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Holography in anomaly flow in orbifold gauge theory

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In orbifold gauge theory and gauge-Higgs unification models, gauge anomaly flows with an Aharonov-Bohm phase θ_H in the fifth dimension. We analyze $SU(2)$ gauge theory with doublet fermions in the flat $M^4 \times (S^1/Z_2)$ spacetime and in the Randall-Sundrum (RS) warped space. With orbifold boundary conditions the $U(1)$ part of gauge symmetry remains unbroken at $\theta_H = 0$ and π. Chiral anomalies smoothly vary with θ_H in the RS space. Anomaly coefficients associated with this anomaly flow are expressed in terms of the values of the wave functions of gauge fields at the UV and IR branes in the RS space and parity conditions of fermion fields. Holography in anomaly flow is observed. Conditions for the anomaly cancellation turn out independent of θ_H.

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1. Introduction

Chiral fermions generally induce chiral anomaly.[1–3] Even for massive fermions chiral anomaly can be generated if gauge couplings are not purely vector-like. Something special happens in gauge-Higgs unification (GHU), in which gauge symmetry is dynamically broken by an Aharonov-Bohm (AB) phase θ_H in the fifth dimension.[4–6] It has been noticed in the GUT inspired $SO(5) \times U(1)_X \times SU(3)_C$ GHU in the Randall-Sundrum (RS) warped space [7] that quarks and leptons are massless and purely chiral at $\theta_H = 0$, become massive at $\theta_H \neq 0$, and smoothly become vector-like at $\theta_H = \pi$. Gauge bosons such as W and Z bosons are massless gauge bosons of $SU(2)_L \times U(1)_Y$ at $\theta_H = 0$, become massive at $\theta_H \neq 0$, and smoothly converted to massless gauge bosons of $SU(2)_R \times U(1)_{Y'}$ at $\theta_H = \pi$. This prompts the following question. What happens to the anomaly generated by chiral quarks and leptons at $\theta_H = 0$? Does it disappear at $\theta_H = \pi$? What is the fate of chiral anomaly?

To pin down what is going on in GHU formulated on orbifolds, the dependence of chiral anomaly on the AB phase θ_H in $SU(2)$ gauge theory with doublet fermions in the flat $M^4 \times (S^1/Z_2)$ space and in the RS warped space has been investigated in Refs. [8, 9]. In the flat $M^4 \times (S^1/Z_2)$ space the mass spectrum of the Kaluza-Klein (KK) modes of gauge bosons and fermions changes linearly in θ_H, namely as $m_n = |n + (\theta_H/\pi)|/R$ or $|n + (\theta_H/2\pi)|/R$, so that the level crossing takes place where θ_H is a multiple of $\frac{1}{2}\pi$ or π. In the RS space there occurs no level crossing in the mass spectrum. The spectrum varies smoothly as θ_H changes from 0 to 2π. The lowest mode remains as the lowest mode for any θ_H. Gauge couplings of right- and left-handed modes of fermions vary smoothly as θ_H, and the magnitude of chiral anomaly also changes as θ_H. The anomaly coefficient (defined below) coming from one doublet fermion changes from 2 at $\theta_H = 0$ to 0 at $\theta_H = \pi$. The anomaly flows with the AB phase θ_H.

Furthermore it was shown that the magnitude of the total anomaly evaluated by summing contributions coming from all KK modes of fermions running along internal loops is expressed in terms of the values of the wave functions of the gauge fields at the two branes (the UV and IR branes in the RS space) and the parity conditions of fermion fields at the two branes. Each fermion field in the RS space is characterized by its own bulk mass parameter c which controls its mass and wave function at general θ_H. Although the anomaly coming from each KK mode depends on the bulk mass parameter c, the total anomaly does not depend on c. There emerges a holographic formula for the total anomaly. This holography becomes crucial to have the cancellation of gauge anomalies in GHU.

2. $SU(2)$ GHU in $M^4 \times (S^1/Z_2)$

Let us first consider $SU(2)$ GHU in the flat $M^4 \times (S^1/Z_2)$ spacetime with coordinate x^M ($M = 0, 1, 2, 3, 5$, $x^5 = y$) whose action is given by

$$I_{\rm flat} = \int d^4x \int_0^L dy\, \mathcal{L}_{\rm flat}\,, \quad \mathcal{L}_{\rm flat} = -\frac{1}{2} {\rm Tr}\, F_{MN} F^{MN} + \overline{\Psi} \gamma^M (\partial_M - i g_A A_M) \Psi\,, \tag{2.1}$$

where $\mathcal{L}_{\rm flat}(x^\mu, y) = \mathcal{L}_{\rm flat}(x^\mu, y + 2L) = \mathcal{L}_{\rm flat}(x^\mu, -y)$. $A_M = \frac{1}{2} \sum_{a=1}^{3} A_M^a \tau^a$ and $F_{MN} = \partial_M A_N - \partial_N A_N - i g_A [A_M, A_N]$ where τ^a's are Pauli matrices. Orbifold boundary conditions are given, with $(y_0, y_1) = (0, L)$ and $P_0 = P_1 = \tau^3$, by

$$\begin{pmatrix} A_\mu \\ A_y \end{pmatrix} (x, y_j - y) = P_j \begin{pmatrix} A_\mu \\ -A_y \end{pmatrix} (x, y_j + y) P_j^{-1} ,$$

$$\Psi(x, y_j - y) = \begin{cases} +P_j \gamma^5 \Psi(x, y_j + y) & \text{type 1A} \\ -P_j \gamma^5 \Psi(x, y_j + y) & \text{type 1B} \\ (-1)^j P_j \gamma^5 \Psi(x, y_j + y) & \text{type 2A} \\ (-1)^{j+1} P_j \gamma^5 \Psi(x, y_j + y) & \text{type 2B} \end{cases} . \tag{2.2}$$

The $SU(2)$ symmetry is broken to $U(1)$. A_μ^3 and $A_y^{1,2}$ are parity even at both y_0 and y_1. The zero mode of A_μ^3 is the 4D $U(1)$ gauge field. The 4D gauge coupling is given by $g_4 = g_A/\sqrt{L}$. The zero modes of $A_y^{1,2}$ may develop nonvanishing expectation values. Without loss of generality we assume $\langle A_y^1 \rangle = 0$. An AB phase θ_H along the fifth dimension is then given by

$$P \exp \left\{ i g_A \int_0^{2L} dy \, \langle A_y \rangle \right\} = e^{i\theta_H \tau^2} , \quad \theta_H = g_4 L \, \langle A_y^2 \rangle . \tag{2.3}$$

When $\theta_H \neq 0$, A_μ^1 and A_μ^3 intertwine with each other. It is straightforward to find mass eigenstates. The KK expansion is given by

$$\begin{pmatrix} A_\mu^1(x, y) \\ A_\mu^3(x, y) \end{pmatrix} = \sum_{n=-\infty}^{\infty} B_\mu^{(n)}(x) \frac{1}{\sqrt{\pi R}} \begin{pmatrix} \sin(ny/R) \\ \cos(ny/R) \end{pmatrix} \tag{2.4}$$

where $R = L/\pi$. The mass of the $B_\mu^{(n)}(x)$ mode is $m_n(\theta_H) = R^{-1} \left| n + \frac{\theta_H}{\pi} \right|$. The spectrum is periodic in θ_H with a period π. The KK expansion of a doublet fermion $\Psi = (u, d)^t$ of type 1A is given by

$$\begin{pmatrix} u_R(x, y) \\ d_R(x, y) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \psi_R^{(n)}(x) \frac{1}{\sqrt{\pi R}} \begin{pmatrix} \cos(ny/R) \\ \sin(ny/R) \end{pmatrix} ,$$
$$\begin{pmatrix} u_L(x, y) \\ d_L(x, y) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \psi_L^{(n)}(x) \frac{1}{\sqrt{\pi R}} \begin{pmatrix} -\sin(ny/R) \\ \cos(ny/R) \end{pmatrix} . \tag{2.5}$$

$\psi_R^{(n)}$ and $\psi_L^{(n)}$ combine to form the $\psi^{(n)}(x)$ mode with a mass given by $m_n(\theta_H) = R^{-1} \left| n + \frac{\theta_H}{2\pi} \right|$. The spectrum is periodic in θ_H with a period 2π. Note that the KK mass scale in the flat space is $m_{\rm KK}^{\rm flat} = 1/R$.

3. $SU(2)$ GHU in the Randall-Sundrum warped space

The metric of the RS warped space is given by [10]

$$ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \tag{3.1}$$

where $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$, $\sigma(y) = \sigma(y + 2L) = \sigma(-y)$ and $\sigma(y) = ky$ for $0 \le y \le L$. It has the same topology as $M^4 \times (S^1/Z_2)$. In the region $0 \le y \le L$ the metric can be written, in terms of the conformal coordinate $z = e^{ky}$, as

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + \frac{dz^2}{k^2}\right) \quad (1 \le z \le z_L = e^{kL}) . \tag{3.2}$$

z_L is called the warp factor of the RS space. The RS space is an anti-de Sitter (AdS) space sandwiched by the UV brane at $y = 0$ ($z = 1$) and the IR brane at $y = L$ ($z = z_L$). The AdS curvature is given by $\Lambda = -6k^2$.

The action in the RS space is given by

$$\begin{aligned}
I_{\rm RS} &= \int d^5x\sqrt{-\det G}\,\mathcal{L}_{\rm RS} , \quad \mathcal{L}_{\rm RS} = -\frac{1}{2}{\rm Tr}\,F_{MN}F^{MN} + \overline{\Psi}\mathcal{D}(c)\Psi ,\\
\mathcal{D}(c) &= \gamma^A e_A{}^M\Big(\partial_M - ig_A A_M + \frac{1}{8}\omega_{MBC}[\gamma^B,\gamma^C]\Big) - c\,\sigma' .
\end{aligned}\tag{3.3}$$

c is a dimensionless bulk mass parameter. Note $\mathcal{L}_{\rm RS}(x^\mu, y) = \mathcal{L}_{\rm RS}(x^\mu, -y) = \mathcal{L}_{\rm RS}(x^\mu, y + 2L)$. Fields A_M and Ψ satisfy the same boundary conditions (2.2) as in the flat spacetime. The AB phase θ_H and the KK mass scale $m_{\rm KK}$ are given by

$$\theta_H = \frac{\langle A_z^{2(0)}\rangle}{f_H} , \quad f_H = \frac{1}{g_4}\sqrt{\frac{2k}{L(z_L^2 - 1)}} , \quad m_{\rm KK} = \frac{\pi k}{z_L - 1} , \tag{3.4}$$

where $A_z^a(x,z) = k^{-1/2}\sum A_z^{a(n)}(x)v_n(z)$ and $v_0(z) = \sqrt{2/(z_L^2 - 1)}\,z$.

The KK expansion of the gauge fields A_μ^1 and A_μ^3 is given by

$$\begin{aligned}
&\begin{pmatrix} A_\mu^1(x,y)\\ A_\mu^3(x,y)\end{pmatrix} = \frac{1}{\sqrt{L}}\sum_{n=0}^\infty Z_\mu^{(n)}(x)\begin{pmatrix} h_n(y)\\ k_n(y)\end{pmatrix},\\
&\begin{pmatrix} h_n(y)\\ k_n(y)\end{pmatrix} = \begin{pmatrix} -h_n(-y)\\ k_n(-y)\end{pmatrix} = \begin{pmatrix} h_n(y+2L)\\ k_n(y+2L)\end{pmatrix}\\
&= \begin{pmatrix} \cos\theta(z) & \sin\theta(z)\\ -\sin\theta(z) & \cos\theta(z)\end{pmatrix}\begin{pmatrix} \tilde{h}_n(z)\\ \tilde{k}_n(z)\end{pmatrix}, \quad \theta(z) = \theta_H\,\frac{z_L^2 - z^2}{z_L^2 - 1} \quad \text{for } 0 \le y \le L .
\end{aligned}\tag{3.5}$$

The mass spectrum $\{m_n = k\lambda_n;\ \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ of the KK modes $\big\{Z_\mu^{(n)}(x)\big\}$ is determined by the zeros of

$$Z_\mu^{(n)} :\ SC'(1;\lambda_n) + \lambda_n \sin^2\theta_H = 0 \tag{3.6}$$

where $S(z;\lambda)$ and $C(z;\lambda)$ are expressed in terms of Bessel functions and are given by (A.1). The wave functions $\tilde{\mathbf{h}}_n(z) \equiv (\tilde{h}_n(z), \tilde{k}_n(z))^t$ are given by (B.1).

For fermion fields we define $\check{\Psi}(x,z) = z^{-2}\,\Psi(x,z)$ for $1 \le z \le z_L$. The KK expansion of $\check{\Psi} = (\check{u}, \check{d})^t$ of type 1A is given by

$$\begin{pmatrix} \check{u}_R(x,y)\\ \check{d}_R(x,y)\end{pmatrix} = \sqrt{k}\sum_{n=0}^\infty \chi_R^{(n)}(x)\begin{pmatrix} f_{Rn}(y)\\ g_{Rn}(y)\end{pmatrix},$$

$$\begin{pmatrix} \check{u}_L(x,y) \\ \check{d}_L(x,y) \end{pmatrix} = \sqrt{k} \sum_{n=0}^{\infty} \chi_L^{(n)}(x) \begin{pmatrix} f_{Ln}(y) \\ g_{Ln}(y) \end{pmatrix}, \tag{3.7}$$

where

$$\begin{aligned}
\begin{pmatrix} f_{Rn}(y) \\ g_{Rn}(y) \end{pmatrix} &= \begin{pmatrix} f_{Rn}(-y) \\ -g_{Rn}(-y) \end{pmatrix} = \begin{pmatrix} f_{Rn}(y+2L) \\ g_{Rn}(y+2L) \end{pmatrix} \\
&= \begin{pmatrix} \cos\frac{1}{2}\theta(z) & -\sin\frac{1}{2}\theta(z) \\ \sin\frac{1}{2}\theta(z) & \cos\frac{1}{2}\theta(z) \end{pmatrix} \begin{pmatrix} \tilde{f}_{Rn}(z) \\ \tilde{g}_{Rn}(z) \end{pmatrix} \quad \text{for } 0 \le y \le L\,, \\
\begin{pmatrix} f_{Ln}(y) \\ g_{Ln}(y) \end{pmatrix} &= \begin{pmatrix} -f_{Ln}(-y) \\ g_{Ln}(-y) \end{pmatrix} = \begin{pmatrix} f_{Ln}(y+2L) \\ g_{Ln}(y+2L) \end{pmatrix} \\
&= \begin{pmatrix} \cos\frac{1}{2}\theta(z) & -\sin\frac{1}{2}\theta(z) \\ \sin\frac{1}{2}\theta(z) & \cos\frac{1}{2}\theta(z) \end{pmatrix} \begin{pmatrix} \tilde{f}_{Ln}(z) \\ \tilde{g}_{Ln}(z) \end{pmatrix} \quad \text{for } 0 \le y \le L\,.
\end{aligned} \tag{3.8}$$

$\chi_R^{(n)}(x)$ and $\chi_L^{(n)}(x)$ combine to form a massive mode $\chi^{(n)}(x)$ for $\theta_H \neq 0$. The mass spectrum $\{m_n = k\lambda_n;\ \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ of the KK modes $\{\chi^{(n)}(x)\}$ is determined by the zeros of

$$\chi^{(n)}:\ S_L S_R(1;\lambda_n,c) + \sin^2\tfrac{1}{2}\theta_H = 0 \tag{3.9}$$

where $S_L(z;\lambda,c)$ and $S_R(z;\lambda,c)$ are given by (A.3). The wave functions $\tilde{\mathbf{f}}_{Rn}(z) = (\tilde{f}_{Rn}(z), \tilde{g}_{Rn}(z))^t$ and $\tilde{\mathbf{f}}_{Ln}(z) = (\tilde{f}_{Ln}(z), \tilde{g}_{Ln}(z))^t$ are given by (B.2).

The mass spectra as functions of θ_H in the flat $M^4 \times (S^1/Z_2)$ space and in the RS warped space are depicted in Fig. 1. In the flat space the mass spectrum of each field changes linearly in θ_H so that the level crossing occurs. In the RS space there is no level crossing so that physical quantities change smoothly as θ_H. It is expected that in the flat space something singular may occur at $\theta_H = 0, \frac{1}{2}\pi, \pi, \cdots$. This is exactly what is going to happen in the anomaly as is seen below.

4. Gauge couplings and anomaly in flat space

Gauge couplings of fermions in the flat space are easily found by inserting the KK expansions (2.4) and (2.5) into the action (2.1). One finds that the $B_\mu^{(n)}$ couplings of the fermion fields are given by

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} B_\mu^{\langle n\rangle}\, j^\mu_{(n)} \\
&= \frac{g_4}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} B_\mu^{\langle n\rangle} \Big\{ s^R_{nm\ell}\, \psi_R^{\langle m\rangle\dagger} \bar{\sigma}^\mu \psi_R^{\langle \ell\rangle} + s^L_{nm\ell}\, \psi_L^{\langle m\rangle\dagger} \sigma^\mu \psi_L^{\langle \ell\rangle} \Big\}\,, \\
&\qquad s^R_{nm\ell} = s^L_{nm\ell} = \delta_{n,m+\ell}\,.
\end{aligned} \tag{4.1}$$

Here we have adopted the two-component notation; $\sigma^\mu = (I_2, \vec{\sigma})$ and $\bar{\sigma}^\mu = (-I_2, \vec{\sigma})$. Chiral anomaly in $\partial_\mu j^\mu_{(n)}$ arises from triangle diagrams in which various combinations of $\psi_{R/L}^{\langle \ell\rangle}$ run;

$$\partial_\mu j^\mu_{(n)} + \cdots = -\Big(\frac{g_4}{2}\Big)^3 \sum_\ell \sum_m \frac{b_{n\ell m}}{16\pi^2}\, B^{(\ell)}_{\mu\nu} \tilde{B}^{(m)\mu\nu} \tag{4.2}$$

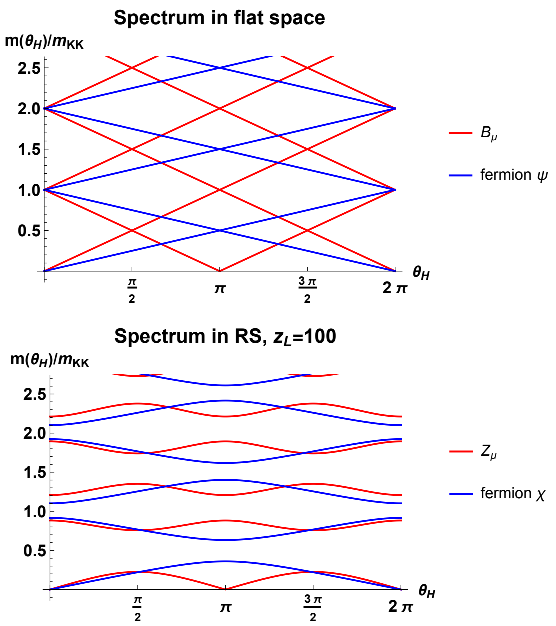

Figure 1: (Top): The mass spectrum of gauge fields $B_\mu^{(n)}$ and fermion fields $\psi^{(n)}$ (type 1A) in the flat $M^4 \times (S^1/Z_2)$ space is displayed. The level crossing occurs at $\theta_H = 0, \frac{1}{2}\pi, \pi, \cdots$. (Bottom): The mass spectrum of gauge fields $Z_\mu^{(n)}$ and fermion fields $\chi^{(n)}$ (type 1A) in the RS warped space is displayed. The warp factor is $z_L = 100$ and the bulk mass parameter of Ψ is $c = 0.25$.

where $B_{\mu\nu}^{(\ell)} = \partial_\mu B_\nu^{(\ell)} - \partial_\nu B_\mu^{(\ell)}$. The anomaly coefficient $b_{n\ell m}$ is found to be

$$
\begin{aligned}
b_{n_1 n_2 n_3} &= \sum_{m,\ell,p} \Big\{ s^R_{n_1 m \ell} s^R_{n_2 \ell p} s^R_{n_3 p m} + s^L_{n_1 m \ell} s^L_{n_2 \ell p} s^L_{n_3 p m} \Big\} \\
&= \begin{cases} 2 & \text{for } n_1 + n_2 + n_3 = \text{even} \\ 0 & \text{for } n_1 + n_2 + n_3 = \text{odd} \end{cases} .
\end{aligned} \tag{4.3}
$$

Chiral anomalies arise even for $j^\mu_{(n\neq 0)}$. A few examples are shown in Fig. 2.

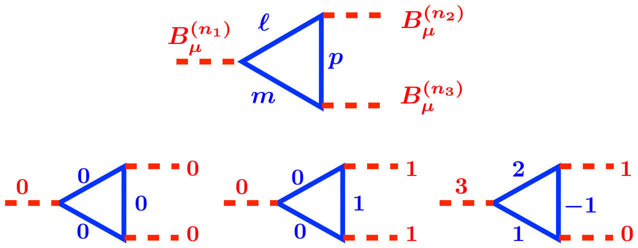

Figure 2: Chiral anomaly in the flat space.

5. Gauge couplings and anomaly in RS

Similarly gauge couplings of fermions in the RS space are expressed as

$$\sum_{n=0}^{\infty} Z_\mu^{\langle n \rangle} \, j_{(n)}^\mu = \frac{g_4}{2} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} Z_\mu^{(n)} \Big\{ t_{n\ell m}^R \, \chi_R^{(\ell)\dagger} \bar\sigma^\mu \chi_R^{(m)} + t_{n\ell m}^L \, \chi_L^{(\ell)\dagger} \sigma^\mu \chi_L^{(m)} \Big\} \,. \tag{5.1}$$

The couplings $t_{n\ell m}^R$ and $t_{n\ell m}^L$ are more involved. They are given, in terms of the wave functions in (3.5) and (3.8), by

$$\begin{aligned} t_{n\ell m}^R &= \frac{k}{2} \int_{-L}^{L} dy \, e^{k|y|} \Big\{ h_n \big(f_{R\ell}^* g_{Rm} + g_{R\ell}^* f_{Rm} \big) + k_n \big(f_{R\ell}^* f_{Rm} - g_{R\ell}^* g_{Rm} \big) \Big\}, \\ t_{n\ell m}^L &= -\frac{k}{2} \int_{-L}^{L} dy \, e^{k|y|} \Big\{ h_n \big(f_{L\ell}^* g_{Lm} + g_{L\ell}^* f_{Lm} \big) + k_n \big(f_{L\ell}^* f_{Lm} - g_{L\ell}^* g_{Lm} \big) \Big\}. \end{aligned} \tag{5.2}$$

Note that $t_{n\ell m}^{R/L}$ depends on θ_H, z_L and c. (It does not depend on k or $m_{\rm KK}$.)

Chiral anomaly in $\partial_\mu j_{(n)}^\mu$ is written as

$$\partial_\mu j_{(n)}^\mu + \cdots = -\Big(\frac{g_4}{2}\Big)^3 \sum_{\ell,m=0}^{\infty} \frac{a_{n\ell m}}{16\pi^2} \, Z_{\mu\nu}^{(\ell)} \tilde Z^{(m)\mu\nu} \tag{5.3}$$

where $Z_{\mu\nu}^{(\ell)} = \partial_\mu Z_\nu^{(\ell)} - \partial_\nu Z_\mu^{(\ell)}$. The anomaly coefficient $a_{n\ell m}$ is found to be

$$\begin{aligned} a_{n_1 n_2 n_3} &= a_{n_1 n_2 n_3}^R + a_{n_1 n_2 n_3}^L \,, \\ a_{n_1 n_2 n_3}^{R/L} &= \sum_{m,\ell,p=0}^{\infty} t_{n_1 m \ell}^{R/L} \, t_{n_2 \ell p}^{R/L} \, t_{n_3 p m}^{R/L} \,. \end{aligned} \tag{5.4}$$

As the couplings $t_{n\ell m}^R$ and $t_{n\ell m}^L$ depend on θ_H, $a_{n\ell m}^R$ and $a_{n\ell m}^L$ also do depend on θ_H. In the RS space the dependence is smooth. For instance, t_{000}^R, t_{000}^L, a_{000}^R, a_{000}^L and a_{000} for $z_L = 10$ and $c = 0.25$ are depicted in Fig. 3. It is seen that a_{000} changes from 2 to 0 as θ_H varies from 0 to π.

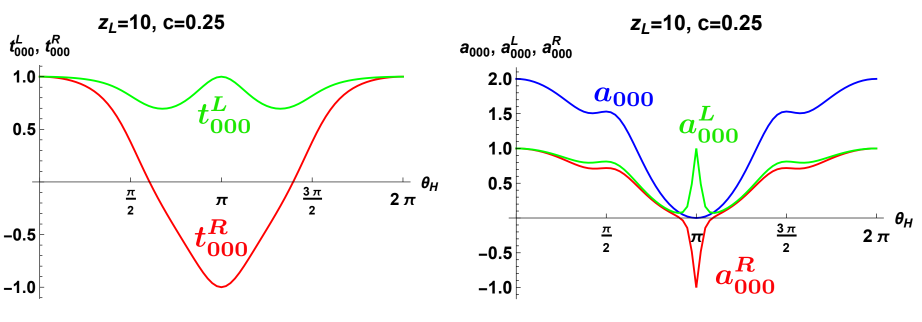

Figure 3: Gauge couplings $t_{000}^{R/L}$ and anomaly coefficients $a_{000}^{R/L}$ and a_{000} for $z_L = 10$ and $c = 0.25$ are shown as functions of θ_H. $a_{000}^{R/L}$ and a_{000} are evaluated by taking account of $t_{0m\ell}^{R/L}$ $(0 \le m, \ell \le 14)$.

6. Anomaly flow

In the flat space the anomaly coefficients $b_{nm\ell}$ in (4.3) are constant, whereas the anomaly coefficients $a_{nm\ell}$ in (5.4) in the RS space depend on θ_H. There is no contradiction between these two facts. Look at the mass spectrum in Fig. 1. In the RS space the lowest mode of the gauge field is always $Z_\mu^{(0)}$ irrespective of the value of θ_H. In the flat space the lowest mode is $B_\mu^{(0)}$ for $0 < \theta_H < \frac{1}{2}\pi$, $B_\mu^{(-1)}$ for $\frac{1}{2}\pi < \theta_H < \frac{3}{2}\pi$, and $B_\mu^{(-2)}$ for $\frac{3}{2}\pi < \theta_H < 2\pi$. The anomaly coefficient b_{nnn} is $+2$ for $n = 0$ and -2, but is 0 for $n = -1$. As the AdS curvature approaches 0, that is, as $k \to 0$, the RS space becomes the flat $M^4 \times (S^1/Z_2)$ space. In other words, a_{000} must flow from 2 to 0 to 2 as θ_H changes from 0 to π to 2π. **The anomaly flows as the AB phase θ_H varies.** In the flat space the behavior of the anomaly becomes singular at the points of the level crossing, namely at $\theta_H = \frac{1}{2}\pi$ and $\frac{3}{2}\pi$.

The phenomenon of the anomaly flow is seen in all anomaly coefficients $a_{nm\ell}$. In Fig. 4 the anomaly coefficients $a_{012}^{R/L}$, a_{012}, $a_{222}^{R/L}$ and a_{222} are plotted for $z_L = 10$ and $c = 0.25$. The anomaly flow is smooth.

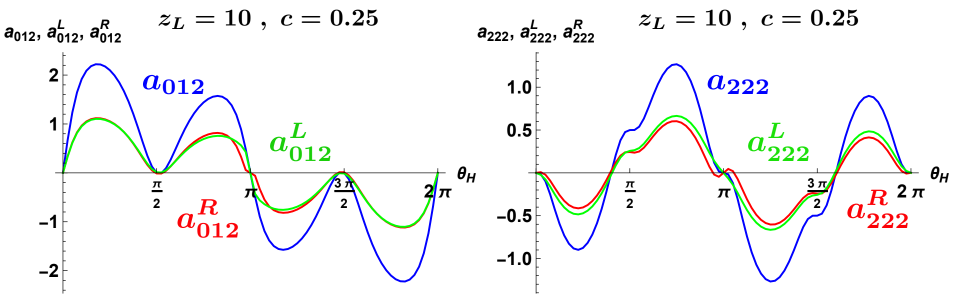

Figure 4: Anomaly coefficients $a_{012}^{R/L}$, a_{012}, $a_{222}^{R/L}$ and a_{222} $a_{012}^{R/L}$, a_{012}, $a_{222}^{R/L}$ and a_{222} for $z_L = 10$ and $c = 0.25$ are shown as functions of θ_H. The coefficients are evaluated by taking account of $t_{jm\ell}^{R/L}$ ($j = 0, 1, 2$, $0 \le m, \ell \le 14$).

One might wonder how the anomaly flow in the RS space reduces to the anomaly flow in the flat space which seems singular at $\theta_H = \frac{1}{2}\pi$ and $\frac{3}{2}\pi$. The flat space limit is obtained by taking the $k \to 0$ limit in the RS space. As $k \to 0$ with $L = \pi R$ kept fixed, $z_L \to 1$. In Fig. 5 the behavior of $a_{000}(\theta_H; z_L)$ is shown.

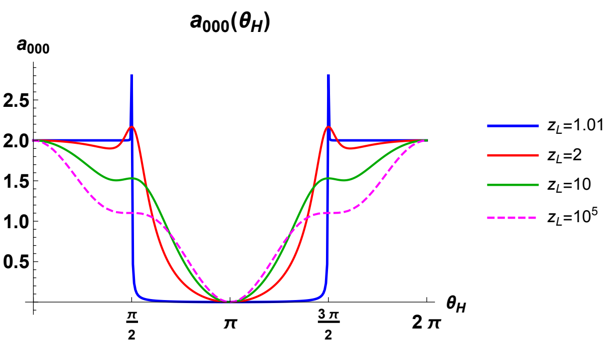

Figure 5: The z_L-dependence of the anomaly coefficient $a_{000}(\theta_H; z_L)$ is displayed for $c = 0.25$.

One sees that $a_{000}(\theta_H; z_L)$ varies smoothly as z_L changes. The $z_L \to 1$ limit is singular at $\theta_H = \frac{1}{2}\pi$ and $\frac{3}{2}\pi$, however.

$$\lim_{z_L \to 1} a_{000}(\theta_H; z_L) = \begin{cases} 2 & \text{for } 0 \le \theta_H < \frac{1}{2}\pi \\ 2\sqrt{2} & \text{for } \theta_H = \frac{1}{2}\pi \\ 0 & \text{for } \frac{1}{2}\pi \le \theta_H < \frac{3}{2}\pi \\ 2\sqrt{2} & \text{for } \theta_H = \frac{3}{2}\pi \\ 2 & \text{for } \frac{3}{2}\pi \le \theta_H \le 2\pi \end{cases} . \tag{6.1}$$

This is precisely the behavior in the flat space as

$$\lim_{z_L \to 1} Z_\mu^{(0)} = \begin{cases} B_\mu^{(0)} & \text{for } 0 \le \theta_H < \frac{1}{2}\pi \\ \frac{1}{\sqrt{2}}(B_\mu^{(0)} + B_\mu^{(-1)}) & \text{for } \theta_H = \frac{1}{2}\pi \\ B_\mu^{(-1)} & \text{for } \frac{1}{2}\pi < \theta_H < \frac{3}{2}\pi \\ \frac{1}{\sqrt{2}}(B_\mu^{(-1)} + B_\mu^{(-2)}) & \text{for } \theta_H = \frac{3}{2}\pi \\ B_\mu^{(-2)} & \text{for } \frac{3}{2}\pi < \theta_H \le 2\pi \end{cases} . \tag{6.2}$$

As a function of θ_H, the anomaly coefficient $a_{000}(\theta_H)$ becomes singular in the flat space limit at the points where level crossing occurs.

7. Holography in anomaly flow

In Figs. 3, 4 and 5, the anomaly coefficients coming from a fermion doublet of type 1A with the bulk mass parameter $c = 0.25$ have been shown. A surprise comes when one investigates the c-dependence of the anomaly coefficients $a_{nm\ell}(\theta_H; z_L, c)$. In the realistic GHU models of electroweak interactions, namely in the $SO(5) \times U(1)_X \times SU(3)_C$ GHU in the RS space, [11] the top quark multiplet has $c \sim 0.3$ whereas the multiplets of other quarks and leptons have $c = 0.6 \sim 1$.

In Fig. 6 gauge couplings $t_{000}^{R/L}$ and anomaly coefficients $a_{000}^{R/L}$ and a_{000} for $z_L = 10$ and $c = 0.8$ are shown. The behavior in Fig. 6 should be compared with that in in Fig. 3 for $c = 0.25$. Although the gauge couplings for $c = 0.8$ are significantly different from those for $c = 0.25$, the total anomaly coefficient $a_{000}(\theta_H)$ turns out universal, being independent of the value of c. There must be a reason for this fact.

Let us go back to the expression for $a_{n_1 n_2 n_3}$ in (5.4) with (5.2).

$$a_{n_1 n_2 n_3} = \sum_{m,\ell,p} \left\{ t^R_{n_1 m\ell}\, t^R_{n_2 \ell p}\, t^R_{n_3 pm} + t^L_{n_1 m\ell}\, t^L_{n_2 \ell p}\, t^L_{n_3 pm} \right\} . \tag{7.1}$$

There are two ways to evaluate $a_{n_1 n_2 n_3}$.

Method 1 (i) First evaluate the couplings $t^{R/L}_{nm\ell}$. (ii) Then do the summation $\sum_{m\ell p}$.

Method 2 (i) First do the summation $\sum_{m\ell p}$. (ii) Then do the integration $\int dy_1 dy_2 dy_3$.

So far we have adopted Method 1 to evaluate $a_{n_1 n_2 n_3}$.

In Method 2 the first step of the summation $\sum_{m\ell p}$ leads to

$$a_{n_1 n_2 n_3} = \left(\frac{k}{2}\right)^3 \int\int\int_{-\beta}^{2L-\beta} dy_1 dy_2 dy_3\, e^{\sigma(y_1)+\sigma(y_2)+\sigma(y_3)}$$

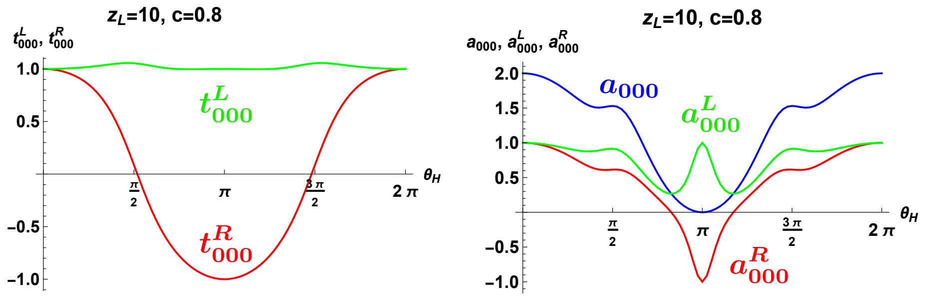

Figure 6: Gauge couplings $t_{000}^{R/L}$ and anomaly coefficients $a_{000}^{R/L}$ and a_{000} for $z_L = 10$ and $c = 0.8$ are shown as functions of θ_H. The anomaly coefficient a_{000} for $c = 0.8$ has the same behavior as a_{000} for $c = 0.25$ in Fig. 3.

$$
\begin{aligned}
&\times \Big[k_1 k_2 k_3 \big\{ A_R(1,2) A_R(2,3) A_R(3,1) - B_R(1,2) B_R(2,3) B_R(3,1) \\
&\qquad\qquad + B_L(1,2) B_L(2,3) B_L(3,1) - A_L(1,2) A_L(2,3) A_L(3,1) \big\} \\
&\quad + k_1 h_2 h_3 \{\cdots\} + h_1 k_2 h_3 \{\cdots\} + h_1 h_2 k_3 \{\cdots\} \Big] , \\
& k_j = k_{n_j}(y_j) , \quad h_j = h_{n_j}(y_j) , \\
& \begin{pmatrix} A_{R/L}(j,k) \\ B_{R/L}(j,k) \end{pmatrix} = \begin{pmatrix} A_{R/L} \\ B_{R/L} \end{pmatrix} (y_j, y_k) = \sum_{n=0}^{\infty} \begin{pmatrix} f_{R/Ln}(y_j) f_{R/Ln}^*(y_k) \\ g_{R/Ln}(y_j) g_{R/Ln}^*(y_k) \end{pmatrix} .
\end{aligned}
\tag{7.2}
$$

Here we have made use of $\sum_{n=0}^{\infty} f_{R/Ln}(y) g_{R/Ln}^*(y') = 0$. β appearing in the integration range in y_j is arbitrary. Finding the explicit form of $A_{R/L}$ and $B_{R/L}$ for general c is a difficult task, however.

It is possible to determine $A_{R/L}$ and $B_{R/L}$ for $c = 0$. One finds, with $\delta_L(x) = \sum_n \delta(x - nL)$, that

$$
\begin{aligned}
& \underline{\text{type 1A}, \ c = 0} \\
& A_R(y,y') = B_L(y,y') = \frac{e^{-\sigma(y)}}{k} \big\{ \delta_{2L}(y - y') + \delta_{2L}(y + y') \big\} , \\
& B_R(y,y') = A_L(y,y') = \frac{e^{-\sigma(y)}}{k} \big\{ \delta_{2L}(y - y') - \delta_{2L}(y + y') \big\} ,
\end{aligned}
\tag{7.3}
$$

Formulas for type 1B are obtained by interchanging R and L. Similarly

$$
\begin{aligned}
& \underline{\text{type 2A}, \ c = 0} \\
& A_R(y,y') = B_L(y,y') = \frac{e^{-\sigma(y)}}{k} \big\{ \hat{\delta}_{2L}(y - y') + \hat{\delta}_{2L}(y + y') \big\} , \\
& B_R(y,y') = A_L(y,y') = \frac{e^{-\sigma(y)}}{k} \big\{ \hat{\delta}_{2L}(y - y') - \hat{\delta}_{2L}(y + y') \big\} , \\
& \hat{\delta}_{2L}(y) = \delta_{4L}(y) - \delta_{4L}(y - 2L) .
\end{aligned}
\tag{7.4}
$$

Formulas for type 2B are obtained by interchanging R and L. When one inserts (7.3) and (7.4) into (7.2), there appear the products of three delta functions in the integrand. With $0 < \beta < L$ the products of delta functions reduce to

$$\begin{aligned} &\left.\begin{array}{l}\delta_{2L}(y_1-y_2)\delta_{2L}(y_2-y_3)\delta_{2L}(y_3+y_1)\\ \delta_{2L}(y_1+y_2)\delta_{2L}(y_2+y_3)\delta_{2L}(y_3+y_1)\end{array}\right\} \\ &\quad \Rightarrow \frac{1}{2}\Big\{\delta(y_1)\delta(y_2)\delta(y_3)+\delta(y_1-L)\delta(y_2-L)\delta(y_3-L)\Big\}~, \\ &\left.\begin{array}{l}\hat\delta_{2L}(y_1-y_2)\hat\delta_{2L}(y_2-y_3)\hat\delta_{2L}(y_3+y_1)\\ \hat\delta_{2L}(y_1+y_2)\hat\delta_{2L}(y_2+y_3)\hat\delta_{2L}(y_3+y_1)\end{array}\right\} \\ &\quad \Rightarrow \frac{1}{2}\Big\{\delta(y_1)\delta(y_2)\delta(y_3)-\delta(y_1-L)\delta(y_2-L)\delta(y_3-L)\Big\}~. \end{aligned} \tag{7.5}$$

Furthermore, as $h_n(0) = h_n(L) = 0$, only the terms proportional to $k_1 k_2 k_3$ in (7.2) survive.

We have arrived at the following formula for the anomaly coefficients;

$$a_{n\ell m}(\theta_H, z_L) = Q_0 k_n(0) k_\ell(0) k_m(0) + Q_1 k_n(L) k_\ell(L) k_m(L)~, \tag{7.6}$$

where

$$(Q_0, Q_1) = \begin{cases} (+1,+1) & \text{for type 1A} \\ (-1,-1) & \text{for type 1B} \\ (+1,-1) & \text{for type 2A} \\ (-1,+1) & \text{for type 2B} \end{cases}~. \tag{7.7}$$

The anomaly coefficients are determined by the values of the wave functions of the gauge fields, $k_n(y)$, at the UV and IR branes and the parity, Q_j, of the right-handed mode of the fermion field.

As observed at the beginning of this section, the anomaly coefficients $a_{n\ell m}(\theta_H)$ do not depend on the bulk mass parameter c of the fermion field. The anomaly formula (7.6) derived for $c = 0$ should apply for other values of c. Indeed one can confirm it explicitly. In Fig. 7 $a_{n\ell m}$'s given by the formula (7.6) and those determined by Method 1 for $c = 0.25$ are displayed in the case $z_L = 10$. Blue curves represent the formula (7.6), whereas red dots represent the values determined from the gauge couplings $t^{R/L}_{nm\ell}$ for $c = 0.25$. It is seen that the red dots are on the blue curves.

We stress that the anomaly coefficients depend solely on $k_n(0)$, $k_n(L)$, Q_0 and Q_1. They depend neither on the behavior of the wave functions of gauge fields and fermion field in the bulk region $0 < y < L$, nor on the bulk mass parameter c of the fermion field. The formula (7.6) represents **holography in anomaly flow**. Although each anomaly generated by specific fermion modes running along the internal triangle loop does depend on the detailed behavior of the wave functions of both gauge and fermion fields in the bulk, the total anomaly coefficients, after summing up all possible loop contributions, are determined by the information of the gauge fields at the UV and IR branes and of the parity conditions for the fermion field there.

The formula for the anomaly coefficients $b_{nm\ell}$ in the flat $M^4 \times (S^1/Z_2)$ spacetime simplifies. As in the RS space, one finds that

$$b_{n\ell m} = Q_0 k_n^{\rm flat}(0) k_\ell^{\rm flat}(0) k_m^{\rm flat}(0) + Q_1 k_n^{\rm flat}(L) k_\ell^{\rm flat}(L) k_m^{\rm flat}(L)~. \tag{7.8}$$

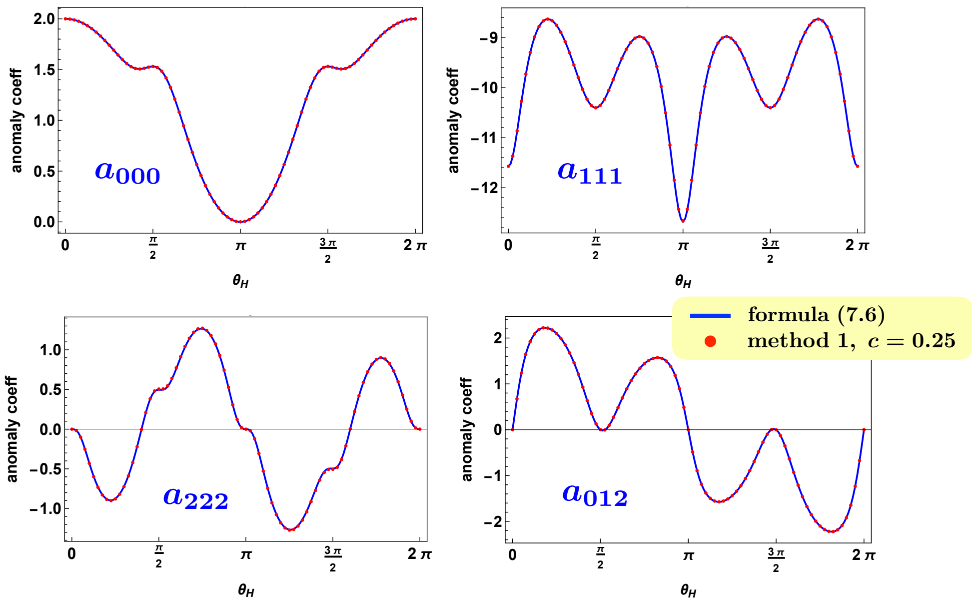

Figure 7: Anomaly coefficients $a_{000}(\theta_H)$, $a_{111}(\theta_H)$, $a_{222}(\theta_H)$, $a_{012}(\theta_H)$ are shown for type 1A fermions with $z_L = 10$. Blue curves represent the formula (7.6), whereas red dots represent the values determined from the gauge couplings $t^{R/L}_{nm\ell}$ for $c = 0.25$ by Method 1.

In the flat space $k_n^{\rm flat}(y) = \cos(n\pi y/L)$ so that

$$b_{n\ell m} = Q_0 + (-1)^{n+\ell+m}\, Q_1 \ , \tag{7.9}$$

which agrees with (4.3) with $Q_0 = Q_1 = 1$ for a doublet fermion of type 1A.

8. Anomaly cancellation

In gauge theory in four dimensions gauge anomalies have to be cancelled for the consistency of the theory. In GHU the holography in anomaly flow becomes crucial to guarantee the cancellation of gauge anomalies. In the SM all gauge anomalies are cancelled among quarks and leptons in each generation.[12, 13]

In the $SU(2)$ GHU under consideration one may introduce several fermion doublets, each of which has its own bulk mass parameter c. Let the numbers of doublet fermions of type 1A, 1B, 2A and 2B be n_{1A}, n_{1B}, n_{2A} and n_{2B}, respectively. It follows from (7.6) that the anomalies are cancelled if

$$n_{1A} = n_{1B} \ , \ \ n_{2A} = n_{2B} \ , \tag{8.1}$$

as the anomaly coefficients do not dependon c. The condition is generalized in the presence of brane fermions, namely fermions living only on the UV or IR brane. Suppose that there are $\hat{n}_R$ right-handed and $\hat{n}_L$ left-handed doublet brane fermions on the UV brane at $y = 0$. As the $Z_\mu^{(n)}$ coupling of each brane fermion is given by $(g_4/2)\, k_n(0)$, the anomaly cancellation conditions become

$$n_{1A} - n_{1B} + n_{2A} - n_{2B} + \hat{n}_R - \hat{n}_L = 0 \,,$$
$$n_{1A} - n_{1B} - n_{2A} + n_{2B} = 0 \,, \tag{8.2}$$

It is important that the conditions (8.1) and (8.2) do not depend on θ_H and z_L. The conditions guarantee that not only the zero mode anomaly a_{000} but also all other anomalies $a_{n\ell m}$ are cancelled at once. In GHU in the RS space the gauge couplings vary as θ_H, which further depends on c, or on the fermion species, but the anomaly cancellation conditions do not depend on θ_H.

9. Summary

We have shown that the anomaly flows with the AB phase θ_H. In the RS space everything changes smoothly with θ_H. In the $SU(2)$ GHU model in the RS space a chiral fermion at $\theta_H = 0$ is transformed to a vector-like fermion at $\theta_H = \pi$. The magnitude of the anomaly coming from one fermion doublet varies with θ_H. The total anomaly coefficients $a_{nm\ell}(\theta_H)$ are given by thec formula (7.6), which represents holography in anomaly flow.

The anomalies can be cancelled among several fermion doublets. The cancellation conditions are given by (8.1) or (8.2). They are independent of θ_H and z_L, which is important to achieve the anomaly cancellation in the realistic GHU models in the RS space. The examination of anomaly cancellation in the $SO(5) \times U(1)_X \times SU(3)_C$ GHU in the RS space is necessary.

In this connection one may worry about the fact that the gauge couplings vary with θ_H, and the couplings of quarks and leptons are not purely chiral at $\theta_H \neq 0$. In the SM only left-handed quark-lepton doublets couple to W. In GHU models in the RS space right-handed components also have small couplings to W at $\theta_H \neq 0$. The detailed study of the GUT inspired $SO(5) \times U(1)_X \times SU(3)_C$ GHU has been done recently.[14]. In the realistic model $\theta_H \sim 0.1$, $m_{\rm KK} \sim 13$ TeV and $z_L \sim 4 \times 10^{11}$. The W couplings of right-handed quarks in units of g_w are $O(10^{-12})$, $O(10^{-9})$, and $O(10^{-5})$ for (u, d), (c, s) and (t, b), respectively. The W couplings of right-handed leptons are much smaller.

Anomaly flow by an Aharonov-Bohm phase is a new phenomenon, which is different from the phenomenon of anomaly inflow.[15–17] Further investigation is desired.

Acknowledgement

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A. Basis functions

Wave functions of gauge fields and fermions in the RS space are expressed in terms of the Bessel functions. For gauge fields we introduce

$$\begin{aligned} C(z;\lambda) &= \frac{\pi}{2}\lambda z z_L F_{1,0}(\lambda z, \lambda z_L) ~, \\ S(z;\lambda) &= -\frac{\pi}{2}\lambda z F_{1,1}(\lambda z, \lambda z_L) ~, \\ C'(z;\lambda) &= \frac{\pi}{2}\lambda^2 z z_L F_{0,0}(\lambda z, \lambda z_L) ~, \\ S'(z;\lambda) &= -\frac{\pi}{2}\lambda^2 z F_{0,1}(\lambda z, \lambda z_L) ~, \\ F_{\alpha,\beta}(u,v) &\equiv J_\alpha(u) Y_\beta(v) - Y_\alpha(u) J_\beta(v) ~, \end{aligned} \tag{A.1}$$

where $J_\alpha(u)$ and $Y_\alpha(u)$ are Bessel functions of the first and second kind. These functions satisfy

$$\begin{aligned} & -z\frac{d}{dz}\frac{1}{z}\frac{d}{dz}\begin{pmatrix} C \\ S \end{pmatrix} = \lambda^2 \begin{pmatrix} C \\ S \end{pmatrix} ~, \quad -\frac{d}{dz} z \frac{d}{dz}\frac{1}{z}\begin{pmatrix} C' \\ S' \end{pmatrix} = \lambda^2 \begin{pmatrix} C' \\ S' \end{pmatrix} ~, \\ & C(z_L;\lambda) = z_L ~, \ \ C'(z_L;\lambda) = 0 ~, \ \ S(z_L;\lambda) = 0 ~, \ \ S'(z_L;\lambda) = \lambda ~, \\ & CS' - SC' = \lambda z ~. \end{aligned} \tag{A.2}$$

For fermion fields with a bulk mass parameter c, we define

$$\begin{aligned} \begin{pmatrix} C_L \\ S_L \end{pmatrix}(z;\lambda,c) &= \pm\frac{\pi}{2}\lambda\sqrt{z z_L} F_{c+\frac{1}{2}, c\mp\frac{1}{2}}(\lambda z, \lambda z_L) ~, \\ \begin{pmatrix} C_R \\ S_R \end{pmatrix}(z;\lambda,c) &= \mp\frac{\pi}{2}\lambda\sqrt{z z_L} F_{c-\frac{1}{2}, c\pm\frac{1}{2}}(\lambda z, \lambda z_L) ~. \end{aligned} \tag{A.3}$$

These functions satisfy

$$\begin{aligned} & D_+(c)\begin{pmatrix} C_L \\ S_L \end{pmatrix} = \lambda \begin{pmatrix} S_R \\ C_R \end{pmatrix}, \\ & D_-(c)\begin{pmatrix} C_R \\ S_R \end{pmatrix} = \lambda \begin{pmatrix} S_L \\ C_L \end{pmatrix}, \ \ D_\pm(c) = \pm\frac{d}{dz} + \frac{c}{z} ~, \\ & C_R = C_L = 1 ~, \ \ S_R = S_L = 0 \quad \text{at } z = z_L ~, \\ & C_L C_R - S_L S_R = 1 ~. \end{aligned} \tag{A.4}$$

B. Wave functions in RS

The wave functions $\tilde{\mathbf{h}}_n(z) \equiv (\tilde{h}_n(z), \tilde{k}_n(z))^t$ in (3.5) for gauge fields are given by

$$\tilde{\mathbf{h}}_0(z) = \bar{\mathbf{h}}_0^a(z) ~,$$

$$[\tilde{\mathbf{h}}_{2\ell-1}(z), \tilde{\mathbf{h}}_{2\ell}(z)] = (-1)^\ell \begin{cases} [\bar{\mathbf{h}}^a_{2\ell-1}(z), \bar{\mathbf{h}}^b_{2\ell}(z)] & \text{for } -\frac{1}{2}\pi < \theta_H < \frac{1}{2}\pi \\ [\bar{\mathbf{h}}^b_{2\ell-1}(z), -\bar{\mathbf{h}}^a_{2\ell}(z)] & \text{for } 0 < \theta_H < \pi \\ [-\bar{\mathbf{h}}^a_{2\ell-1}(z), -\bar{\mathbf{h}}^b_{2\ell}(z)] & \text{for } \frac{1}{2}\pi < \theta_H < \frac{3}{2}\pi \\ [-\bar{\mathbf{h}}^b_{2\ell-1}(z), \bar{\mathbf{h}}^a_{2\ell}(z)] & \text{for } \pi < \theta_H < 2\pi \\ [\bar{\mathbf{h}}^a_{2\ell-1}(z), \bar{\mathbf{h}}^b_{2\ell}(z)] & \text{for } \frac{3}{2}\pi < \theta_H < \frac{5}{2}\pi \end{cases}$$

$$(\ell = 1, 2, 3, \cdots) ,$$

$$\bar{\mathbf{h}}^a_n(z) = \frac{1}{\sqrt{r^a_n}} \begin{pmatrix} -\sin\theta_H \hat{S}(z;\lambda_n) \\ \cos\theta_H C(z;\lambda_n) \end{pmatrix}, \quad \bar{\mathbf{h}}^b_n(z) = \frac{1}{\sqrt{r^b_n}} \begin{pmatrix} \cos\theta_H S(z;\lambda_n) \\ \sin\theta_H \check{C}(z;\lambda_n) \end{pmatrix},$$

$$\hat{S}(z;\lambda) = \frac{C(1;\lambda)}{S(1;\lambda)} S(z;\lambda) , \quad \check{C}(z;\lambda) = \frac{S'(1;\lambda)}{C'(1;\lambda)} C(z;\lambda) ,$$

$$r_n = \frac{1}{kL} \int_1^{z_L} \frac{dz}{z} \{ |\hat{h}_n(z)|^2 + |\hat{k}_n(z)|^2 \} \quad \text{for} \begin{pmatrix} \hat{h}_n(z) \\ \hat{k}_n(z) \end{pmatrix}. \tag{B.1}$$

In (B.1) two expressions in an overlapping region in θ_H are the same.

The wave functions $\tilde{\mathbf{f}}_{Rn}(z) = (\tilde{f}_{Rn}(z), \tilde{g}_{Rn}(z))^t$ and $\tilde{\mathbf{f}}_{Ln}(z) = (\tilde{f}_{Ln}(z), \tilde{g}_{Ln}(z))^t$ in (3.8) for fermion fields of type 1A with $c \geq 0$ are given by

$$[\tilde{\mathbf{f}}_{R,2\ell}(z), \tilde{\mathbf{f}}_{R,2\ell+1}(z)] = \begin{cases} [\bar{\mathbf{f}}^a_{R,2\ell}(z), \bar{\mathbf{f}}^c_{R,2\ell+1}(z)] & \text{for } -\pi < \theta_H < \pi \\ [\bar{\mathbf{f}}^b_{R,2\ell}(z), \bar{\mathbf{f}}^d_{R,2\ell+1}(z)] & \text{for } 0 < \theta_H < 2\pi \\ [-\bar{\mathbf{f}}^a_{R,2\ell}(z), -\bar{\mathbf{f}}^c_{R,2\ell+1}(z)] & \text{for } \pi < \theta_H < 3\pi \\ [-\bar{\mathbf{f}}^b_{R,2\ell}(z), -\bar{\mathbf{f}}^d_{R,2\ell+1}(z)] & \text{for } 2\pi < \theta_H < 4\pi \\ [\bar{\mathbf{f}}^a_{R,2\ell}(z), \bar{\mathbf{f}}^c_{R,2\ell+1}(z)] & \text{for } 3\pi < \theta_H < 5\pi \end{cases}$$

$$(\ell = 0, 1, 2, \cdots) ,$$

$$\tilde{\mathbf{f}}_{L0}(z) = \bar{\mathbf{f}}^a_{L0}(z),$$

$$[\tilde{\mathbf{f}}_{L,2\ell-1}(z), \tilde{\mathbf{f}}_{L,2\ell}(z)] = \begin{cases} [\bar{\mathbf{f}}^a_{L,2\ell-1}(z), \bar{\mathbf{f}}^c_{L,2\ell}(z)] & \text{for } -\pi < \theta_H < \pi \\ [\bar{\mathbf{f}}^b_{L,2\ell-1}(z), \bar{\mathbf{f}}^d_{L,2\ell}(z)] & \text{for } 0 < \theta_H < 2\pi \\ [-\bar{\mathbf{f}}^a_{L,2\ell-1}(z), -\bar{\mathbf{f}}^c_{L,2\ell}(z)] & \text{for } \pi < \theta_H < 3\pi \\ [-\bar{\mathbf{f}}^b_{L,2\ell-1}(z), -\bar{\mathbf{f}}^d_{L,2\ell}(z)] & \text{for } 2\pi < \theta_H < 4\pi \\ [\bar{\mathbf{f}}^a_{L,2\ell-1}(z), \bar{\mathbf{f}}^c_{L,2\ell}(z)] & \text{for } 3\pi < \theta_H < 5\pi \end{cases}$$

$$(\ell = 1, 2, 3, \cdots) . \tag{B.2}$$

Here

$$\bar{\mathbf{f}}^a_{Rn}(z) = \frac{1}{\sqrt{r^a_n}} \begin{pmatrix} \cos\frac{1}{2}\theta_H C_R(z;\lambda_n,c) \\ -\sin\frac{1}{2}\theta_H \hat{S}_R(z;\lambda_n,c) \end{pmatrix}, \quad \bar{\mathbf{f}}^b_{Rn}(z) = \frac{1}{\sqrt{r^b_n}} \begin{pmatrix} \sin\frac{1}{2}\theta_H C_R(z;\lambda_n,c) \\ \cos\frac{1}{2}\theta_H \check{S}_R(z;\lambda_n,c) \end{pmatrix},$$

$$\bar{\mathbf{f}}^c_{Rn}(z) = \frac{1}{\sqrt{r^c_n}} \begin{pmatrix} \sin\frac{1}{2}\theta_H \hat{C}_R(z;\lambda_n,c) \\ \cos\frac{1}{2}\theta_H S_R(z;\lambda_n,c) \end{pmatrix}, \quad \bar{\mathbf{f}}^d_{Rn}(z) = \frac{1}{\sqrt{r^d_n}} \begin{pmatrix} -\cos\frac{1}{2}\theta_H \check{C}_R(z;\lambda_n,c) \\ \sin\frac{1}{2}\theta_H S_R(z;\lambda_n,c) \end{pmatrix},$$

$$\bar{\mathbf{f}}^a_{Ln}(z) = \frac{1}{\sqrt{r^a_n}} \begin{pmatrix} \sin \frac{1}{2}\theta_H \hat{S}_L(z;\lambda_n,c) \\ \cos \frac{1}{2}\theta_H C_L(z;\lambda_n,c) \end{pmatrix}, \ \bar{\mathbf{f}}^b_{Ln}(z) = \frac{1}{\sqrt{r^b_n}} \begin{pmatrix} -\cos \frac{1}{2}\theta_H \check{S}_L(z;\lambda_n,c) \\ \sin \frac{1}{2}\theta_H C_L(z;\lambda_n,c) \end{pmatrix},$$

$$\bar{\mathbf{f}}^c_{Ln}(z) = \frac{1}{\sqrt{r^c_n}} \begin{pmatrix} \cos \frac{1}{2}\theta_H S_L(z;\lambda_n,c) \\ -\sin \frac{1}{2}\theta_H \hat{C}_L(z;\lambda_n,c) \end{pmatrix}, \ \bar{\mathbf{f}}^d_{Ln}(z) = \frac{1}{\sqrt{r^d_n}} \begin{pmatrix} \sin \frac{1}{2}\theta_H S_L(z;\lambda_n,c) \\ \cos \frac{1}{2}\theta_H \check{C}_L(z;\lambda_n,c) \end{pmatrix},$$

$$r_n = \int_1^{z_L} dz \big\{ |\hat{f}_n(z)|^2 + |\hat{g}_n(z)|^2 \big\} \quad \text{for} \ \begin{pmatrix} \hat{f}_n(z) \\ \hat{g}_n(z) \end{pmatrix}, \tag{B.3}$$

and

$$\begin{pmatrix} \hat{S}_L \\ \hat{C}_R \end{pmatrix}(z;\lambda,c) = \frac{C_L(1;\lambda,c)}{S_L(1;\lambda,c)} \begin{pmatrix} S_L \\ C_R \end{pmatrix}(z;\lambda,c), \quad \begin{pmatrix} \hat{C}_L \\ \hat{S}_R \end{pmatrix}(z;\lambda,c) = \frac{C_R(1;\lambda,c)}{S_R(1;\lambda,c)} \begin{pmatrix} C_L \\ S_R \end{pmatrix}(z;\lambda,c),$$

$$\begin{pmatrix} \check{S}_L \\ \check{C}_R \end{pmatrix}(z;\lambda,c) = \frac{S_R(1;\lambda,c)}{C_R(1;\lambda,c)} \begin{pmatrix} S_L \\ C_R \end{pmatrix}(z;\lambda,c), \quad \begin{pmatrix} \check{C}_L \\ \check{S}_R \end{pmatrix}(z;\lambda,c) = \frac{S_L(1;\lambda,c)}{C_L(1;\lambda,c)} \begin{pmatrix} C_L \\ S_R \end{pmatrix}(z;\lambda,c). \tag{B.4}$$

In (B.2) two expressions in an overlapping region in θ_H are the same.

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