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# Scattering, IR dynamics and entanglement

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We consider scattering processes in perturbative QED and compute the entanglement entropy between the hard and the soft particles in the final state. The leading perturbative entanglement entropy diverges logarithmically with respect to the IR cutoff. The coefficient of the divergence is proportional to the cusp anomalous dimension in QED, irrespective of the precise details of the initial state. For two-electron scattering processes, the computations can be extended to all orders in perturbation theory. The Renyi entropies (per unit time, per particle flux) turn out to be proportional to the total inclusive cross-section in the initial state, and so they are free of any IR divergences. Nonetheless, the entanglement entropy exhibits non-analyticity with respect to the IR cutoff. This logarithmic behavior is induced by the small eigenvalues of the hard density matrix, which scale inversely proportional with the size of the box providing the IR cutoff.

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## 1. Introduction

Symmetry forbids purely hard scattering processes in QED and gravity [1–3]. Indeed, conservation laws and Ward identities associated with large gauge transformations require that the hard asymptotic particles be accompanied by infinite clouds of soft photons and gravitons [2, 3]. An infinite number of soft particles should appear in the final scattering state, together with the final hard particles. These hard and soft particles are highly correlated. Therefore, it is important to understand the entanglement between them, and to quantify the information carried by the soft particles [4–11].

Over the years, we have gained many insights from studying the entanglement between local degrees of freedom across regions of space [12–15]. Given a region $\mathcal{A}$ of a Cauchy slice $\Sigma$ and its complement $\mathcal{A}^c$, most of the entanglement arises from local field theoretic degrees of freedom near the two sides of the boundary surface $\partial\mathcal{A}$. The entanglement entropy diverges, with the singular terms acquiring universal forms that depend on the geometrical properties of the entangling surface $\partial\mathcal{A}$:

$$S_{\mathcal{A}} \;=\; c_1 \frac{\text{Area}(\partial\mathcal{A})}{\epsilon^2} \;+\; c_2 K \log\epsilon + \ldots, \tag{1}$$

where $\epsilon$ is a short distance cutoff and $K$ depends on the extrinsic curvature of $\partial\mathcal{A}$ [12–14, 16, 17]. The coefficients $c_1$ and $c_2$ scale with the number of degrees of freedom of the quantum field theory and contain useful information [12–14]. In particular, they decrease along RG flows.

In holographic gauge/gravity dualities, such as the AdS/CFT correspondence, the entangling surface $\partial\mathcal{A}$ grows into an extremal (codimension 2) surface $\tilde{\mathcal{A}}$ in the bulk [18, 19]. Here, the CFT state gives rise to a bulk geometry. The bulk surface $\tilde{\mathcal{A}}$ is homologous to $\mathcal{A}$ on the boundary and minimizes the area functional. The entanglement entropy is equal to the area of $\tilde{\mathcal{A}}$ in Planck units:

$$S_{\mathcal{A}} \;=\; \frac{\text{Area}(\tilde{\mathcal{A}})}{4G_N^{(5)}}, \tag{2}$$

a formula that is reminiscent of the area law for black hole entropy [18, 19]. This holographic prescription allows us to compute entanglement entropy in strongly coupled CFTs and other quantum field theory systems. More importantly, it underlies a deep connection between quantum entanglement and geometry. Indeed, it has been understood that it is quantum entanglement that builds geometry and allows for a smooth bulk manifold to emerge [13, 14, 20]. The classic example of such an emergent geometry is the eternal AdS black hole [21]. This geometry has two causally disconnected, asymptotically AdS regions, which are separated by the black hole interior. As a result, we get two copies of the holographic dual CFT, each leaving on one of the disconnected components of the boundary. The Hilbert space takes the form of a tensor product $\mathcal{H}_1 \times \mathcal{H}_2$, with each factor standing for the Hilbert space of a CFT copy. The black hole is described by the thermofield double state

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\beta E_n/2} |E_n\rangle_1 \times |E_n\rangle_2, \tag{3}$$

where $|E_n\rangle$ are the energy eigenstates (with energy eigenvalues $E_n$); $Z$ is the thermal CFT partition function and $\beta = 1/T$ is the inverse temperature [21]. The temperature $T$ is equal to the Hawking temperature of the black hole. Notice that this is a non-product state in $\mathcal{H}_1 \times \mathcal{H}_2$, and so the degrees of freedom from each CFT copy are entangled. This is despite the fact that the two CFTs are not interacting. Taking a partial trace over the degrees of freedom of one copy leads to a thermal density matrix for the other copy, with the Von - Neumann entropy corresponding to the black hole entropy. Quantum entanglement bridges the two asymptotic regions. This bridging is achieved via a geometrical wormhole, through the black hole interior, in accordance with the ER=EPR relation [22]. Destroying the entanglement destroys the connectivity of spacetime, leading to a disconnected bulk manifold with naked singularities.

Armed with these insights concerning the entanglement of degrees of freedom across regions of space, it would be interesting to investigate patterns of entanglement in momentum space [23, 24]. The goal is to uncover properties of the S-matrix of the soft degrees of freedom in the deep infrared. In particular, the soft part of the S-matrix in gauge theories and gravity is highly constrained by an infinite number of asymptotic gauge symmetries, giving rise to specific patterns of entanglement between the soft and the hard particles in the final state. It has been argued that these structures may shed light on the black hole information paradox [25, 26]. Indeed, since the black hole formation and evaporation must be accompanied by the production of an infinite number of soft photons and gravitons, these could play a role in purifying the final state of Hawking radiation. The Hilbert space admits a decomposition into hard and soft factors

$$\mathcal{H}_H \times \mathcal{H}_S,$$

with $\mathcal{H}_S$ comprising states with energy less than an infrared scale $E_{IR} \ll T_H$ (where $T_H$ is the Hawking temperature). Suppose that the black hole evaporation is a unitary process, giving rise to a final state in $\mathcal{H}_H \times \mathcal{H}_S$ of the form

$$|\Psi\rangle_{in} \to |\Psi\rangle_{out} = S|\Psi\rangle_{in} = \sum_a c_a \left(|H_a\rangle \times |S_a\rangle\right). \tag{4}$$

This should be a non-trivial superposition of a large number of correlated pairs of hard and soft quanta, signalling a high degree of entanglement. Tracing over the final soft quanta should produce Hawking's thermal density matrix for black hole radiation:

$$\rho_{Hawk.} = Tr_S\,|\Psi\rangle_{out}\langle\Psi|_{out}, \tag{5}$$

placing specific constraints on the coefficients $c_a$ in terms of thermal occupation numbers [26]. The thermal entropy of the radiation can be understood as entanglement entropy between the soft and the hard degrees of freedom

$$S_{ent} = -Tr\rho_{Hawk.}\log\rho_{Hawk.} = S_{Thermal} \geq \frac{A_H}{4G}. \tag{6}$$

The second law of thermodynamics requires that this entropy is greater or equal to the Bekenstein-Hawking entropy of the black hole, and should remain large and finite in the limit $E_{IR} \to 0$.

In this work, we will study a simpler problem associated with scattering processes in QFT. Soft particles with energy below some IR energy scale $E$ escape detection, and we would like to quantify the information carried by these states. In particular, we will study the behavior of the entanglement entropy as a function of $E$ and $\lambda$, the IR cutoff of the theory [4–11]. We focus on scattering processes in perturbative QED, but the results are expected to generilize to perturbative gravity.

## 2. Scattering in QED and entanglement

There are infinitely many conserved charges in QED associated with large gauge transformations [2, 27, 28]. The gauge parameters do not vanish at infinity but instead asymptote to angle dependent constants. Indeed, observe that for any scalar gauge function $\epsilon(x)$, the corresponding Noether current

$$J^{\mu}_{\epsilon} = \partial_{\nu}(\epsilon F^{\mu\nu}) = F^{\mu\nu}\partial_{\nu}\epsilon + J^{\mu}\epsilon, \tag{7}$$

where $J^{\mu}$ is the usual electromagnetic current associated with the charge matter fields, is conserved. The first term in the last equation is the soft part of the epsilon current, which is linear in the electromagnetic field tensor, while the second term, which is quadratic in the charged matter fields, is the hard part. The soft part depends on the Lienard-Wiechert field produced by the charged particles. The conservation of this epsilon current leads to an infinite number of conservation laws when the gauge parameters $\epsilon$ tend to angle dependent constants at infinity. For example in massless QED, one can define the following charges by integrating the Hodge dual of the epsilon current (which is a 3-form) on the future and past null infinity, respectively:

$$Q^{+}_{\epsilon} = \int_{\mathcal{I}^{+}} * J_{\epsilon}\ , \quad Q^{-}_{\epsilon} = \int_{\mathcal{I}^{-}} * J_{\epsilon}\ . \tag{8}$$

If the gauge function $\epsilon$ obeys suitable boundary conditions at the common boundary of $\mathcal{I}^{+}, \mathcal{I}^{-}$ [2], we obtain a conservation law

$$Q^{+}_{\epsilon} = Q^{-}_{\epsilon}. \tag{9}$$

The epsilon charge $Q^{-}_{\epsilon}$ associated with an initial configuration massless particles on $\mathcal{I}^{-}$ must be equal to $Q^{+}_{\epsilon}$ associated with the final configuration of massless particles on $\mathcal{I}^{+}$.

These conservation laws give rise to Ward identities constraining the S-matrix

$$\langle out|Q^{+}_{\epsilon}S - SQ^{-}_{\epsilon}|in\rangle = 0. \tag{10}$$

By suitably choosing $\epsilon$, it can be shown that the Ward identities become equivalent to the Soft Theorems, which govern the IR structure of QED [2].

The first Soft Theorem concerns scattering processes involving the emission of an arbitrary number of $N$ soft photons with momenta $\vec{q}_r, r \in \{1, \dots, N\}$ [1, 29–35]:

$$\alpha \to \beta + \gamma_1,\ \gamma_2, \dots, \gamma_N, \tag{11}$$

where $\alpha$ and $\beta$ stand for generic configurations of incoming and outgoing hard particles, respectively. The corresponding S-matrix amplitude factorizes in the limit of small photon momenta $\vec{q}_r$:

$$S_{\beta\alpha}(\gamma_1 \dots \gamma_N) = S_{\beta\alpha} \prod_{r=1}^{N} \sum_{n} \frac{\eta_n e_n [p_n \cdot \epsilon^*(\vec{q}_r, h_r)]}{[p_n \cdot q_r]}, \tag{12}$$

where the sums in the soft factor run over all incoming and outgoing particles, $e_n$ is the charge of the $n$th particle and $\eta_n = +1$ for outgoing particles and $\eta_n = -1$ for incoming particles. Observe that the soft factor exhibits poles as $\vec{q}_r \to 0$, leading to IR divergences in the amplitudes involving real soft photon emission.

The second Soft Theorem concerns the exponentiation of IR divergences associated with virtual soft photons. Indeed to all orders, the amplitude square for the exclusive hard process $\alpha \to \beta$ is given by [1]

$$|S_{\beta\alpha}|^2 = |S^{(\Lambda)}_{\beta\alpha}|^2 \left(\frac{\lambda}{\Lambda}\right)^A , \tag{13}$$

where $\lambda$ is the IR cutoff scale; $\Lambda > \lambda$ is a fixed IR scale, characterizing soft virtual photons and

$$A = 2\mathcal{B}_{\beta\alpha} = -\frac{1}{8\pi^2} \sum_{ij} \eta_i \eta_j e_i e_j v_{ij}^{-1} \ln\left(\frac{1+v_{ij}}{1-v_{ij}}\right) \tag{14}$$

is a positive real constant. Here $v_{ij}$ is the magnitude of the relative velocity between particles $i$ and $j$ given by

$$v_{ij} = \left[1 - \frac{m_i^2 m_j^2}{(p_i \cdot p_j)^2}\right]^{1/2} . \tag{15}$$

Since $A$ is a positive constant, the amplitude $S_{\beta\alpha}$ vanishes in the strict limit $\lambda \to 0$. On the other hand, at any finite order in the QED coupling constant $e$, the amplitude is plagued by logarithmic IR divergences.

A typical hard process $\alpha \to \beta$ violates the $Q_\epsilon$ conservation laws, leading to the vanishing of the corresponding amplitude $S_{\beta\alpha}$. Purely hard processes are forbidden. To satisfy these laws, an infinite number of soft photons must be produced in the final state. One may restrict to inclusive cross-sections, which can be shown to be finite, free of any IR divergences, order by order in perturbation theory [1]. E.g. the cross-section for the transition $\alpha \to \beta$ + any number of soft photons with $E_{TOT} \leq E$ is IR finite order by order in perturbation theory, with IR divergences arising from real photon emission cancelling against IR divergences from virtual soft photons in the Feynman diagram loops. Alternatively, we may dress the incoming and outgoing charged particles with coherent clouds of soft photons, in accordance with the Faddeev-Kulish prescription, leading to non-vanishing, IR finite S-matrix elements [36–39]. In this latter case as well, an infinite number of soft photons appears in the final state.

In this work, we shall focus on $e^- + e^-$ scattering in perturbative QED, ignoring the Faddeev-Kulish dressings of the asymptotic particles[1]. We will work in a large box of finite volume $L^3$. So the momenta are discrete and the states of definite momentum normalizable. There is a natural IR cutoff in momentum given by the inverse size of the box, $\lambda = 1/L$, which will be taken to zero at the end of the computations. Our goal is to compute the entanglement entropy between the hard and the soft particles produced in the final state.

The initial state is a two-electron state with no photons

$$|\psi\rangle_{in} = |e_i e_j\rangle = |\alpha\rangle, \tag{16}$$

where $i, j$ stand for the momenta and the polarization indices of the initial particles. The final state is given by the action of the S-matrix on the initial sate

$$|\psi\rangle_{out} = S|\alpha\rangle. \tag{17}$$

[1]As it was shown in [10, 11], the leading entanglement entropy is independent of the Faddeev-Kulish dressing function.

As usual, we set $S = 1 + iT$ and impose the unitarity relation

$$i(T - T^\dagger) = -T^\dagger T. \tag{18}$$

The final state takes the following superposition form

$$|\Psi\rangle_{out} = |\alpha\rangle + T_{\beta\alpha}|\beta\rangle + T_{\beta\gamma,\alpha}|\beta\gamma\rangle + \ldots, \tag{19}$$

where $|\beta\rangle$ stands for a generic two electron state, $|\beta\gamma\rangle$ for a two electron state plus an additional photon and $T_{\beta\alpha} = \langle\beta|iT|\alpha\rangle$ and $T_{\beta\gamma,\alpha} = \langle\beta\gamma|iT|\alpha\rangle$ are the corresponding S-matrix elements. Summation over repeated indices is implied. (The energy and momenta of the final particles are constrained by energy-momentum conservation.) The hard and soft particles in the final state are entangled.

We then choose an IR reference scale $E < m_e$, where $m_e$ is the mass of the electron[2], and decompose the Hilbert space into hard and soft factors:

$$\mathcal{H}_H \times \mathcal{H}_S. \tag{20}$$

The hard factor $\mathcal{H}_H$ comprises electron/positron and photon states with energy greater than $E$, while the soft factor $\mathcal{H}_S$ comprises photon states with total energy less than $E$. The reduced density matrix for the hard particles can be obtained by taking a partial trace over the soft photons

$$\rho_H = Tr_S\, |\psi\rangle_{out}\langle\psi|_{out} = \rho_0 + \epsilon, \;\; \rho_0 = |\alpha\rangle_H\langle\alpha|_H. \tag{21}$$

Recall that $\rho_H$ is a hermitial operator acting on $\mathcal{H}_H$, with non-negative eigenvalues that add to unity. The density matrix $\rho_0$ associated with the initial two-electron state is pure. The variation matrix $\epsilon$ admits a perturbative expansion in terms of the coupling constant $e$ at fixed cutoff scale $\lambda$, which is set by the size of the box. Unitarity ensures that $\mathrm{Tr}\epsilon = 0$, as expected [10, 11].

The structure of the density matrix is very interesting. The diagonal elements are free of any IR divergences, order by order in perturbation theory, since they are given in terms of inclusive Bloch - Nordsieck rates associated with box states. Notice that these scale inversely proportional with powers of the volume of the box. The non-diagonal elements contain IR divergences at any finite order in perturbation theory. To all orders, however, these divergences exponentiate leading to the (faster) vanishing of the non-diagonal elements as $\lambda \to 0$, and to decoherence [4–11]. For example, the coefficient of the $|\beta\rangle_H\langle\beta|_H = |e_k e_l\rangle_H\langle e_k e_l|_H$ term in the reduced density matrix is the Bloch - Nordsieck rate associated with the inclusive rate $e_i e_j \to e_k e_l$ + any number of photons with $E_{TOT} \le E$:

$$D_{\beta,\beta} = T_{\beta\alpha}T^*_{\beta\alpha} + \sum_{\omega_\gamma < E} T_{\beta\gamma,\,\alpha}T^*_{\beta\gamma,\,\alpha} + \ldots. \tag{22}$$

The information carried by the soft photons scales with the entanglement entropy, which is given by

$$S_{ent} = -Tr\rho_H \log \rho_H. \tag{23}$$

To calculate this, one can first compute the Renyi entropies for integer $m$

$$S_m = \frac{1}{1-m}\log Tr(\rho_H)^m, \tag{24}$$

[2]More precisely the reference scale $E$ should be chosen according to the sensitivity of the detector.

and then take the limit

$$S_{ent} = \lim_{m\to 1} S_m. \tag{25}$$

The Renyi entropies can be also used to quantify the degree of non-purity for the density matrix. When the density matrix is pure, the Renyi entropies for integer $m$ vanish.

We first expand to a fixed order in perturbation theory, keeping the volume of the box finite, and take the continuum, $\lambda \to 0$ limit in the end. Soft photon production appears at order $e^3$, and so the leading Renyi entropies are of order $e^6$. Defining the following superposition of multiparticle states, which do not contain any soft photons,

$$|\Phi\rangle = |\alpha\rangle_H \; + \; \sum_\beta T_{\beta\alpha}|\beta\rangle_H \; + \; \sum_\beta \sum_{\omega_\gamma > E} T_{\beta\gamma,\alpha}|\beta\gamma\rangle_H \; + \; \ldots, \tag{26}$$

the reduced density matrix $\rho_H$ takes the form [11]

$$\rho_H = |\Phi\rangle\langle\Phi| + G, \tag{27}$$

where $G$ is an order $e^6$ matrix that annihilates $|\Phi\rangle$ (to this order). The precise expression for $G$ can be found in [11]. As a result, $|\Phi\rangle$ is an eigenstate of $\rho_H$ with a *large* eigenvalue:

$$\lambda_\Phi = \langle\Phi|\Phi\rangle = 1 - \Delta, \tag{28}$$

where

$$\Delta = \sum_\beta \sum_{\omega_\gamma < E} T_{\beta\gamma,\alpha} T^*_{\beta\gamma,\alpha} \tag{29}$$

is an order $e^6$ quantity, depending on the amplitude for single real photon emission in the energy range $\lambda < \omega_\gamma < E$. All the rest of the eigenvalues of $\rho_H$ are of order $e^6$ or higher. Their sum must be equal to $\Delta$ by unitarity:

$$\lambda_i = e^6 a_i, \;\; i \neq \Phi, \quad \sum_{i\neq\Phi} \lambda_i = \Delta, \tag{30}$$

where the $a_i$'s are order one quantities (or zero to this order).

The large eigenvalue governs the behavior of the Renyi entropies at leading order:

$$Tr(\rho_H)^m = 1 - m\Delta, \quad m \geq 2, \tag{31}$$

giving

$$S_m = -\frac{1}{m-1}\ln[1 - m\Delta] = \frac{m}{m-1}\Delta, \quad m \geq 2. \tag{32}$$

The expression above for the traces breaks down for sufficiently high $m$ [11] (since they must be positive, less than unity). From these results, we can also deduce the leading entanglement entropy

$$S_{ent} = -\sum_i \lambda_i \ln \lambda_i = -\Delta \ln e^6 + O(e^6). \tag{33}$$

which exhibits non-analytic behavior in the coupling constant.

The coefficient of the non-analytic term is found to be equal to $\Delta$, which is singular in the limit $\lambda \to 0$ [10, 11]. Indeed, using the soft theorems for single soft photon emission, we get for the singular part of the leading perturbative entanglement entropy

$$S_{ent,sing} = -\ln e^6 \sum_\beta (T_{\beta\alpha} T^*_{\beta\alpha}) \times$$
$$\left( \sum_{\omega_\gamma < E} \frac{1}{2V\omega_\gamma} \sum_{ss' \in \{\alpha,\beta\}} e_s e_{s'} \eta_s \eta_{s'} \frac{p_s p_{s'}}{(p_s q_\gamma)(p_{s'} q_\gamma)} \right). \tag{34}$$

This is a universal formula, applicable for generic initial states, with arbitrary number of electrons and positrons [11]. It is also independent of the FK dressing of the asymptotic particles [10, 11].

### 2.1 The leading entanglement entropy in the continuum limit

To take the continuum limit, we use the following relation between box and continuum states

$$|\vec{p}\rangle_{Box} \to \frac{1}{(2E_{\vec{p}}\, V)^{1/2}} |\vec{p}\rangle. \tag{35}$$

We end up with the following expression [10, 11]

$$S_{ent,\,sing} = -\frac{T\, v_{rel}}{16\, V} \ln\left(\frac{E}{\lambda}\right) \int \frac{d^2\hat{k}}{(2\pi)^2} \frac{\ln e^6}{E^2_{cm}} \left| i\mathcal{M}^{ij}_{kl} \right|^2 \mathcal{B}_{kl,\,ij}, \tag{36}$$

where $T$ is the time-scale of the experiment, $i\mathcal{M}^{ij}_{kl}$ is the invariant amplitude for the process $e_i + e_j \to e_k + e_l$ at tree level (Moller scattering) and $v_{rel}$ is the relative velocity between the incoming particles. There is a logarithmic IR divergence in the limit $\lambda \to 0$ with the reference energy scale $E$ kept fixed. In particular, the entanglement entropy, per unit time, per particle flux, is logarithmically divergent in this limit:

$$s_{ent,\,sing} = -\frac{1}{16} \ln\left(\frac{E}{\lambda}\right) \int \frac{d^2\hat{k}}{(2\pi)^2} \frac{\ln e^6}{E^2_{cm}} \left| i\mathcal{M}^{ij}_{kl} \right|^2 \mathcal{B}_{kl,\,ij}. \tag{37}$$

The integrand diverges for forward ($\theta = 0$) and backward ($\theta = \pi$) scattering, where $\theta$ is the scattering angle in the center of mass frame. To regulate this, we introduce an effective lower and upper cutoff on the scattering angle, $\theta_0 \le \theta \le \pi - \theta_0$ (where $\theta_0$ is a small angle). We can uncover a very interesting behavior in the high energy limit $p = E_{cm}/2 \to \infty$, $\theta_0 \to 0$, keeping the quantity $\xi = 4p^2 \sin^2(\theta_0/2)$ fixed and large. In this limit, the kinematical factor $\mathcal{B}_{kl,\,ij}$ becomes

$$\mathcal{B}_{kl,ij} = \frac{e^2}{4\pi^2} \ln\left(\frac{\xi}{4m^2}\right), \tag{38}$$

and so it is proportional to the cusp anomalous dimension of QED, $e^2\Gamma(\varphi)/4\pi^2$, via the relation $\xi = 2m^2(\cosh\varphi - 1)$, which controls the vacuum expectation value of a Wilson loop with a cusp of angle $\varphi$ [40]. The dominant differential entanglement entropy (per unit time, per particle flux) at the cutoff angles $\theta_0$ and $\pi - \theta_0$ becomes

$$\frac{ds_{ent,sing}}{d\Omega}|_{\theta=\theta_0,\,\pi-\theta_0} = -\frac{e^6 \ln e^6}{2\pi^2 \sin^4\theta_0} \ln\left(\frac{E}{\lambda}\right) \frac{1}{E^2_{cm}} \Gamma(\varphi)/4\pi^2. \tag{39}$$

We emphasize that this behavior is universal, exhibited irrespective of the details of the initial state (any number of electrons/positrons) [11]. The coefficient of the IR logarithmic singularity contains physical information since it is proportional to the cusp anomalous dimension of QED.



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## 3. The Renyi and entanglement entropies to all orders

The considerations above are valid in a particular limit, where we fix the size of the box $L$, or the infrared cutoff $\lambda$, and take the coupling constant $e$ to be sufficiently small. We can proceed now to carry out a calculation to all orders in the coupling $e$, taking $\lambda \to 0$ without truncating to a given order in the perturbative expansion. For this purpose, let us define the following superposition of multiparticle states

$$|\Phi\rangle = |\alpha\rangle_H + \sum_{\beta} T_{\beta\alpha}|\beta\rangle_H + \sum_{\beta}\sum_{|\gamma\rangle\in\mathcal{H}_H} T_{\beta\gamma,\alpha}|\beta\gamma\rangle_H, \tag{40}$$

where $|\gamma\rangle = |\gamma_1\gamma_2,\ldots,\gamma_n\rangle$ denotes a generic multiphoton state. We require the multiphoton states $|\gamma\rangle$ appearing in $|\Phi\rangle$ to lie in the hard part of the Hilbert space. In [11] we showed that to all orders in the coupling, the reduced density matrix $\rho_H$ takes the form

$$\rho_H = |\Phi\rangle\langle\Phi| + G, \tag{41}$$

where $G$ annihilates $|\Phi\rangle$ (taking into account energy-momentum conservation). So $|\Phi\rangle$ is an exact eigenstate of $\rho_H$ with eigenvalue $\lambda_\Phi = \langle\Phi|\Phi\rangle$. In terms of box amplitudes this eigenvalue is given by the expression

$$\lambda_\Phi = 1 + T_{\alpha\alpha} + T^*_{\alpha\alpha} + \sum_{\beta} T_{\beta\alpha}T^*_{\beta\alpha} + \sum_{\beta}\sum_{|\gamma\rangle\in\mathcal{H}_H} T_{\beta\gamma,\alpha}T^*_{\beta\gamma,\alpha}. \tag{42}$$

It follows that the sum of the rest of the eigenvalues is given by

$$\Delta = \sum_{i\neq\Phi}\lambda_i = -\Big(T_{\alpha\alpha} + T^*_{\alpha\alpha} + \sum_{\beta} T_{\beta\alpha}T^*_{\beta\alpha} + \sum_{\beta}\sum_{|\gamma\rangle\in\mathcal{H}_H} T_{\beta\gamma,\alpha}T^*_{\beta\gamma,\alpha}\Big). \tag{43}$$

The amplitudes $T_{\beta\alpha}, T_{\beta\gamma,\alpha}$ appearing in the above expressions *are hard amplitudes*. At any finite order in perturbation theory, they are plagued by IR logarithmic divergences due to virtual soft photons running in the loops. To all orders however, these divergences exponentiate leading to the vanishing of these amplitudes. On the other hand $T_{\alpha\alpha} + T^*_{\alpha\alpha}$ is free of any IR divergences, order by order in perturbation theory. This is because it is related to the *total inclusive cross-section* in the state $\alpha$ by unitarity:

$$T_{\alpha\alpha} + T^*_{\alpha\alpha} = -\sum_{\beta} T_{\beta\alpha}T^*_{\beta\alpha} - \sum_{\beta}\sum_{|\gamma\rangle} T_{\beta\gamma,\alpha}T^*_{\beta\gamma,\alpha} = -\frac{Tv_{rel}}{V}\Sigma_\alpha. \tag{44}$$

Taking into account the vanishing of the hard amplitudes in the continuum limit and the scaling of the box amplitudes with the size of the box, we obtain to all orders

$$\lambda_\Phi = 1 + T_{\alpha\alpha} + T^*_{\alpha\alpha} = 1 - \frac{Tv_{rel}}{V}\Sigma_\alpha \tag{45}$$

and

$$\Delta = \frac{Tv_{rel}}{V}\Sigma_\alpha. \tag{46}$$

Therefore, the reduced density matrix is dominated by a large eigenvalue, $\lambda_\Phi$, which is free of any IR divergences in $\lambda$. The rest of the eigenvalues should tend to zero as $L \to \infty$. For sufficiently small $m$, we get

$$Tr(\rho_H)^m = 1 - m\Delta = 1 - m\,\frac{Tv_{rel}}{V}\,\Sigma_\alpha \tag{47}$$

and so

$$S_m = -\frac{1}{m-1}\ln(1-m\Delta) = \frac{m}{m-1}\,\frac{Tv_{rel}}{V}\,\Sigma_\alpha. \tag{48}$$

The Renyi entropies per unit time, per particle flux, remain finite in the limit:

$$s_m = \frac{m}{(m-1)}\,\Sigma_\alpha. \tag{49}$$

They are proportional to the total cross-section in the two-electron state $\alpha$ [11].

The entanglement entropy, however, is not analytic in the volume of the box. The non-analytic behavior is induced by the small eigenvalues, via the expressions $-\lambda_i \ln \lambda_i$, for $i \neq \Phi$. Ignoring the spin polarization structure, we may approximate the small eigenvalues with the diagonal elements of $\rho_H$ (in the momentum indices), for example:

$$D_{\beta\gamma,\beta\gamma} = \sum_{|\gamma'\rangle \in \mathcal{H}_S} T_{\beta\gamma\gamma',\alpha} T^*_{\beta\gamma\gamma',\alpha} \tag{50}$$

In the continuum limit, taking into account the relative normalization between box and continuum amplitudes, we find that this diagonal element scales with $T^2/V^{N_f}$ in the large volume limit, where $N_f$ is the number of particles in the final state. It induces a logarithmically diverging contribution to the entanglement entropy, per unit time, per particle flux, in terms of the size of the box (or the IR cutoff $\lambda$), of the form $\ln(V^{N_f}/T^2)$. So the non-analytic behavior in the IR cutoff $\lambda$ persists to all orders in the continuum, large volume limit.

## 4. Conclusions

Generic scattering processes in QED lead to the appearance of an infinite number of soft photons in the final state, which may evade detection. We have argued in this work that it is possible to study quantitatively the entanglement between the soft and hard particles produced, and in particular to study the flow of information from the initial hard particles to the final soft degrees of freedom. At the leading perturbative level, the Renyi and entanglement entropies are governed by a large eigenvalue, which is logarithmically divergent with respect to the IR cutoff $\lambda$. The coefficient of the logarithmic divergence in the leading entanglement entropy, per unit time, per particle flux, exhibits certain universality properties, irrespective of the details of the initial sate, and it is proportional to the cusp anomalous dimension in QED. These perturbative results are strictly valid when the size of the box regulating infinite space is kept fixed and the QED coupling constant is taken to be sufficiently small.

For two-electron scattering processes, we can obtain the behavior of the Renyi and entanglement entropies to all orders in the QED coupling, in the continuum limit. The IR divergences appearing in the expression of the large eigenvalue of the density matrix at finite orders in perturbation theory exponentiate, leading to a non-vanishing, finite result. The Renyi entropies, per unit time, per



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particle flux, turn out to be proportional to the total, inclusive cross-section in the initial state, and so they are free of any IR divergences. The entanglement entropy though retains non-analytic behavior with respect to the volume of the box. This non analytic behavior appears when we keep the reference scale $E$ fixed in the limit $\lambda \to 0$. It would be interesting to obtain an exact expression for the entanglement entropy, per unit time, per particle flux, to all orders. It may help that for Fock basis states, the reduced density matrix is almost diagonal.

It would be also interesting to extend the analysis to gravitational scattering processes, and in particular to processes involving the formation and evaporation of a black hole. It would be nice to understand the order of limits in the black hole case and to investigate whether the information carried by the soft photons and gravitons accompanying the Hawking quanta can help ameliorate the information paradox along the lines of [25, 26].

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## References

[1] S. Weinberg, "Infrared photons and gravitons," Phys. Rev. **140**, B516 (1965). doi:10.1103/PhysRev.140.B516

[2] A. Strominger, "Lectures on the Infrared Structure of Gravity and Gauge Theory," arXiv:1703.05448 [hep-th].

[3] D. Kapec, M. Perry, A. M. Raclariu and A. Strominger, "Infrared Divergences in QED, Revisited," Phys. Rev. D **96**, no. 8, 085002 (2017) doi:10.1103/PhysRevD.96.085002 [arXiv:1705.04311 [hep-th]].

[4] D. Carney, L. Chaurette and G. Semenoff, "Scattering with partial information," arXiv:1606.03103 [hep-th].

[5] D. Carney, L. Chaurette, D. Neuenfeld and G. W. Semenoff, "Infrared quantum information," Phys. Rev. Lett. **119**, no. 18, 180502 (2017) doi:10.1103/PhysRevLett.119.180502 [arXiv:1706.03782 [hep-th]].

[6] D. Carney, L. Chaurette, D. Neuenfeld and G. W. Semenoff, "Dressed infrared quantum information," Phys. Rev. D **97**, no. 2, 025007 (2018) doi:10.1103/PhysRevD.97.025007 [arXiv:1710.02531 [hep-th]].

[7] D. Carney, L. Chaurette, D. Neuenfeld and G. Semenoff, "On the need for soft dressing," JHEP **1809**, 121 (2018) doi:10.1007/JHEP09(2018)121 [arXiv:1803.02370 [hep-th]].

[8] C. Gomez, R. Letschka and S. Zell, “Infrared Divergences and Quantum Coherence,” Eur. Phys. J. C **78**, no. 8, 610 (2018) doi:10.1140/epjc/s10052-018-6088-2 [arXiv:1712.02355 [hep-th]].

[9] C. Gomez, R. Letschka and S. Zell, “The Scales of the Infrared,” JHEP **1809**, 115 (2018) doi:10.1007/JHEP09(2018)115 [arXiv:1807.07079 [hep-th]].

[10] T. N. Tomaras and N. Toumbas, “IR dynamics and entanglement entropy,” Phys. Rev. D **101**, no.6, 065006 (2020) doi:10.1103/PhysRevD.101.065006 [arXiv:1910.07847 [hep-th]].

[11] A. Irakleous, T. N. Tomaras and N. Toumbas, “Soft photon radiation and entanglement,” Eur. Phys. J. C **81**, no.8, 745 (2021) doi:10.1140/epjc/s10052-021-09539-1 [arXiv:2105.11858 [hep-th]].

[12] P. Calabrese and J. L. Cardy, “Entanglement entropy and quantum field theory,” J. Stat. Mech. **0406**, P06002 (2004) doi:10.1088/1742-5468/2004/06/P06002 [arXiv:hep-th/0405152 [hep-th]].

[13] M. Van Raamsdonk, “Lectures on Gravity and Entanglement,” doi:10.1142/9789813149441_0005 [arXiv:1609.00026 [hep-th]].

[14] M. Rangamani and T. Takayanagi, “Holographic Entanglement Entropy,” Lect. Notes Phys. **931**, pp.1-246 (2017) Springer, 2017, doi:10.1007/978-3-319-52573-0 [arXiv:1609.01287 [hep-th]].

[15] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, “The entropy of Hawking radiation,” Rev. Mod. Phys. **93**, no.3, 035002 (2021) doi:10.1103/RevModPhys.93.035002 [arXiv:2006.06872 [hep-th]].

[16] S. N. Solodukhin, “Entanglement entropy, conformal invariance and extrinsic geometry,” Phys. Lett. B **665**, 305-309 (2008) doi:10.1016/j.physletb.2008.05.071 [arXiv:0802.3117 [hep-th]].

[17] H. Dorn, “On the logarithmic divergent part of entanglement entropy, smooth versus singular regions,” Phys. Lett. B **763**, 134-138 (2016) doi:10.1016/j.physletb.2016.10.029 [arXiv:1608.04900 [hep-th]].

[18] S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” Phys. Rev. Lett. **96**, 181602 (2006) doi:10.1103/PhysRevLett.96.181602 [arXiv:hep-th/0603001 [hep-th]].

[19] V. E. Hubeny, M. Rangamani and T. Takayanagi, “A Covariant holographic entanglement entropy proposal,” JHEP **07**, 062 (2007) doi:10.1088/1126-6708/2007/07/062 [arXiv:0705.0016 [hep-th]].

[20] M. Van Raamsdonk, “Comments on quantum gravity and entanglement,” [arXiv:0907.2939 [hep-th]].

[21] J. M. Maldacena, “Eternal black holes in anti-de Sitter,” JHEP **04**, 021 (2003) doi:10.1088/1126-6708/2003/04/021 [arXiv:hep-th/0106112 [hep-th]].



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[22] J. Maldacena and L. Susskind, "Cool horizons for entangled black holes," Fortsch. Phys. **61**, 781-811 (2013) doi:10.1002/prop.201300020 [arXiv:1306.0533 [hep-th]].

[23] V. Balasubramanian, M. B. McDermott and M. Van Raamsdonk, "Momentum-space entanglement and renormalization in quantum field theory," Phys. Rev. D **86**, 045014 (2012) doi:10.1103/PhysRevD.86.045014 [arXiv:1108.3568 [hep-th]].

[24] S. Seki, I. Y. Park and S. J. Sin, "Variation of Entanglement Entropy in Scattering Process," Phys. Lett. B **743**, 147 (2015) doi:10.1016/j.physletb.2015.02.028 [arXiv:1412.7894 [hep-th]]; R. Peschanski and S. Seki, "Entanglement Entropy of Scattering Particles," Phys. Lett. B **758**, 89 (2016) doi:10.1016/j.physletb.2016.04.063 [arXiv:1602.00720 [hep-th]]; R. Peschanski and S. Seki, "Evaluation of Entanglement Entropy in High Energy Elastic Scattering," Phys. Rev. D **100**, no.7, 076012 (2019) doi:10.1103/PhysRevD.100.076012 [arXiv:1906.09696 [hep-th]].

[25] S. W. Hawking, M. J. Perry and A. Strominger, "Soft Hair on Black Holes," Phys. Rev. Lett. **116**, no. 23, 231301 (2016) doi:10.1103/PhysRevLett.116.231301 [arXiv:1601.00921 [hep-th]].

[26] A. Strominger, "Black Hole Information Revisited," doi:10.1142/9789811203961 0010 arXiv:1706.07143 [hep-th].

[27] T. He, P. Mitra, A. P. Porfyriadis and A. Strominger, "New Symmetries of Massless QED," JHEP **1410**, 112 (2014) doi:10.1007/JHEP10(2014)112 [arXiv:1407.3789 [hep-th]].

[28] B. Gabai and A. Sever, "Large gauge symmetries and asymptotic states in QED," JHEP **1612**, 095 (2016) doi:10.1007/JHEP12(2016)095 [arXiv:1607.08599 [hep-th]].

[29] D. R. Yennie, S. C. Frautschi and H. Suura, "The infrared divergence phenomena and high-energy processes," Annals Phys. **13**, 379 (1961). doi:10.1016/0003-4916(61)90151-8

[30] F. Bloch and A. Nordsieck, "Note on the Radiation Field of the electron," Phys. Rev. **52**, 54 (1937). doi:10.1103/PhysRev.52.54

[31] F. E. Low, "Scattering of light of very low frequency by systems of spin 1/2," Phys. Rev. **96**, 1428 (1954). doi:10.1103/PhysRev.96.1428

[32] M. Gell-Mann and M. L. Goldberger, "Scattering of low-energy photons by particles of spin 1/2," Phys. Rev. **96**, 1433 (1954). doi:10.1103/PhysRev.96.1433

[33] F. E. Low, "Bremsstrahlung of very low-energy quanta in elementary particle collisions," Phys. Rev. **110**, 974 (1958). doi:10.1103/PhysRev.110.974

[34] T. H. Burnett and N. M. Kroll, "Extension of the low soft photon theorem," Phys. Rev. Lett. **20**, 86 (1968). doi:10.1103/PhysRevLett.20.86

[35] V. Del Duca, "High-energy Bremsstrahlung Theorems for Soft Photons," Nucl. Phys. B **345**, 369 (1990). doi:10.1016/0550-3213(90)90392-Q

[36] P. P. Kulish and L. D. Faddeev, "Asymptotic conditions and infrared divergences in quantum electrodynamics," Theor. Math. Phys. **4**, 745 (1970) [Teor. Mat. Fiz. **4**, 153 (1970)]. doi:10.1007/BF01066485

[37] V. Chung, "Infrared Divergence in Quantum Electrodynamics," Phys. Rev. **140**, B1110 (1965). doi:10.1103/PhysRev.140.B1110

[38] T. W. B. Kibble, "Coherent soft-photon states and infrared divergences. ii. mass-shell singularities of green's functions," Phys. Rev. **173**, 1527 (1968). doi:10.1103/PhysRev.173.1527

[39] T. W. B. Kibble, "Coherent Soft-Photon States and Infrared Divergences. I. Classical Currents," J. Math. Phys. **9**, no. 2, 315 (1968). doi:10.1063/1.1664582

[40] G. P. Korchemsky and A. V. Radyushkin, "Infrared factorization, Wilson lines and the heavy quark limit," Phys. Lett. B **279**, 359 (1992) doi:10.1016/0370-2693(92)90405-S [hep-ph/9203222].