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A 3d perspective on de Sitter quantum field theory

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A basic introduction to scalar field theory in de Sitter spacetime is presented. The focus of this presentation takes place in three-dimensions. I will review the representation theory of $SO(1,3)$, the isometry group of three-dimensional de Sitter space. I will use this representation theory to build scalar fields in the inflationary patch and reproduce their two-point function in the Bunch Davies vacuum. I will then describe analogous calculations in Euclidean signature utilizing the sphere heat-kernel which is exact in three-dimensions. I will also cast the heat-kernel in an alternate description as the quantum mechanics of a particle worldline; in this context, computations will make explicit use of worldlines that wrap the compact Euclidean space, a phenomenon that does not occur in analogous Anti-de Sitter computations. Lastly, I will describe recent results for coupling quantum matter to three-dimensional quantum gravity in the Chern-Simons formalism by bootstrapping intuition from worldline quantum mechanics to Wilson loop operators. These proceedings are an expanded form of a talk given in the introductory session of a workshop on "Features of a Quantum de Sitter Universe."

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1. Introduction

de Sitter (dS) spacetime has long played an enigmatic role in theoretical physics. It is notoriously difficult to realize meta-stable dS vacua in string theory [1–3] and the "holographic dictionary" in de Sitter space [4, 5] is much less developed than in Anti-de Sitter (AdS) space. These difficulties make finding a unified framework for a "quantum de Sitter universe" elusive[1]. And yet, de Sitter spacetime has obvious relevance to our own universe: both the inflationary era as well as for our current era of weak accelerated expansion are well described by a positive cosmological constant. Understanding how quantum fields propagate on de Sitter space is then a necessary element in understanding cosmological physics, regardless of the ultimate status of de Sitter quantum gravity.

It is with this ethos in mind that these proceedings have been prepared. The content of these proceedings is not new, however I will attempt to condense, package, and summarize results into a friendly introduction into quantum field theory on a dS background. A partial aim of these notes is to highlight the ways in which quantum field theory in dS differs from that in AdS. In the course of this, I will provide an basic overview of the representation theory of the dS isometry group; representation theory is a useful paradigm for organizing the particle content on de Sitter but it also provides us with additional instructive points: we will be able to emphasize the many similarities of the dS isometry group and the Euclidean conformal group while also pointing out important differences.

To ground this discussion I will focus almost entirely on three-dimensional de Sitter spacetime, dS_3. This "3d perspective" will provide us with some simplifications with regards to representation theory and as well as some additional advantages. For one, it will be illuminating to discuss quantum field theory in the Euclidean signature: Euclidean de Sitter space is compact and so highlights in a geometric manner differences with AdS and the difficulty in adapting a holographic dictionary (due to a lack of a conformal boundary). While these statements are true in any dimension, we will see that many computations in quantum field theory can be performed exactly in three-dimensions using Euclidean heat kernel techniques. These heat kernel techniques are intimately related to the worldline quantum mechanics of a massive particle and make explicit use of the compactness of the Euclidean de Sitter: essentially particle worldlines can wrap the compact space many times.

While the primary purpose of these notes is to provide a friendly entry into quantum field theory on a fixed dS background, our 3d perspective will provide us one more conceptual advantage: a framework for describing de Sitter quantum gravity. This relies heavily on the topological nature of 3d gravity and ability to recast gravity in terms of Chern-Simons theory [9–12]. The relevant Chern-Simons theory for Euclidean gravity with a positive cosmological constant is solvable and provides exact answers for the gravity path-integral about a fixed saddle-point. However, again, since the focal point of this talk is quantum field theory in de Sitter, I will also briefly summarize recent results for incorporating quantum matter into this theory of quantum gravity [13]. The solubility of the Chern-Simons theory not only allows one to reproduce our earlier Euclidean computations (on a fixed classical de Sitter background), but also to compute controlled quantum gravity corrections to these quantities in a perturbation theory organized by Newton's constant.

[1]See, e.g., [6–8] for further reflections on a quantum de Sitter spacetime.

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2. $SO(1,3)$ representation theory

Following the footsteps of Wigner [14], we will classify single particle states by the unitary irreducible representations (UIRs) of the background isometry group[2]. de Sitter space, realized as a hyperboloid in $\mathbb{R}^{1,3}$ (what we will call "embedding spae") with radius ℓ (the *de Sitter radius*),

$$\eta_{AB}X^AX^B = \ell^2 \qquad \eta_{AB} = \text{diag}(-1,1,1,1) \tag{1}$$

has $SO(1,3)$ as its isometry group. From here-on we will choose units where $\ell = 1$. The generating algebra, $\mathfrak{so}(1,3) = \text{span}\{\boldsymbol{L}_{AB}\}$ is given by

$$[\boldsymbol{L}_{AB}, \boldsymbol{L}_{CD}] = \eta_{BC}\boldsymbol{L}_{AD} - \eta_{AC}\boldsymbol{L}_{BD} + \eta_{AD}\boldsymbol{L}_{BC} - \eta_{BD}\boldsymbol{L}_{AC}. \tag{2}$$

These generators have a natural action on the hyperboloid as the Killing vectors $\boldsymbol{L}_{AB} = X_A\partial_B - X_B\partial_A$. This is a real basis and so UIRs will realize the reality condition $\boldsymbol{L}_{AB}^\dagger = -\boldsymbol{L}_{AB}$. It useful to build representations in a way that mimics the Euclidean conformal group by going to the basis

$$D = \boldsymbol{L}_{03}, \qquad M = \boldsymbol{L}_{12}, \qquad P_i = \boldsymbol{L}_{3i} + \boldsymbol{L}_{0i}, \qquad K_i = \boldsymbol{L}_{3i} - \boldsymbol{L}_{0i} \tag{3}$$

obeying the algebra

$$\begin{aligned} [D,\vec{P}] &= \vec{P} & [M,P_i] &= -\,\varepsilon_{ij}P^j & [K_i,P_j] &= 2\delta_{ij}D - 2\varepsilon_{ij}M \\ [D,\vec{K}] &= -\,\vec{K} & [M,K_i] &= -\,\varepsilon_{ij}K^j\ . \end{aligned} \tag{4}$$

Note that in this basis $\vec{P}$ and $\vec{K}$ act as conformal ladder operators and so we can attempt to build UIRs as lowest weight representations. To this end we will label a *primary state* (lowest weight state) by (Δ, s), its eigenvalues under D and $-iM$, respectively. This state will be annihilated by $\vec{K}$:

$$D|\Delta,\mathsf{s};0\rangle = \Delta|\Delta,\mathsf{s};0\rangle \qquad M|\Delta,\mathsf{s};0\rangle = i\mathsf{s}|\Delta,\mathsf{s};0\rangle \qquad \vec{K}|\Delta,\mathsf{s};0\rangle = 0. \tag{5}$$

The quadratic Casimir of $\mathfrak{so}(1,3)$ acts on this basis as

$$c_2|\Delta,\mathsf{s};0\rangle = D(2-D) + \sum_{i=1}^{2} P_iK_i + M^2|\Delta,\mathsf{s};0\rangle = c_{\Delta,\mathsf{s}}|\Delta,\mathsf{s};0\rangle, \qquad c_{\Delta,\mathsf{s}} = \Delta(2-\Delta) - \mathsf{s}^2. \tag{6}$$

However unlike the Euclidean conformal representation theory, say relevant to AdS_3, reality of the generators here are realized as[3]

$$D^\dagger = -D, \qquad M^\dagger = -M, \qquad P_i^\dagger = -P_i, \qquad K_i^\dagger = -K_i\ , \tag{8}$$

and so the typical method of generating descendants as through the finite action of ladder operators fails here. For instance we could consider the example state

$$P_1^n|\Delta,\mathsf{s};0\rangle\ . \tag{9}$$

[2]See [15] and references there-in for a modern and concise resource.

[3]We remind the reader that Hermiticity relevant for the Euclidean conformal group is

$$D^\dagger = D, \qquad M^\dagger = -M, \qquad P_i^\dagger = K_i, \qquad K_i^\dagger = P_i\ . \tag{7}$$

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Hermiticity, (8), implies that in order for this state to have non-zero norm then the primary state has non-zero overlap with an infinite number of descendant states:

$$\left|P_1^n|\Delta,\mathsf{s};0\rangle\right|^2\neq 0\qquad\Rightarrow\qquad -\langle\Delta,\mathsf{s};0|P_1^{2n}|\Delta,\mathsf{s};0\rangle\neq 0. \tag{10}$$

So unlike the conformal representation theory relevant to AdS_3, representations here cannot be a direct sum of a discrete number of descendant states with the standard inner product. A more useful basis is given in "position space"

$$|\Delta,\mathsf{s},\vec{x}\rangle=e^{\vec{x}\cdot\vec{P}}|\Delta,\mathsf{s},0\rangle \tag{11}$$

with $\vec{x}\in\mathbb{R}^2$. The other generators act on this basis as

$$\begin{aligned}
P_i|\Delta,\mathsf{s},\vec{x}\rangle=&\partial_i^{(x)}|\Delta,\mathsf{s},\vec{x}\rangle\\
D|\Delta,\mathsf{s},\vec{x}\rangle=&(x\cdot\partial^{(x)}+\Delta)|\Delta,\mathsf{s},\vec{x}\rangle\\
M|\Delta,\mathsf{s},\vec{x}\rangle=&(-\varepsilon_{ij}x^i\partial^j+\mathsf{s})|\Delta,\mathsf{s},\vec{x}\rangle\\
K_i|\Delta,\mathsf{s},\vec{x}\rangle=&(2x_i(x\cdot\partial^{(x)}+\Delta)-x^2\partial_i-2\mathsf{s}\,\varepsilon_{ij}x^j)|\Delta,\mathsf{s},\vec{x}\rangle.
\end{aligned} \tag{12}$$

Unitarity further constrains the conformal weight and the spin, Δ and s, by imposing consistency of (8) inside $\langle\Delta,\mathsf{s},\vec{x}|\ldots|\Delta,\mathsf{s},\vec{y}\rangle$ as well as positivity of the norm. This procedure is wholly analogous to the procedure constructed in [15] and we will leave filling in the fine details to the reader. There are two unitary branches of representations

$$\begin{array}{lll}
\textbf{Case I: Complementary Series} & \Delta=1+\nu,\quad \nu\in(-1,1) & \mathsf{s}=0\\
\textbf{Case II: Principal Series} & \Delta=1-i\mu,\quad \mu\in\mathbb{R} & \mathsf{s}\in\mathbb{Z}\,.
\end{array} \tag{13}$$

Expressing a generic state of a given representation as

$$|\psi\rangle=\int d^2x\,\psi(\vec{x})|\Delta,\mathsf{s},\vec{x}\rangle \tag{14}$$

then these two branches of representations induce inner products on wave-functions as

$$\begin{array}{ll}
\textbf{Case I: Complementary Series} & \langle\psi_1|\psi_2\rangle=\dfrac{\Gamma(\Delta)}{\pi\Gamma(1-\Delta)}\displaystyle\int d^2x d^2y\,\psi_1^*(\vec{x})\frac{1}{|x-y|^{2\Delta}}\psi_2(\vec{y})\\
\textbf{Case II: Principal Series} & \langle\psi_1|\psi_2\rangle=\displaystyle\int d^2x\,\psi_1^*(\vec{x})\psi_2(\vec{x})
\end{array} \tag{15}$$

where $|\cdot|$ above is given by the Euclidean $\mathbb{R}^2$ norm. Thus these two types of representations induce very different inner products at the level of wave-functions. It is useful to note the existence of a representation isomorphism between representations with conformal weight Δ and $\bar{\Delta}=2-\Delta$. This isomorphism, $\mathbb{S}$, is called the *shadow map* and acts on states as

$$\mathbb{S}_\Delta|\Delta,\mathsf{s},x\rangle=\frac{\Gamma(\Delta)}{\pi\Gamma(1-\Delta)}\int d^2y\,\frac{1}{|x-y|^{2\Delta}}|\bar{\Delta},\mathsf{s},y\rangle\,. \tag{16}$$

This is a representation intertwiner, i.e. $[\mathbb{S},\boldsymbol{L}_{AB}]=0$ acting on this basis, and so is a true isomorphism. Note that in AdS_3, fields with conformal weight Δ and $2-\Delta$ lie in different representations. In dS_3 these representations are equivalent and so observables will be organized in such a way to utilize both conformal weights.

3. Scalar field theory

Now let use this representation theory to help us construct a field theory in de Sitter space. For the rest of this note I will focus on scalar field theory (and so for the principal series representation I will set $\mathsf{s} = 0$). Much of this section mirrors the discussions in [16, 17] however with details adapted to suit our conventions. Let us look at fields lying in the *inflationary patch* of de Sitter space, indicated in orange inside the Penrose diagram depicted in Figure 1. This is realized by parameterizing the embedding coordinates as

$$\begin{aligned} X^0 &= -\sinh t - \frac{1}{2}e^t\vec{x}^2 \\ X^i &= e^t\, x^i \\ X^3 &= \cosh t - \frac{1}{2}e^t\vec{x}^2 \end{aligned} \tag{17}$$

which induces the metric

$$ds^2 = -dt^2 + e^{2t}\,\delta_{ij}dx^i dx^j\,. \tag{18}$$

Thus each time slice of this patch is a Euclidean $\mathbb{R}^2$ which then expands to infinite size at future

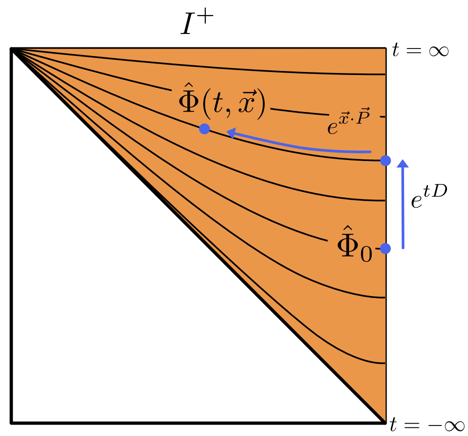

Figure 1: A cartoon of the Penrose diagram of dS_3 with the inflationary patch depicted in orange. Constant time slices (in black) are flat geometries that expand to infinite size at I^+. A primary scalar operator, $\hat{\Phi}_0$, inserted at $(t=0,\vec{x}=\vec{0})$ can be translated to another point in the patch by acting with e^{tD} followed by $e^{\vec{x}\cdot\vec{P}}$.

infinity, I^+. Expressing $\boldsymbol{L}_{AB} = X_A\partial_B - X_B\partial_A$ in the conformal basis, (3), we find that the conformal generators have a natural geometric action in this patch

$$\begin{aligned} D &= \partial_t - \vec{x}\cdot\vec{\partial} && \text{Time translation + dilitation} \\ P_i &= \partial_i && \mathbb{R}^2 \text{ translations} \\ M &= \epsilon^{ij}x_i\partial_j && \mathbb{R}^2 \text{ rotations} \\ K_i &= e^{-2t}\partial_i - 2x^i(\partial_t - \vec{x}\cdot\vec{\partial}) - x^2\partial_i && \text{Special conformal transformations.} \end{aligned} \tag{19}$$

These generators also act naturally on scalar field operators. For instance, we can then start with a scalar primary field, $\hat{\Phi}_0 \equiv \hat{\Phi}(t=0,\vec{x}=\vec{0})$, placed at the origin of the inflationary patch and satisfying $[K_i, \hat{\Phi}_0] = [M, \hat{\Phi}_0] = 0$. We can translate it to a generic point by conjugating with the above vector fields

$$\hat{\Phi}(t, \vec{x}) = e^{\vec{x}\cdot\vec{P}} e^{tD} \hat{\Phi}_0 e^{-tD} e^{-\vec{x}\cdot\vec{P}} \,. \tag{20}$$

Acting on $\Phi(t, \vec{x})$ via adjoint action, it is easy to verify that the generators of $\mathfrak{so}(1,3)$ act as the appropriate vector fields, (19). The quadratic Casimir, expressed as a second order differential operator through (19), acts on $\Phi(t, \vec{x})$ as the Klein-Gordon operator associated to the metric (18)

$$c_2 \circ \Phi(t, \vec{x}) = \left(-\partial_t^2 - 2\partial_t + e^{-2t}\partial_i^2\right)\Phi(t, \vec{x}) = \nabla^2\Phi(t, \vec{x}) \,. \tag{21}$$

Since this is a constant of the representation, we find that Φ satisfies the Klein-Gordon equation with a mass (measured in units of the Hubble scale, ℓ^{-1}) related to the conformal weight as

$$(\nabla^2 - m^2)\Phi = 0 \qquad \Rightarrow \qquad \Delta(2 - \Delta) = m^2. \tag{22}$$

Once again we see two branches of solutions corresponding to “light" scalar fields with mass less than the Hubble scale and “heavy" scalar fields with mass greater than the Hubble scale:

$$\begin{array}{llll} \textbf{Case I: } m^2 < 1 & \textbf{Complementary Series} & \Delta = 1 + \nu, & \nu = \sqrt{1 - m^2} \\ \textbf{Case II: } m^2 > 1 & \textbf{Principal Series} & \Delta = 1 - i\mu, & \mu = \sqrt{m^2 - 1} \end{array} \tag{23}$$

We can proceed to quantize the free scalar field theory in the inflationary patch through the standard Fock quantization:

$$\hat{\Phi}(t, \vec{x}) = \int \frac{d^2p}{(2\pi)^2} \left\{\hat{a}^\dagger_{\vec{p}}\, \Psi_{\vec{p}}(t, \vec{x}) + \text{h.c.}\right\} \tag{24}$$

with

$$[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] = \begin{cases} \delta^2(\vec{p} - \vec{p}') & m^2 > 1 \\ |p|^{2\nu}\, \delta^2(\vec{p} - \vec{p}') & m^2 < 1 \end{cases} \tag{25}$$

where the difference in the canonical commutation relations arises from the difference in representation norm (appropriately Fourier transformed), (15), between the complementary and principal series [16, 17]. The basis wave-functions are given by solving the Klein-Gordon equation, (22):

$$\Psi_{\vec{p}}(t, \vec{x}) = \alpha_\Delta\, e^{i\vec{p}\cdot\vec{x}}\, |p|^{1-\Delta}\, e^{-t}\, J_{\Delta-1}\left(|p|e^{-t}\right) + \alpha_{\bar{\Delta}}\, e^{i\vec{p}\cdot\vec{x}}\, |p|^{1-\Delta}\, e^{-t}\, J_{\bar{\Delta}-1}\left(|p|\, e^{-t}\right). \tag{26}$$

where $J_\nu(z)$ are Bessel functions and α_Δ and $\alpha_{\bar{\Delta}}$ are complex numbers that parameterize a two-parameter family of Fock quantizations. Equivalently there are a two-parameter family of de Sitter-invariant vacua, known as the *α-vacua* [18]. We can distinguish these vacua, for instance, through the vacuum Wightmann function:

$$\langle\Phi(t_1, \vec{x}_1)\Phi(t_2, \vec{x}_2)\rangle = |\alpha_\Delta|^2\, \mathcal{G}_{\Delta-1,\Delta-1} + \alpha^*_\Delta \alpha_{\bar{\Delta}} \mathcal{G}_{\Delta-1,\bar{\Delta}-1} + \alpha^*_{\bar{\Delta}}\alpha_\Delta \mathcal{G}_{\bar{\Delta}-1,\Delta-1} + \left|\alpha_{\bar{\Delta}}\right|^2 \mathcal{G}_{\bar{\Delta}-1,\bar{\Delta}-1} \tag{27}$$

where

$$\mathcal{G}_{a,b} = \frac{e^{-t_1-t_2}}{2\pi}\int_0^\infty dp\, p\, J_0\left(p|x_2 - x_1|\right)\, J_a\left(pe^{-t_1}\right)^*\, J_b\left(pe^{-t_2}\right) \,. \tag{28}$$

Let us focus on the *Bunch-Davies vacuum*, distinguished by enforcing Hadamard behavior of the scalar two-point function [19]:

$$\lim_{t_1\to t_2,\vec{x}_1\to\vec{x}_2}\langle\Phi(t_1,\vec{x}_1)\Phi(t_2,\vec{x}_2)\rangle_{\text{BD}}=\frac{1}{4\pi\left|\sigma(t_1,\vec{x}_1;t_2,\vec{x}_2)\right|}\sim\frac{1}{4\pi\left|(t_1-t_2)^2-e^{t_1+t_2}|x_1-x_2|^2\right|^{1/2}} \tag{29}$$

where σ is the geodesic distance between two points. By $\sim$ we mean the leading term as $(t_1,\vec{x}_1)\to(t_2,\vec{x}_2)$. At short distances, $|x_2-x_1|\ll 1$, due to the fall-off of J_0, the above integral is dominated by large p and so we can approximate $\mathcal{G}$ by replacing $J^*_{\nu_1}J_{\nu_2}$ with their large argument asymptotics:

$$J_\nu(z)\overset{z\to\infty}{\sim}\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)+\ldots \tag{30}$$

to find

$$2\pi^2\mathcal{G}_{\nu_1,\nu_2}\sim\frac{\cos\frac{\pi}{2}(\nu_1^*-\nu_2)}{\left(e^{t_1+t_2}|x_2-x_1|^2-4\sinh^2(\frac{t_1-t_2}{2})\right)^{1/2}}-\frac{\sin\frac{\pi}{2}(\nu_1^*+\nu_2)}{\left(e^{t_1+t_2}|x_2-x_1|^2-4\cosh^2(\frac{t_1-t_2}{2})\right)^{1/2}}\,. \tag{31}$$

Taking the $t_1\longrightarrow t_2$ limit of $\mathcal{G}$ inside (27), we find we can reproduce the correct Hadamard behavior when

$$\alpha_\Delta|_{\text{BD}}=\sqrt{\frac{\pi}{4}}\frac{e^{i\frac{\pi}{2}(\Delta-1)}}{\sin\pi(\Delta-1)},\qquad\qquad \alpha_{\bar{\Delta}}|_{\text{BD}}=\sqrt{\frac{\pi}{4}}\frac{e^{i\frac{\pi}{2}(\bar{\Delta}-1)}}{\sin\pi(\bar{\Delta}-1)}\,. \tag{32}$$

The integral[4] inside of $\mathcal{G}_{a,b}$ can be evaluated in terms of the Appell F_4 function [21] however simplifies greatly in the cases of interest, $b=\pm a$ [22]. After some tedious but otherwise straightforward massaging this leads to the known expression for the Bunch-Davies two-point function [20]:

$$\langle\Phi(t_1,\vec{x}_1)\Phi(t_2,\vec{x}_2)\rangle_{\text{BD}}=\frac{\Gamma(\Delta)\Gamma(\bar{\Delta})}{(4\pi)^{3/2}\Gamma\left(\frac{3}{2}\right)}{}_2F_1\left(\Delta,\bar{\Delta};\frac{3}{2};\frac{1+X_1\cdot X_2}{2}\right) \tag{33}$$

where $X_1\cdot X_2$ is the invariant length in embedding space. Expressed in inflationary patch coordinates:

$$X_1\cdot X_2=\eta_{AB}X_1^AX_2^B=\cosh(t_2-t_1)-\frac{1}{2}e^{t_1+t_2}|x_2-x_1|^2\,. \tag{34}$$

4. Euclidean methods

In the rest of this note, I will focus on *Euclidean methods* applicable to scalar field theory on dS_3. This will allow us to highlight some of the differences between AdS and dS. While the Euclidean rotation of AdS_3 retains its topology (a cylinder), the Euclidean rotation of dS_3 is much more dramatic: Euclidean dS_3 is a geometric three-sphere, S^3, which can easily be seen through Wick rotating the embedding space relation (1) through $X^0=-iX^4$:

$$\sum_{a=1}^{4}(X^a)^2=\ell^2. \tag{35}$$

[4]Strictly speaking this integral converges for $|x_2-x_1|>e^{-t_1}+e^{-t_2}$ or $X_1\cdot X_2<-1$, and all manipulations done in this regime. However the final answer, (33), can be analytically continued and remains correct at all separations [20].

In this section we will restore the de Sitter radius, ℓ. Wick rotating de Sitter effectively compactifies spacetime. Thus Euclidean signature casts many of the differences between field theory on AdS and dS in stark and geometric terms. Lastly, the methods of this section this will set the stage for discussing quantizing the background geometry in section 5 which we will perform through the Euclidean path-integral.

For the purposes of performing calculations in Euclidean signature it is useful to define the heat kernel, $K_{m^2}(x,y;\beta)$, of a scalar particle of mass, m, as the solution to

$$(\nabla^2_{(x)}-m^2\ell^2)K_{m^2}(x,y;\beta)=\frac{d}{d\beta}K_{m^2}(x,y;\beta)\qquad\qquad \lim_{\beta\to 0}K_{m^2}(x,y;\beta)=\frac{1}{\sqrt{g(x)}}\delta^3(x-y)\,. \quad (36)$$

K_{m^2} can be used to assign formal definitions to functional inverses

$$G_{m^2}(x,y)=\frac{1}{-\nabla^2+m^2\ell^2}:=\int_0^\infty d\beta\,K_{m^2}(x,y;\beta) \quad (37)$$

and functional determinants

$$\log\det(-\nabla^2+m^2\ell^2)=-\int_\times^\infty\frac{d\beta}{\beta}\int d^2x\,\sqrt{g(x)}K_{m^2}(x,x;\beta). \quad (38)$$

These expression for the one-loop determinant, (38), is only defined after regulating a UV divergence at $\beta\sim 0$, which I indicate schematically by the notation $\int_\times^\infty d\beta$.

There are many standard techniques for solving $K_{m^2}(x,y;\beta)$ perturbatively about $\beta\sim 0$ which are particularly valid for large $m^2\ell^2$ (this is equivalent to a *geodesic approximation*), however to our benefit, in three-dimensions we can solve for K_{m^2} exactly. Since its defining equation and initial condition, (36), are spherically symmetric, K_{m^2} can only depend on the arclength between x and y (most conveniently expressed in embedding space)

$$\Theta=\arccos(X\cdot Y). \quad (39)$$

We can reduce the heat equation to

$$\frac{1}{\sin^2\Theta}\partial_\Theta\left(\sin^2\Theta\,\partial_\Theta K_{m^2}(\Theta,\beta)\right)-m^2\ell^2K_{m^2}(\Theta,\beta)=\partial_\beta K_{m^2}(\Theta,\beta). \quad (40)$$

For small β, the initial condition tells us that K_{m^2} localizes to $\Theta\sim 0$. In a neighborhood of that point, (40) is the same as the $\mathbb{R}^3$ heat equation in polar coordinates with Θ playing acting as the radius. Thus we can use the $\mathbb{R}^3$ heat-kernel as an ansatz for small-β perturbation theory:

$$K_{m^2}(\Theta,\beta)=\frac{e^{-m^2\ell^2\beta}}{(4\pi\beta)^{3/2}}e^{-\frac{\Theta^2}{4\beta}-\alpha_0(\Theta)-\beta\alpha_1(\Theta)-\beta^2\alpha_2(\Theta)-\ldots}. \quad (41)$$

Applying the heat equation, (40), and collecting in powers of β, the functions α_n can be solved recursively with integration constants chosen to mimic the Euclidean heat-kernel at $\Theta\sim 0$. This provides a tractable method for calculating “heat kernel coefficients" which govern K_{m^2}'s small β expansion. For S^3 however, amazingly, this recursive procedure truncates and admits a solution setting $\alpha_{n\geq 2}=0$:

$$K_{m^2}=\frac{e^{-m^2\ell^2\beta}}{(4\pi\beta)^{3/2}}\frac{\Theta}{\sin\Theta}e^{-\frac{\Theta^2}{4\beta}+\beta}. \quad (42)$$

One may recognize this as a simple analytic continuation of the heat kernel on three-dimensional hyperbolic space, $\mathbb{H}_3$ [23]. However (42) is not exactly correct: it is not periodic in Θ! This is not surprising as we have only arrived at it from perturbation theory about $\beta \sim 0$; there are non-perturbative effects in β which correct (42). These effects remember the compactness of S^3. Namely if we interpret $K_{m^2}(x, y; \beta)$ as the probability of a particle localized at y diffusing to the point x after time β, then we should include the possibility for that particle to travel around the compact space any number of times before arriving at x (although such probabilities should be exponentially suppressed). This leads to the correct result:

$$K_{m^2}(x, y; \beta) = \frac{e^{-\mu^2\beta}}{(4\pi\beta)^{3/2}} \sum_{n\in\mathbb{Z}} \frac{\Theta + 2\pi n}{\sin\Theta} e^{-\frac{(\Theta+2\pi n)^2}{4\beta}}$$
$$= -\frac{1}{2\pi}\frac{e^{-\mu^2\beta}}{\sqrt{2\pi\beta}} \frac{\partial_\Theta\left(e^{\frac{-\Theta^2}{4\beta}}\vartheta_3\left(i\frac{\pi}{2\beta}\Theta, i\frac{\pi}{\beta}\right)\right)}{\sin\Theta}, \tag{43}$$

where $\mu^2 = m^2\ell^2 - 1$ and $\vartheta_3(z, \tau)$ is the Jacobi theta function

$$\vartheta_3(z, \tau) = \sum_{n\in\mathbb{Z}} e^{i2\pi z n + i\pi\tau n^2}. \tag{44}$$

It is worth reiterating: although perturbatively K_{m^2} is the analytic continuation of $\mathbb{H}_3 \simeq$ Euclidean AdS_3, the exact result is very different due to the compactness of Euclidean de Sitter.

4.1 The Green's function

We can now integrate the heat-kernel, (43) to arrive at the Euclidean Green's function for a scalar field

$$G_{m^2}(\Theta, m^2) = \int_0^\infty d\beta\, K_{m^2}(\Theta, \beta) \tag{45}$$

This integral is easily done on each term in the sum when $\mu^2 = m^2\ell^2 - 1 > 0$ to yield

$$G_{m^2}(\Theta) = \frac{1}{4\pi} \sum_{n\in\mathbb{Z}} \frac{\text{sgn}(\Theta + 2\pi n)}{\sin\Theta} e^{-\mu|\Theta+2\pi n|}. \tag{46}$$

(we have, without loss of generality, chosen $\mu > 0$). By choosing a representative of Θ to lie in between $[0, 2\pi)$ this can be massaged into the form

$$G_{m^2}(\Theta) = \frac{\sinh\left(\mu(\pi - \Theta)\right)}{4\pi\sin\Theta\sinh(\pi\mu)} = \frac{\Gamma(\Delta)\Gamma(\bar{\Delta})}{(4\pi)^{3/2}\Gamma(3/2)} {}_2F_1\left(\Delta, \bar{\Delta}; \frac{3}{2}, \frac{1+\cos\Theta}{2}\right). \tag{47}$$

where, again, $\Delta = 1 - i\mu$ and $\bar{\Delta} = 2 - \Delta$. This indeed Wick rotates to the two-point function of a massive scalar field in the Bunch-Davies vacuum of dS_3, (33). This was expected: the $\beta \to 0$ initial condition on the K_{m^2} forces G_{m^2} to have a Hadamard singularity structure. The corresponding Green's function for $m^2\ell^2 < 1$ can be found by the analytic continuation, $\mu = i\nu$ with $\nu = \sqrt{1 - m^2\ell^2}$.

4.2 The one-loop determinant

Additionally we can integrate (43) via (38) to arrive at the one-loop determinant of a massive scalar on S^3:

$$\log Z_{\text{scalar}} = \frac{V_{S^3}}{2}\int_0^\infty \frac{d\beta}{\beta}\lim_{\Theta\to 0} K_{m^2}(\Theta,\beta) \tag{48}$$

where $V_{S^3} = 2\pi^2$ is the volume of the three-sphere. Again this is easily done on each term of the sum appearing in (43) assuming $\mu^2 > 0$. However, importantly the integral over the $n = 0$ term of that sum diverges at $\beta \sim 0$ behavior. We can regulate this divergence by including a $R_\epsilon(\beta) = e^{-\frac{\epsilon^2}{4\beta}}$ regulator and taking the $\epsilon \to 0$ limit. The result of this is

$$\log Z_{\text{scalar}} = \frac{\pi}{2\epsilon^3} - \frac{\pi\mu^2}{4\epsilon} + \frac{\pi\mu^3}{6} - \frac{1}{4\pi^3}\left(\text{Li}_3\left(e^{-2\pi\mu}\right) + 2\pi\mu\text{Li}_2\left(e^{-2\pi\mu}\right) + 2\pi^2\mu^2\text{Li}_1\left(e^{-2\pi\mu}\right)\right) \tag{49}$$

where $\text{Li}_q(x)$ is the polylogarithm

$$\text{Li}_q(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^q}. \tag{50}$$

The one-loop partition function is UV divergent however we can minimally subtract these divergences to arrive at a finite part:

$$\log Z^{(\text{finite})}_{\text{scalar}} = \frac{\pi\mu^3}{6} - \frac{1}{4\pi^3}\left(\text{Li}_3\left(e^{-2\pi\mu}\right) + 2\pi\mu\text{Li}_2\left(e^{-2\pi\mu}\right) + 2\pi^2\mu^2\text{Li}_1\left(e^{-2\pi\mu}\right)\right). \tag{51}$$

This matches the one-loop determinant written down, e.g. in [24]. This also can be analytically continued, $\mu = i\nu$, for light scalars. Note that in the above result it was essential to include the sum over arclengths wrapping S^3 to find the polylogarithms. This has no analogue for fields in AdS_3. We give this sum a stark geometric interpretation below.

4.3 S^3 worldline quantum mechanics

We can cast the sum appearing in the exact heat kernel, (43), in explicitly geometric terms by first expressing K_{m^2} as a worldline path-integral. We will then see that this sum emerges as a sum over path-integral saddle-points that wind S^3. Let us begin by expressing the solution to (36) as a sum over eigenfunctions of ∇^2 weighted by their eigenvalue:

$$K_{m^2}(x,y;\beta) = e^{-\beta m^2\ell^2}\sum_{l=0}^{\infty}\sum_{\vec{m}} Y_{l,\vec{m}}(x)e^{-\beta l(l+2)}Y_{l,\vec{m}}(y) \tag{52}$$

where $Y_{l,\vec{m}}(x)$ are a set of hyper-spherical harmonics. In order to write this as a continuous path-integral we need to take care of this discrete sum. To that degree we will "integrate in" a continuous variable, p, such that the Gaussian weighting is continuous, as well as a variable α to collapse p to a discrete set:

$$K_{m^2}(x,y;\beta) = e^{-\beta\mu^2}\sum_{l,\vec{m}} Y^*_{l,\vec{m}}(x)\left(\int\frac{d\alpha}{2\pi}\int dp\, e^{i\alpha(p-l-1)}e^{-\beta p^2}\right)Y_{l,\vec{m}}(y). \tag{53}$$

The sums $\sum_{l,\vec{m}} Y^*_{l,\vec{m}} e^{-i\alpha l} Y_{l,\vec{m}}$ can now be performed to write the heat kernel as

$$K_{m^2}(\Theta,\beta) = \frac{e^{-\beta\mu^2}}{8\pi^2}\int\frac{d\alpha}{2\pi i}\int dp\,\frac{\cos\alpha/2\,\sin\alpha/2}{\sin^2\left(\frac{\alpha+\Theta}{2}\right)\sin^2\left(\frac{\alpha-\Theta}{2}\right)}e^{i\alpha p}e^{-\beta p^2}. \tag{54}$$

where again, Θ is the invariant arclength between x and y. So far this is not a path-integral. However we can introduce an auxiliary worldline, parameterized by a coordinate $\tau\in[0,1]$ and promote $p\to P(\tau)$ to a function on this worldline (with boundary conditions $P(0)=P(1)=p$). We will then introduce a function $\mathcal{X}^1(\tau)$ to act as a Lagrange multiplier forcing $\partial_\tau P(\tau)=0$:

$$K_{m^2}(\Theta,\beta) = \frac{e^{-\beta\mu^2}}{8\pi^2}\int\frac{d\alpha}{2\pi i}\int DPD\mathcal{X}^1\,\frac{\cos\alpha/2\,\sin\alpha/2}{\sin^2\left(\frac{\alpha+\Theta}{2}\right)\sin^2\left(\frac{\alpha-\Theta}{2}\right)}e^{i\alpha\int_0^1 d\tau P-\beta\int_0^1 P(\tau)^2-i\int_0^1\mathcal{X}^1(\tau)\partial_\tau P(\tau)}, \tag{55}$$

where $\mathcal{X}^1$ has boundary conditions $\mathcal{X}^1(0)=\mathcal{X}^1(1)=0$. We can now integrate out $P(\tau)$ to arrive at a path-integral for $\mathcal{X}^1$:

$$K_{m^2}(\Theta,\beta) = \frac{e^{-\beta\mu^2}}{8\pi^2}\int\frac{d\alpha}{2\pi i}\int D\mathcal{X}^1\,\frac{\cos\alpha/2\,\sin\alpha/2}{\sin^2\left(\frac{\alpha+\Theta}{2}\right)\sin^2\left(\frac{\alpha-\Theta}{2}\right)}e^{-\frac{1}{4\beta}\int_0^1\left(\partial_\tau\mathcal{X}^1+\alpha\right)^2}. \tag{56}$$

We now see something interesting: recall that α which began its life reminding us that the eigenvalues of ∇^2 are discrete. However now its role is to change the boundary conditions on $\mathcal{X}^1$ to realize saddle-points of its path-integral that wind the three-sphere. To see this we can pick up the residues of α's second order poles at $\alpha=\pm\Theta+2\pi n$ where $n\in\mathbb{Z}$. We will also trivially "integrate in" two other bosonic variables, $\mathcal{X}^2(\tau)$ and $\mathcal{X}^3(\tau)$ to soak up factors of β and arrive at

$$K_{m^2}(\Theta,\beta) = e^{-\beta\mu^2}\sum_{n\in\mathbb{Z}}\int D\vec{\mathcal{X}}_{(n)}\left(\frac{\Theta+2\pi n}{\sin\Theta}\right)e^{-\frac{1}{4\beta}\int_0^1 d\tau\,\partial_\tau\vec{\mathcal{X}}_{(n)}\cdot\partial_\tau\vec{\mathcal{X}}_{(n)}} \tag{57}$$

where $\vec{\mathcal{X}}_{(n)}=(\mathcal{X}^1_{(n)},\mathcal{X}^2,\mathcal{X}^3)$ and $\mathcal{X}^1_{(n)}=\mathcal{X}^1+(\Theta+2\pi n\tau)$. This is precisely the worldline path-integral of a particle moving on S^3 in a set of Riemann normal coordinates[5] [25]. Importantly this path-integral includes saddles[6] that wind the S^3, which distinguishes it from similar path-integral expressions for Euclidean AdS$_3$ [26].

As a final note, we can return to (54) and instead integrate out p directly leading to the expression of K_{m^2} as an integral over α. This leads ultimately to following integral expression of the one-loop determinant of a scalar field (after performing the $\int\frac{d\beta}{\beta}$ integral and deforming the α contour in the complex plane):

$$\log Z_{\text{scalar}} = \int_0^\infty\frac{d\alpha}{2\pi\alpha}\frac{\cosh\alpha/2}{\sinh\alpha/2}\chi_\Delta(\alpha) \tag{58}$$

[5]It is perhaps surprising that this path-integral, apart from its one-loop pre-factor, looks effectively flat. As shown in [25], wordline path-integrals on spheres can be simplified by using Riemann normal coordinates where they can be expanded in a series of $1/m^2\ell^2$ about the flat path-integral, up to a one-loop pre-factor. Coincidentally in $d=3$ this expansion terminates after one term.

[6]These inclusions were missed by [25].

where

$$\chi_\Delta(\alpha) = \frac{e^{-i\mu\alpha} + e^{i\mu\alpha}}{\sinh^2 \alpha/2} \tag{59}$$

is the *Harish-Chandra character* corresponding to the principal series representation of $SO(1,3)$ with $\Delta = 1 - i\mu$ [27]. The connection between Euclidean one-loop determinants and Lorentzian representation characters is generic [24] and hints at interesting connections between entropy, horizon scattering, and quasi-normal modes [28].

5. Outlook on a 3d quantum de Sitter universe

I've tried to provide a friendly introduction to scalar quantum field theory on de Sitter space that (i) makes clear the role of the representation theory of the dS isometry group and (ii) emphasizes important differences with scalar fields on AdS. Some of the most surprising differences in fact happen in Euclidean signature where compactness of Euclidean dS has clear consequences on scalar Green's functions and one-loop determinants. Focusing on three dimensions has allowed us to explore these differences with concrete and, often, exact examples.

As I mentioned in the introduction, focusing on three dimensions also affords us a window into a quantum de Sitter spacetime. In three dimensions there is no propagating graviton and so gravity is a topological theory. In fact, Euclidean dS_3, at least at the level of the action, is equivalent to two $SU(2)$ Chern-Simons theories [11, 12]

$$ik_L S_{\rm CS}[A_L] + ik_R S_{\rm CS}[A_R] = -I_{\rm EH} \tag{60}$$

where

$$S_{\rm CS}[A] = \frac{1}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A^3\right) \tag{61}$$

(with Tr taken in the fundamental representation of $SU(2)$) and

$$I_{\rm EH} = -\frac{1}{16\pi G_N}\int d^3x\sqrt{g}\left(R - 2\ell^{-2}\right). \tag{62}$$

This follows from writing the Chern-Simons connections as

$$A_L = i(\omega^a + e^a/\ell)L_a \qquad\qquad A_R = i(\omega^a - e^a/\ell)\bar{L}_a \tag{63}$$

where e^a and ω^a are the co-frame and the dualized spin-connection and $\{L_a\}$ and $\{\bar{L}_a\}$ generate the respective $SU(2)$'s. The equivalence (60) requires that the Chern-Simons levels are imaginary and inverse to Newton's constant[7]:

$$k_L = \frac{i}{4G_N} \qquad\qquad k_R = -\frac{i}{4G_N}. \tag{65}$$

[7] I am omitting a potential *gravitational Chern-Simons* action

$$I_{\rm GCS} = \frac{1}{2\pi}\int \mathrm{Tr}\left(\omega\wedge d\omega + \frac{2}{3}\omega^3\right). \tag{64}$$

with integer coefficient. Either omitting or including $I_{\rm GCS}$ does not alter the basic points made in this section.

It is tempting to extend this equivalence of actions to an equivalence of path-integrals:

$$\int Dg_{\mu\nu}\, e^{-I_{\rm EH}[g]} \stackrel{?}{=} \int DA_L DA_R\, e^{ik_L S_{\rm CS}[A_L]+ik_R S_{\rm CS}[A_R]} \tag{66}$$

This identification is subtle at the non-perturbative level. For instance, a sum over saddle-points relevant for Euclidean dS_3 in (66) diverges [29]. However, restricting the left-hand side of (66) to a saddle-point geometry, M, and the right-hand side to a fixed topology and background connection, Chern-Simons theory provides a successful framework for perturbative quantum gravity:

$$\int Dg_{\mu\nu}\big|_M\, e^{-I_{EH}[g]} = \int DA_L DA_R\big|_M\, e^{ik_L S_{CS}[A_L]+ik_R S_{CS}[A_R]}\,. \tag{67}$$

For instance by verifying that celebrated “exact techniques" (such as Abelianization [30, 31] and supersymmetric localization [32]) continue working in the context of Chern-Simons gravity [13] one can extract the gravity path-integral, $Z_{\rm grav}$, about the S^3 saddle from well-known $SU(2)$ Chern-Simons results [33]:

$$\begin{aligned} Z_{\rm grav} =& i e^{S_{\rm dS}}\left(\sqrt{\frac{2}{k_L+2}}\sin\left(\frac{\pi}{k_L+2}\right)\right)\left(\sqrt{\frac{2}{k_R+2}}\sin\left(\frac{\pi}{k_R+2}\right)\right)\\ =& i e^{S_{\rm dS}}\frac{8G_N}{\ell}\sinh^2\left(\frac{4\pi G_N}{\ell}\right) \end{aligned} \tag{68}$$

where $S_{\rm dS} = \frac{\pi\ell}{2G_N}$ is the tree-level de Sitter entropy. This result matches, to one-loop order, the graviton determinant independently calculated from the metric formulation [24]. Analytically continuing similar exact-results extends this one-loop matching to Lens space saddles in the dS_3 gravity path-integral [29].

However these proceedings are about quantum field theory in de Sitter spacetime. Can we couple matter into this theory? Can we do more in quantum gravity than compute one number: $Z_{\rm grav}$? We find hints of how to do this from the vantage point of effective field theory: Chern-Simons theories often arise after integrating out massive degrees of freedom. The stand-ins of those massive degrees of freedom in this effective theory are its *Wilson lines* which are morally regarded as the worldlines of massive charged particles.

Here we find that the lessons learned from expressing Euclidean computations as worldline path-integrals in Section 4.3, are echoed in constructing the effective coupling of matter to Chern-Simons gravity. In particular, for a scalar field minimally coupled to background geometry determined by Chern-Simons connections, A_L and A_R, we can express its one-loop determinant as a integral over Wilson loops:

$$\begin{aligned} \log Z_{\rm scalar} =& \frac{1}{4}\mathbb{W}_j[A_L, A_R]\\ \equiv& \frac{i}{4}\int \frac{d\alpha}{\alpha}\frac{\cos\alpha/2}{\sin\alpha/2}{\rm Tr}_{R_j} P\exp\left(\frac{\alpha}{2\pi}\oint_\gamma A_L\right){\rm Tr}_{R_j} P\exp\left(-\frac{\alpha}{2\pi}\oint_\gamma A_R\right)\,, \end{aligned} \tag{69}$$

where the α integration contour runs along imaginary axis to just to left and right of the poles at $\alpha = 0$ and packages the effect of Wilson loops wrapping the topological S^3 arbitrarily many times.

This object, coined the *Wilson spool* in [13], captures the physics of the worldline path-integrals of Section 4.3 in the Chern-Simons formalism.

It is not within the scope of these proceedings to undertake the detailed construction (69) however I will mention that in order for (69) to correctly correspond to massive matter on de Sitter, the representations appearing there have be constructed from the ground up using an alternative inner product on the $\mathfrak{su}(2)$ algebra [13, 34]. This is necessary so that highest-weight representations, R_j, can admit continuous (and possibly complex) highest-weights

$$j = -\frac{1}{2}\left(1+\sqrt{1-m^2\ell^2}\right) \tag{70}$$

and can correspond to the representations discussed in Section 2.

Since (69) provides an effective expression for the one-loop determinant of a massive field as an gauge-invariant object we can now insert it into the gravitational path-integral (again restricted to a given saddle-point)

$$\Big\langle \log Z_{\text{scalar}}[M]\Big\rangle_{\text{grav}} := \int Dg_{\mu\nu}\big|_M\, e^{-I_{EH}[g]} \log Z_{\text{scalar}}[g_{\mu\nu}] \tag{71}$$

by inserting the right-hand side of (69) into the Chern-Simons path-integral:

$$\Big\langle \log Z_{\text{scalar}}[M]\Big\rangle_{\text{grav}} = \frac{1}{4}\int DA_L DA_R|_M\, e^{ik_L S_{\text{CS}}[A_L]+ik_R S_{\text{CS}}[A_R]}\mathbb{W}_j[A_L,A_R]\,. \tag{72}$$

This provides a principle for calculating perturbative quantum gravity corrections to $\log Z_{\text{scalar}}[M]$ about the saddle-point, M. This is more than schematic, it is effective: one can verify that the exact techniques we employed to evaluate the partition function extend to the inclusion of Wilson loops carrying non-standard $SU(2)$ representations [13]. These techniques reduce the quantum expectation value of Wilson loops to an ordinary integral. The leading contribution to this integral as $G_N \to 0$ is in fact the exact one-loop determinant of scalar field that we derived in Section 4.2:

$$\lim_{G_N\to 0}\frac{\Big\langle \log Z_{\text{scalar}}[S^3]\Big\rangle_{\text{grav}}}{Z_{\text{grav}}} = \frac{\pi\mu^3}{6} - \frac{1}{4\pi^3}\left(\text{Li}_3\left(e^{-2\pi\mu}\right) + 2\pi\mu\text{Li}_2\left(e^{-2\pi\mu}\right) + 2\pi^2\mu^2\text{Li}_1\left(e^{-2\pi\mu}\right)\right) \tag{73}$$

where recall $\mu = \sqrt{m^2\ell^2-1}$. This is without any need to minimally subtract any divergences: within the gravity path-integral, (73) is completely finite.

Moving away from $G_N \to 0$, one can utilize exact methods to evaluate (72) order by order in G_N perturbation theory. At each order the Wilson spool provides finite and computable quantum gravity corrections to $\log Z_{\text{scalar}}$. These can be naturally matched to a renormalization of the mass of the scalar and so provide predictive statements about quantum gravity in de Sitter space.

Wrapping up this 3d perspective, we see that moving down a dimension gives novel and powerful effective methods for coupling matter to quantum gravity. The full utility of Chern-Simons for describe the interplay between quantum matter and quantum gravity has yet to be explored. It will be very interesting to generalize the construction of the Wilson spool, (69), to include massive spinning fields. Additionally, the spool is a natural object in AdS_3 gravity for describing one-loop determinants, at least at the classical ($G_N \to 0$) level [35]; this provides a non-trivial check of the proposal, (69), beyond the context of de Sitter, however it would be nice to proceed further and calculate (at least perturbatively) the AdS spool in the gravity path-integral. This could provide a promising framework for organizing $1/c$ corrections to CFT correlators.

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