge Invariance and the Gell-Mann - Low Eigenvalue*, *Nucl. Phys.* **B77** (1974) 413.

[36] K. Fujikawa and W. Lang, *Perturbation Calculations for the Scalar Multiplet in a Superfield Formulation*, *Nucl. Phys.* **B88** (1975) 61.

[37] A. Parkes and P.C. West, *Finiteness in Rigid Supersymmetric Theories*, Phys. Lett. **B138** (1984) 99.

[38] P.C. West, *The Yukawa beta Function in N=1 Rigid Supersymmetric Theories*, Phys. Lett. **B137** (1984) 371.

[39] D.R.T. Jones and A.J. Parkes, *Search for a three loop finite chiral theory*, Phys. Lett. **B160** (1985) 267.

[40] D.R.T. Jones and L. Mezincescu, *The Chiral Anomaly and a Class of Two Loop Finite Supersymmetric Gauge Theories*, Phys. Lett. **B138** (1984) 293.

[41] A.J. Parkes, *Three loop finiteness conditions in N=1 Superyang-Mills*, Phys. Lett. **B156** (1985) 73.

[42] R. Oehme, K. Sibold and W. Zimmermann, *Renormalization Group Equations With Vanishing Lowest Order of the Primary Beta Function*, Phys. Lett. **147B** (1984) 115.

[43] T.P. Cheng, E. Eichten and L.-F. Li, *Higgs Phenomena in Asymptotically Free Gauge Theories*, Phys. Rev. **D9** (1974) 2259.

[44] W. Zimmermann, *Electroweak corrections to reduction solutions in the Standard Model*, Phys. Lett. **B311** (1993) 249.

[45] O. Piguet and K. Sibold, *Reduction of couplings in the presence of parameters*, Phys. Lett. **B229** (1989) 83.

[46] J. Kubo, M. Mondragon and G. Zoupanos, *Perturbative unification of soft supersymmetry-breaking terms*, Phys. Lett. **B389** (1996) 523 [hep-ph/9609218].

[47] P. Breitenlohner and D. Maison, *Gauge and mass parameter dependence of renormalized Green's functions*, Commun. Math. Phys. **219** (2001) 179.

[48] W. Zimmermann, *Scheme independence of the reduction principle and asymptotic freedom in several couplings*, Commun.Math.Phys. **219** (2001) 221.

[49] I. Jack and D.R.T. Jones, *Renormalization group invariance and universal soft supersymmetry breaking*, Phys. Lett. **B349** (1995) 294 [hep-ph/9501395].

[50] Y. Kawamura, T. Kobayashi and J. Kubo, *Soft scalar-mass sum rule in gauge-Yukawa unified models and its superstring interpretation*, Phys. Lett. **B405** (1997) 64 [hep-ph/9703320].

[51] R. Delbourgo, *Superfield Perturbation Theory and Renormalization*, Nuovo Cim. **A25** (1975) 646.

[52] A. Salam and J.A. Strathdee, *Feynman Rules for Superfields*, Nucl. Phys. **B86** (1975) 142.

[53] M.T. Grisaru, W. Siegel and M. Rocek, *Improved Methods for Supergraphs*, Nucl. Phys. **B159** (1979) 429.

[54] L. Girardello and M.T. Grisaru, *Soft Breaking of Supersymmetry*, Nucl. Phys. **B194** (1982) 65.

[55] Y. Yamada, *Two loop renormalization group equations for soft SUSY breaking scalar interactions: Supergraph method*, Phys. Rev. **D50** (1994) 3537 [hep-ph/9401241].

[56] D.I. Kazakov, *Finiteness of soft terms in finite N = 1 SUSY gauge theories*, Phys. Lett. **B421** (1998) 211 [hep-ph/9709465].

[57] I. Jack and D.R.T. Jones, *The gaugino beta-function*, Phys. Lett. **B415** (1997) 383 [hep-ph/9709364].

[58] J. Hisano and M.A. Shifman, *Exact results for soft supersymmetry breaking parameters in supersymmetric gauge theories*, Phys. Rev. **D56** (1997) 5475 [hep-ph/9705417].

[59] I. Jack, D.R.T. Jones and A. Pickering, *Renormalisation invariance and the soft beta functions*, Phys. Lett. **B426** (1998) 73 [hep-ph/9712542].

[60] L.V. Avdeev, D.I. Kazakov and I.N. Kondrashuk, *Renormalizations in softly broken SUSY gauge theories*, Nucl. Phys. **B510** (1998) 289 [hep-ph/9709397].

[61] D.I. Kazakov, *Exploring softly broken SUSY theories via Grassmannian Taylor expansion*, Phys. Lett. **B449** (1999) 201 [hep-ph/9812513].

[62] A. Karch, T. Kobayashi, J. Kubo and G. Zoupanos, *Infrared behavior of softly broken SQCD and its dual*, Phys. Lett. **B441** (1998) 235 [hep-th/9808178].

[63] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Instanton Effects in Supersymmetric Theories*, Nucl. Phys. **B229** (1983) 407.

[64] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Beta Function in Supersymmetric Gauge Theories: Instantons Versus Traditional Approach*, Phys. Lett. **B166** (1986) 329.

[65] M.A. Shifman, *Little miracles of supersymmetric evolution of gauge couplings*, Int. J. Mod. Phys. **A11** (1996) 5761 [hep-ph/9606281].

[66] T. Kobayashi, J. Kubo and G. Zoupanos, *Further all-loop results in softly-broken supersymmetric gauge theories*, Phys. Lett. **B427** (1998) 291 [hep-ph/9802267].

[67] L.E. Ibanez and D. Lust, *Duality anomaly cancellation, minimal string unification and the effective low-energy Lagrangian of 4-D strings*, Nucl. Phys. **B382** (1992) 305 [hep-th/9202046].

[68] A. Brignole, L.E. Ibanez, C. Munoz and C. Scheich, *Some issues in soft SUSY breaking terms from dilaton / moduli sectors*, Z.Phys. **C74** (1997) 157 [hep-ph/9508258].

[69] I. Jack and D. Jones, *RG invariant solutions for the soft supersymmetry breaking parameters*, Phys.Lett. **B465** (1999) 148 [hep-ph/9907255].

[70] T. Kobayashi, J. Kubo, M. Mondragon and G. Zoupanos, *Finite and gauge-Yukawa unified theories: Theory and predictions*, *AIP Conf. Proc.* **490** (1999) 279.

[71] T. Kobayashi, J. Kubo, M. Mondragon and G. Zoupanos, *Finite unification*, *Surveys High Energ. Phys.* **16** (2001) 87.

[72] S. Rajpoot and J.G. Taylor, *On finite quantum field theories*, *Phys. Lett.* **B147** (1984) 91.

[73] S. Rajpoot and J.G. Taylor, *Towards finite quantum field theories*, *Int. J. Theor. Phys.* **25** (1986) 117.

[74] D.R.T. Jones, L. Mezincescu and Y.P. Yao, *Soft breaking of two loop finite N=1 supersymmetric gauge theories*, *Phys. Lett.* **B148** (1984) 317.

[75] I. Jack and D.R.T. Jones, *Soft supersymmetry breaking and finiteness*, *Phys. Lett.* **B333** (1994) 372 [hep-ph/9405233].

[76] L. O'Raifeartaigh, *Spontaneous Symmetry Breaking for Chiral Scalar Superfields*, *Nucl. Phys.* **B96** (1975) 331.

[77] P. Fayet and J. Iliopoulos, *Spontaneously Broken Supergauge Symmetries and Goldstone Spinors*, *Phys. Lett.* **B51** (1974) 461.

[78] C. Lucchesi, O. Piguet and K. Sibold, *Vanishing beta functions in N=1 supersymmetric gauge theories*, *Helv. Phys. Acta* **61** (1988) 321.

[79] C. Lucchesi, O. Piguet and K. Sibold, *Necessary and sufficient conditions for all order vanishing beta functions in supersymmetric Yang-Mills theories*, *Phys. Lett.* **B201** (1988) 241.

[80] S. Ferrara and B. Zumino, *Transformation Properties of the Supercurrent*, *Nucl. Phys.* **B87** (1975) 207.

[81] O. Piguet and K. Sibold, *The supercurrent in N=1 supersymmetrical Yang-Mills theories. 1. The classical case*, *Nucl. Phys.* **B196** (1982) 428.

[82] O. Piguet and K. Sibold, *The supercurrent in N=1 supersymmmetrical Yang-Mills theories. 2. Renormalization*, *Nucl. Phys.* **B196** (1982) 447.

[83] O. Piguet and K. Sibold, *Nonrenormalization theorems of chiral anomalies and finiteness in supersymmetric Yang-Mills theories*, *Int. J. Mod. Phys.* **A1** (1986) 913.

[84] O. Piguet and K. Sibold, *Nonrenormalization theorems of chiral anomalies and finiteness*, *Phys. Lett.* **B177** (1986) 373.

[85] P. Ensign and K.T. Mahanthappa, *The supercurrent and the Adler-Bardeen theorem in coupled supersymmetric Yang-Mills theories*, *Phys. Rev.* **D36** (1987) 3148.

[86] C. Lucchesi and G. Zoupanos, *All-order Finiteness in N=1 SYM Theories: Criteria and Applications*, *Fortschr. Phys.* **45** (1997) 129 [hep-ph/9604216].

[87] O. Piguet, *Supersymmetry, ultraviolet finiteness and grand unification*, hep-th/9606045.

[88] A.V. Ermushev, D.I. Kazakov and O.V. Tarasov, *Finite N=1 supersymmetric Grand Unified Theories*, Nucl. Phys. **B281** (1987) 72.

[89] D.I. Kazakov, *Finite N=1 susy gauge field theories*, Mod. Phys. Lett. **A2** (1987) 663.

[90] D.R.T. Jones, *Coupling Constant Reparametrization and Finite Field Theories*, Nucl. Phys. **B277** (1986) 153.

[91] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory*, Nucl. Phys. **B447** (1995) 95 [hep-th/9503121].

[92] S. Heinemeyer, J. Kalinowski, W. Kotlarski, M. Mondragón, G. Patellis, N. Tracas et al., *Probing Unified Theories with Reduced Couplings at Future Hadron Colliders*, Eur. Phys. J. **C81** (2021) 185 [2011.07900].

[93] G. Patellis, *Phenomenological Consequences of Supersymmetric Theories from Dimensional Reduction and Reduction of Couplings*, Ph.D. thesis, Natl. Tech. U., Athens, 2020. 2102.01476.

[94] S. Heinemeyer, M. Mondragón, N. Tracas and G. Zoupanos, *Reduction of Couplings and its application in Particle Physics*, Phys. Rept. **814** (2019) 1 [1904.00410].

[95] M. Mondragón, N. Tracas and G. Zoupanos, *Reduction of Couplings in the MSSM*, Phys.Lett. **B728** (2014) 51 [1309.0996].

[96] S. Heinemeyer, M. Mondragon, N. Tracas and G. Zoupanos, *Reduction of the Parameters in MSSM*, 1712.02729.

[97] J. Leon, J. Perez-Mercader, M. Quiros and J. Ramirez-Mittelbrunn, *A sensible finite SU(5) susy GUT?*, Phys. Lett. **B156** (1985) 66.

[98] S. Hamidi and J.H. Schwarz, *A realistic finite Unified Theory?*, Phys. Lett. **B147** (1984) 301.

[99] D.R.T. Jones and S. Raby, *A two loop finite supersymmetric SU(5) theory: towards a theory of fermion masses*, Phys. Lett. **B143** (1984) 137.

[100] K.S. Babu, T. Enkhbat and I. Gogoladze, *Finite grand unified theories and the quark mixing matrix*, Phys. Lett. **B555** (2003) 238 [hep-ph/0204246].

[101] A. De Rújula, H. Georgi and G.S. L.

[102] G. Lazarides, C. Panagiotakopoulos and Q. Shafi, *Supersymmetric unification without proton decay*, Phys. Lett. **B315** (1993) 325 [hep-ph/9306332].

[103] G. Lazarides and C. Panagiotakopoulos, *MSSM from SUSY trinification*, Phys. Lett. **B336** (1994) 190 [hep-ph/9403317].

[104] E. Ma, *Particle Dichotomy and Left-Right Decomposition of E(6) Superstring Models*, Phys. Rev. **D36** (1987) 274.

[105] N. Irges and G. Zoupanos, *Reduction of $N = 1$, E_8 SYM over $SU(3)/U(1) \times U(1) \times Z_3$ and its four-dimensional effective action*, Phys.Lett. **B698** (2011) 146 [1102.2220].

[106] N. Irges, G. Orfanidis and G. Zoupanos, *Dimensional Reduction of N = 1, E_8 SYM over $SU(3)/U(1) \times U(1) \times \mathbb{Z}_3$ and its four-dimensional effective action*, PoS **CORFU2011** (2011) 105 [1205.0753].

[107] Particle Data Group collaboration, *Review of Particle Physics*, Phys. Rev. **D98** (2018) 030001.

[108] S. Heinemeyer, O. Stal and G. Weiglein, *Interpreting the LHC Higgs Search Results in the MSSM*, Phys.Lett. **B710** (2012) 201 [1112.3026].

[109] P. Bechtle, S. Heinemeyer, O. Stal, T. Stefaniak, G. Weiglein and L. Zeune, *MSSM Interpretations of the LHC Discovery: Light or Heavy Higgs?*, Eur. Phys. J. **C73** (2013) 2354 [1211.1955].

[110] P. Bechtle, H.E. Haber, S. Heinemeyer, O. Stål, T. Stefaniak, G. Weiglein et al., *The Light and Heavy Higgs Interpretation of the MSSM*, Eur. Phys. J. **C77** (2017) 67 [1608.00638].

[111] G. Degrassi, S. Heinemeyer, W. Hollik, P. Slavich and G. Weiglein, *Towards high-precision predictions for the MSSM Higgs sector*, Eur. Phys. J. **C28** (2003) 133 [hep-ph/0212020].

[112] H. Bahl, S. Heinemeyer, W. Hollik and G. Weiglein, *Reconciling EFT and hybrid calculations of the light MSSM Higgs-boson mass*, 1706.00346.

[113] S. Heinemeyer, W. Hollik and G. Weiglein, *FeynHiggs: A program for the calculation of the masses of the neutral CP-even Higgs bosons in the MSSM*, Comput. Phys. Commun. **124** (2000) 76 [hep-ph/9812320].

[114] S. Heinemeyer, W. Hollik and G. Weiglein, *The masses of the neutral CP-even Higgs bosons in the MSSM: Accurate analysis at the two-loop level*, Eur. Phys. J. **C9** (1999) 343 [hep-ph/9812472].

[115] M. Frank et al., *The Higgs boson masses and mixings of the complex MSSM in the Feynman-diagrammatic approach*, JHEP **02** (2007) 047 [hep-ph/0611326].

[116] T. Hahn, S. Heinemeyer, W. Hollik, H. Rzehak and G. Weiglein, *High-precision predictions for the light CP-even Higgs Boson Mass of the MSSM*, 1312.4937.

[117] H. Bahl and W. Hollik, *Precise prediction for the light MSSM Higgs boson mass combining effective field theory and fixed-order calculations*, Eur. Phys. J. **C76** (2016) 499 [1608.01880].

[118] H. Bahl, T. Hahn, S. Heinemeyer, W. Hollik, S. Paßehr, H. Rzehak et al., *Precision calculations in the MSSM Higgs-boson sector with FeynHiggs 2.14*, *Comput. Phys. Commun.* **249** (2020) 107099 [1811.09073].

[119] E. Bagnaschi et al., *Supersymmetric Models in Light of Improved Higgs Mass Calculations*, *Eur. Phys. J.* **C79** (2019) 149 [1810.10905].

[120] H. Bahl, S. Heinemeyer, W. Hollik and G. Weiglein, *Theoretical uncertainties in the MSSM Higgs boson mass calculation*, *Eur. Phys. J.* **C80** (2020) 497 [1912.04199].

[121] ATLAS collaboration, *Search for heavy Higgs bosons decaying into two tau leptons with the ATLAS detector using pp collisions at $\sqrt{s} = 13$ TeV*, *Phys. Rev. Lett.* **125** (2020) 051801 [2002.12223].

[122] F. Staub, *SARAH 4 : A tool for (not only SUSY) model builders*, *Comput. Phys. Commun.* **185** (2014) 1773 [1309.7223].

[123] W. Porod, *SPheno, a program for calculating supersymmetric spectra, SUSY particle decays and SUSY particle production at e+ e- colliders*, *Comput. Phys. Commun.* **153** (2003) 275 [hep-ph/0301101].

[124] W. Porod and F. Staub, *SPheno 3.1: Extensions including flavour, CP-phases and models beyond the MSSM*, *Comput. Phys. Commun.* **183** (2012) 2458 [1104.1573].

[125] J. Alwall, R. Frederix, S. Frixione, V. Hirschi, F. Maltoni, O. Mattelaer et al., *The automated computation of tree-level and next-to-leading order differential cross sections, and their matching to parton shower simulations*, *JHEP* **07** (2014) 079 [1405.0301].

[126] Heavy Flavor Averaging Group collaboration, *Averages of b-hadron, c-hadron, and τ-lepton properties*, 1010.1589.

[127] LHCb Collaboration collaboration, *First Evidence for the Decay $B_s^0 \to \mu^+\mu^-$*, *Phys.Rev.Lett.* **110** (2013) 021801 [1211.2674].

[128] A.J. Buras, P. Gambino, M. Gorbahn, S. Jager and L. Silvestrini, *epsilon-prime / epsilon and rare K and B decays in the MSSM*, *Nucl. Phys.* **B592** (2001) 55 [hep-ph/0007313].

[129] LHCb collaboration, *Precision measurement of the B_s^0-$\bar{B}_s^0$ oscillation frequency with the decay $B_s^0 \to D_s^-\pi^+$*, *New J. Phys.* **15** (2013) 053021 [1304.4741].

[130] H. Goldberg, *Constraint on the photino mass from cosmology*, *Phys. Rev. Lett.* **50** (1983) 1419.

[131] J.R. Ellis, J.S. Hagelin, D.V. Nanopoulos, K.A. Olive and M. Srednicki, *Supersymmetric relics from the big bang*, *Nucl. Phys.* **B238** (1984) 453.

[132] Planck collaboration, *Planck 2018 results. VI. Cosmological parameters*, *Astron. Astrophys.* **641** (2020) A6 [1807.06209].

[133] G. Belanger, F. Boudjema, A. Pukhov and A. Semenov, *micrOMEGAs: A program for calculating the relic density in the MSSM*, *Comput. Phys. Commun.* **149** (2002) 103 [hep-ph/0112278].

[134] G. Belanger, F. Boudjema, A. Pukhov and A. Semenov, *MicrOMEGAs: Version 1.3*, *Comput. Phys. Commun.* **174** (2006) 577 [hep-ph/0405253].

[135] D. Barducci, G. Belanger, J. Bernon, F. Boudjema, J. Da Silva, S. Kraml et al., *Collider limits on new physics within micrOMEGAs_4.3*, *Comput. Phys. Commun.* **222** (2018) 327 [1606.03834].

[136] S. Heinemeyer, M. Mondragon and G. Zoupanos, *Finite Theories after the discovery of a Higgs-like boson at the LHC*, *Phys.Lett.* **B718** (2013) 1430 [1211.3765].

[137] S. Heinemeyer, M. Mondragon and G. Zoupanos, *Futs and the Higgs-boson*, *Int.J.Mod.Phys.Conf.Ser.* **13** (2012) 118.

[138] S. Heinemeyer, M. Mondragon and G. Zoupanos, *Finite unified theories and their predicitions*, *Phys.Part.Nucl.* **44** (2013) 299.

[139] S. Heinemeyer, M. Mondragón, G. Patellis, N. Tracas and G. Zoupanos, *The LHC Higgs Boson Discovery: Updated implications for Finite Unified Theories and the SUSY breaking scale*, *Symmetry* **10** (2018) 62 [1802.04666].

[140] S. Heinemeyer, M. Mondragon, G. Patellis, N. Tracas and G. Zoupanos, *The Higgs boson discovery: recent implications for the Finite Unified Theories and SUSY breaking scale*, *PoS* **CORFU2017** (2018) 081.

[141] S. Heinemeyer, M. Mondragon, G. Patellis, N. Tracas and G. Zoupanos, *Reduction of Couplings: Finite Unified Theories and the reduced MSSM*, *PoS* **CORFU2018** (2019) 077.

[142] S. Heinemeyer, M. Mondragón, G. Patellis, N. Tracas and G. Zoupanos, *New Results in Theories with Reduced Couplings*, *PoS* **CORFU2019** (2020) 066.

[143] S. Heinemeyer, M. Mondragon and G. Zoupanos, *Finite Unification: Theory and Predictions*, *SIGMA* **6** (2010) 049 [1001.0428].

[144] S. Heinemeyer, M. Mondragón, N. Tracas and G. Zoupanos, *Reduction of couplings in a finite GUT and the MSSM*, *Phys. Part. Nucl. Lett.* **11** (2014) 910.

[145] S. Heinemeyer, M. Mondragon and G. Zoupanos, *Finite Theories Before and After the Discovery of a Higgs Boson at the LHC*, *Fortsch. Phys.* **61** (2013) 969 [1305.5073].

[146] S. Heinemeyer, M. Mondragón, N. Tracas and G. Zoupanos, *Reduction of parameters in Finite Unified Theories and the MSSM*, *Nucl. Phys.* **B927** (2018) 319.

[147] M. Mondragon and G. Zoupanos, *Finite unified theories: Theoretical basis and phenomenological implications*, *Phys. Part. Nucl. Lett.* **8** (2011) 173.

[148] J.W.F. Valle, *Neutrino mass: Theory, data and interpretation*, hep-ph/9907222.

[149] S. Heinemeyer, E. Ma, M. Mondragon and G. Zoupanos, *Finite SU(3)**3 model*, *AIP Conf. Proc.* **1200** (2010) 568 [0910.0501].

[150] S. Heinemeyer, E. Ma, M. Mondragon and G. Zoupanos, *Finite unification: Phenomenology*, *J. Phys. Conf. Ser.* **259** (2010) 012097.

[151] S. Heinemeyer, W. Hollik and G. Weiglein, *FeynHiggs: A Program for the calculation of the masses of the neutral CP even Higgs bosons in the MSSM*, Comput. Phys. Commun. **124** (2000) 76 [hep-ph/9812320]; S. Heinemeyer, W. Hollik and G. Weiglein, *The Masses of the neutral CP - even Higgs bosons in the MSSM: Accurate analysis at the two loop level*, Eur. Phys. J. C **9** (1999) 343 [hep-ph/9812472]; M. Frank, T. Hahn, S. Heinemeyer, W. Hollik, H. Rzehak and G. Weiglein, *The Higgs Boson Masses and Mixings of the Complex MSSM in the Feynman-Diagrammatic Approach*, JHEP **0702** (2007) 047 [hep-ph/0611326]; T. Hahn, S. Heinemeyer, W. Hollik, H. Rzehak and G. Weiglein, *FeynHiggs: A program for the calculation of MSSM Higgs-boson observables - Version 2.6.5*, Comput. Phys. Commun. **180** (2009) 1426. T. Hahn, S. Heinemeyer, W. Hollik, H. Rzehak and G. Weiglein, *High-Precision Predictions for the Light CP -Even Higgs Boson Mass of the Minimal Supersymmetric Standard Model*, Phys. Rev. Lett. **112** (2014) no.14, 141801 [arXiv:1312.4937 [hep-ph]]; H. Bahl and W. Hollik, Eur. Phys. J. C **76** (2016) no.9, 499 [arXiv:1608.01880 [hep-ph]]; H. Bahl, T. Hahn, S. Heinemeyer, W. Hollik, S. Paßehr, H. Rzehak and G. Weiglein, arXiv:1811.09073 [hep-ph]; See http://www.feynhiggs.de.

[152] M. Misiak *et al.*, Phys. Rev. Lett. **98** (2007) 022002 [hep-ph/0609232];
M. Ciuchini, G. Degrassi, P. Gambino and G. F. Giudice, Nucl. Phys. B **534** (1998) 3 [hep-ph/9806308];
G. Degrassi, P. Gambino and G. F. Giudice, JHEP **0012** (2000) 009 [hep-ph/0009337];
M. Carena, D. Garcia, U. Nierste and C. E. M. Wagner, Phys. Lett. B **499** (2001) 141 [hep-ph/0010003];
G. D'Ambrosio, G. F. Giudice, G. Isidori and A. Strumia, Nucl. Phys. B **645** (2002) 155 [hep-ph/0207036].

[153] C. Bobeth, M. Gorbahn, T. Hermann, M. Misiak, E. Stamou and M. Steinhauser, Phys. Rev. Lett. **112** (2014) 101801 [arXiv:1311.0903 [hep-ph]];
T. Hermann, M. Misiak and M. Steinhauser, JHEP **1312** (2013) 097 [arXiv:1311.1347 [hep-ph]];
C. Bobeth, M. Gorbahn and E. Stamou, Phys. Rev. D **89** (2014) no.3, 034023 [arXiv:1311.1348 [hep-ph]].

[154] A. J. Buras, *Relations between $\Delta M(s, d)$ and $B(s, d) \to \mu\bar{\mu}$ in models with minimal flavor violation*, Phys. Lett. B **566** (2003) 115 [hep-ph/0303060]; G. Isidori and D. M. Straub, *Minimal Flavour Violation and Beyond*, Eur. Phys. J. C **72** (2012) 2103 [arXiv:1202.0464 [hep-ph]].

[155] CMS and LHCb Collaborations [CMS and LHCb Collaborations], *Combination of results on the rare decays $B^0_{(s)} \to \mu^+\mu^-$ from the CMS and LHCb experiments*, CMS-PAS-BPH-13-007, LHCb-CONF-2013-012, CERN-LHCb-CONF-2013-012.

[156] G. Isidori and P. Paradisi, *Hints of large tan(beta) in flavour physics*, Phys. Lett. B **639** (2006) 499 [hep-ph/0605012]; G. Isidori, F. Mescia, P. Paradisi and D. Temes, *Flavour physics at large tan(beta) with a Bino-like LSP*, Phys. Rev. D **75** (2007) 115019 [hep-ph/0703035 [hep-ph]].

[157] K. A. Olive *et al.* [Particle Data Group], *Review of Particle Physics*, Chin. Phys. C **38** (2014) 090001.

[158] D. Asner *et al.* [Heavy Flavor Averaging Group], *Averages of b-hadron, c-hadron, and τ-lepton properties*, arXiv:1010.1589 [hep-ex].

[159] H. Goldberg, *Constraint on the Photino Mass from Cosmology*, Phys. Rev. Lett. **50** (1983) 1419 Erratum: [Phys. Rev. Lett. **103** (2009) 099905]; J. R. Ellis, J. S. Hagelin, D. V. Nanopoulos, K. A. Olive and M. Srednicki, *Supersymmetric Relics from the Big Bang*, Nucl. Phys. B **238** (1984) 453.

[160] M. A. Diaz, M. Hirsch, W. Porod, J. C. Romao and J. W. F. Valle, *Solar neutrino masses and mixing from bilinear R parity broken supersymmetry: Analytical versus numerical results*, Phys. Rev. D **68** (2003) 013009 Erratum: [Phys. Rev. D **71** (2005) 059904] [hep-ph/0302021].

[161] H. K. Dreiner, *An Introduction to explicit R-parity violation*, Adv. Ser. Direct. High Energy Phys. **21** (2010) 565 [hep-ph/9707435]; G. Bhattacharyya, *A Brief review of R-parity violating couplings*, In *Tegernsee 1997, Beyond the desert 1997* 194-201 [hep-ph/9709395]; B. C. Allanach, A. Dedes and H. K. Dreiner, *Bounds on R-parity violating couplings at the weak scale and at the GUT scale*, Phys. Rev. D **60** (1999) 075014 [hep-ph/9906209]; J. C. Romao and J. W. F. Valle, *Neutrino masses in supersymmetry with spontaneously broken R parity*, Nucl. Phys. B **381** (1992) 87.

[162] https://twiki.cern.ch/twiki/bin/view/AtlasPublic/SupersymmetryPublicResults, https://twiki.cern.ch/twiki/bin/view/CMSPublic/PhysicsResultsSUS

[163] D. H. Lyth and E. D. Stewart, *Thermal inflation and the moduli problem*, Phys. Rev. D **53** (1996) 1784 [hep-ph/9510204].

[164] G. B. Gelmini and P. Gondolo, *Neutralino with the right cold dark matter abundance in (almost) any supersymmetric model*, Phys. Rev. D **74** (2006) 023510 [hep-ph/0602230].

[165] CMS-PAS-FTR-18-017, 2018;
M. Cepeda *et al. Report from Working Group 2: Higgs Physics at the HL-LHC and HE-LHC,'* CERN Yellow Rep. Monogr. **7** (2019), 221-584 [arXiv:1902.00134 [hep-ph]].

[166] M. Mangano, *Physics at the FCC-hh, a 100 TeV pp collider*, CERN Yellow Report CERN 2017-003-M [arXiv:1710.06353 [hep-ph]].

[167] N. Craig, J. Hajer, Y. Y. Li, T. Liu and H. Zhang, *Heavy Higgs bosons at low* $\tan\beta$*: from the LHC to 100 TeV,'* JHEP **01** (2017), 018 [arXiv:1605.08744 [hep-ph]].

[168] J. Hajer, Y. Y. Li, T. Liu and J. F. H. Shiu, *Heavy Higgs Bosons at 14 TeV and 100 TeV,'* JHEP **11** (2015), 124 [arXiv:1504.07617 [hep-ph]].

[169] T. Golling *et al. Physics at a 100 TeV pp collider: beyond the Standard Model phenomena,* CERN Yellow Rep. (2017) no.3, 441-634 [arXiv:1606.00947 [hep-ph]].

[170] S. Chatrchyan *et al.* [CMS Collaboration], *Measurement of the B(s) to mu+ mu- branching fraction and search for B0 to mu+ mu- with the CMS Experiment*, Phys. Rev. Lett. **111** (2013) 101804 [arXiv:1307.5025 [hep-ex]].

[171] R. L. Workman *et al.* [Particle Data Group], PTEP **2022** (2022), 083C01 doi:10.1093/ptep/ptac097