PROCEEDINGS OF SCIENCE

Tensions with cosmological singularities: Should we try to avoid their appearance?

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The appearance of an initial (and sometimes also a final) cosmological singularity in practically all realistic models of the evolution of the universe is a distinguishing feature of modern cosmology. Tensions concerning this question are always present. There are attempts to construct cosmological models where the geometry is always regular. However, some approaches based on the description of the possible passage through the singularities were also developed recently. We have tried to elaborate a general formalism, telling when it is possible and when the singularity present an unsurmountable obstacle.

Corfu Summer Institute 2022 "School and Workshops on Elementary Particle Physics and Gravity", 28 August - 1 October, 2022 Corfu, Greece

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PoS(CORFU2022)198

1. Introduction

Appearance of singularities is one of the most important phenomena in General Relativity and its generalizations and modifications. The singularities were first discovered in such simple geometries as those of the Friedmann universes and the Schwarzschild black holes and later their general character was established [1, 2]. The investigation of the oscillatory approach to the cosmological singularity [3] known also as Mixmaster universe [4] has opened the way to the birth of a new branch of mathematical physics connecting chaos in cosmology and hyperbolic Kac-Moody algebras [5].

Should we try to avoid the singularities and to construct models without them? There are attempts to construct histories of the universe without singularities (see e.g. [6]). One can construct also regular black holes (see e.g. [7–10]).

However, one can try also to cross a singularity. Sometimes one can suggest and justify a prescription to match the geometry and matter field configurations in the regions separated by a singularity. This can be called singularity crossing.

In the case of the so called soft or sudden singularities, the curvature is divergent but the Christoffel symbols are finite [11]. The geodesics are well defined and the geometry can be reconstructed. The crossing of the Big Bang - Big Crunch singularities looks more counterintuitive [12]. However, it can be sometimes described by using the reparametrization of fields, including the metric. One can say that to do this, it is necessary to resort to one of two ideas, or a combination thereof. One of these ideas is to employ a reparameterization of the field variables which makes the singular geometrical invariant non-singular.

Another idea is to find such a parameterization of the fields, including, naturally, the metric, that gives enough information to describe consistently the crossing of the singularity even if some of the curvature invariants diverge.

The application of these ideas looks in a way as a craftsman work (see e.g. [13–15]). We have tried to develop a general formalism to distinguish "dangerous" and "non-dangerous" singularities, considering the field variable space of the model under consideration [16–18]. Field redefinitions play prominent roles in physics. They are primarily used at the linear and perturbative level in high-energy physics, but their nonlinear generalizations are being met with increasing interest, particularly in the study of gravity. Fields are dummy variables in the path integral, hence it is generally expected that quantities computed from functional integrals, hence physics, should remain invariant under field reparameterizations. From this viewpoint, fields are coordinates in the infinite-dimensional configuration space, field redefinitions are changes of coordinates in this space, and path integrals take on a more geometrical taste, generalizing the usual theory of integration on manifolds. This has led to the Vilkovisky-DeWitt effective action [19–21] where the coupling to the background field is made invariant by the introduction of a Levi-Civita connection in configuration space. Thus, we shall take the position that physics at the fundamental level should not depend on the way fields are parameterized.

The structure of the paper is as follows: in the second section we discuss standard spacetime singularities; in the third section we give some examples of spacetime singularities removable by field redefinitions; in the fourth section we consider the singularities in the space of field configurations; in the fifth section we treat quantum effective action and topological classification of functional singularities; the final section contains concluding remarks.

2. Spacetime singularities

There exist singularities in the metric that are connected with the unhappy choice of coordinates. Such singularities are called "coordinate singularities". Some of them are trivial like the singularity in the origin of the spherical coordinate system of the flat space $r = 0$. It is removed by the transition to the Cartesian coordinates. The coordinate singularity at the horizon in the Schwarzschild metric is much more involved:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

One can eliminate it by the transition to the Kruskal coordinates, but the horizon is physically significant. As is well known one can cross it only in one direction. Mathematically, there exists no parameter which smoothly connects the Kruskal change of coordinates to the identity. Such changes of coordinates are called "large".

In the center of the Schwarzschild geometry $r = 0$ one has the singularity of the Kretschmann invariant

$$R_{ijmn}R^{ijmn},$$

which cannot be eliminated by a coordinate change.

In the Friedmann universe

$$ds^2 = dt^2 - a^2(t)dl^2$$

the Ricci scalar R diverges at $t = 0$.

The question arises what can we do with the singularities of this kind?

3. Examples of spacetime singularities removable by field redefinitions

Let us consider as an example the Hawking-Turok instanton [22].

$$ds^2 = d\sigma^2 + b^2(\sigma)\left[d\chi^2 + \sin^2(\chi)\, d\Omega^2\right],$$

$$b(\sigma) \approx \begin{cases} \sigma\,, & \text{for} \quad \sigma \sim 0\,, \\ (\sigma_f - \sigma)^{1/3}\,, & \text{for} \quad \sigma \sim \sigma_f\,, \end{cases}$$

$$\phi(\sigma) \approx \begin{cases} \frac{1}{2}\,\sigma^2\,, & \text{for} \quad \sigma \sim 0\,, \\ -\sqrt{\frac{2}{3}}\ln(\sigma_f - \sigma), & \text{for} \quad \sigma \sim \sigma_f\,. \end{cases}$$

On Wick rotating χ, one obtains an open universe. The Ricci scalar is

$$R \sim \frac{1}{(\sigma_f - \sigma)^2}\,,$$

there is a spacetime singularity at $\sigma = \sigma_f$.

Changing spacetime coordinates to $d\bar{\sigma} = b^{-1}\, d\sigma$, followed by a Weyl transformation $\bar{g}_{\mu\nu} = b^{-2}\, g_{\mu\nu}$ gives us a non-singular geometry.

Let us make another Weyl transformation

$$\bar{g}_{\mu\nu} = \Omega^2\, \tilde{g}_{\mu\nu},$$

$$\Omega = 1 + \beta\, e^{-\alpha\sqrt{2/3}\,\phi},$$

where α and β are free parameters. Introducing a new, canonically normalized scalar field

$$d\tilde{\phi}^2 = 6\, e^{-\sqrt{2/3}\,\phi}\, \Omega^2\, \frac{\partial \ln \Omega}{\partial \phi} \left(\sqrt{\frac{2}{3}} - \frac{\partial \ln \Omega}{\partial \phi} \right) d\phi^2,$$

we come to the situation when both the geometry and the scalar field are regular.

As a second example we consider a flat Friedmann universe with a scalar field

$$ds^2 = dt^2 - a^2(t) dl^2.$$

In such a universe there is a Big Bang - Big Crunch singularity. One can prescribe the rules for its crossing making conformal transformations between the Einstein and Jordan frames, combined with the transformation of the scalar field, which leaves it canonically normalized [14].

4. Field space and singularities

Let us try to answer the question: When the spacetime singularities can be removed by a reparametrization of the field variables? We can put forward the following hypothesis: it can be done when the geometry of the space of the field variables is non-singular. The notion of the field space $\mathcal{S}$ was developed in order to treat on the same (geometrical) footing both changes of coordinates in the spacetime $\mathcal{M}$ and field redefinitions in the functional approach to quantum field theory. This approach requires introducing a local metric G in field space $\mathcal{S}$ and computing the associated geometric scalars by defining a covariant derivative which is compatible with G. The metric G can be determined by the kinetic part of the action and its dimension depends on the field content of the latter [19–21].

First of all we would like to consider the geometry of field space for pure gravity. For pure gravity theories there is a unique one-parameter family of field-space metrics

$$G_{ab} = G_{AB}\, \delta(x, x')\,,$$

where

$$G_{AB} = \frac{1}{2} \left(g_{\mu\rho}\, g_{\sigma\nu} + g_{\mu\sigma}\, g_{\rho\nu} + c\, g_{\mu\nu}\, g_{\rho\sigma} \right)$$

called DeWitt super-metric. It involves a dimensionless parameter c. Following [19] and [20], we introduce also the Christoffel symbols, covariant derivatives and curvature tensor in the field space.

For the DeWitt functional metric, the Ricci scalar is

$$\mathcal{R} = \frac{n}{4} - \frac{n^2}{8} - \frac{n^3}{8},$$

where n is the dimensionality of the spaceime.

We shall define the functional Kretschmann scalar of the underlying field space $\mathcal{S}$ as

$$\mathcal{K} = \mathcal{R}_{ABCD}\,\mathcal{R}^{ABCD}.$$

Rather cumbersome calculations give

$$\mathcal{K} = \frac{n}{8}\left(\frac{n^3}{4} + \frac{3n^2}{4} - 1\right).$$

This shows that $\mathcal{K}$ is smooth for any spacetime metric g in any spacetime dimension n. Besides, $\mathcal{K}$ does not depend on the DeWitt parameter c. Therefore, every theory of pure gravity is free of curvature singularities in the field space $\mathcal{G}$.

5. Quantum effective action and topological classification of functional singularities

At some field configurations the quantum effective action and the corresponding path integral can become ill-defined. These configurations can correspond to the appearance of the gravitational singularities.

It is somewhat surprising that both the functional Kretschmann scalar and the path-integral measure is non-singular in four spacetime dimensions for the DeWitt metric. This suggests that $n = 4$ stands at a special place from the perspective of the geometry of field space.

Let us introduce the functional

$$\psi[\varphi] = e^{i\,\Gamma[\varphi]}.$$

We shall call $\psi[\varphi]$ the functional order parameter because ψ plays the analogous role of an order parameter in the theory of phase transitions in ordered media or cosmology (see e.g. [23]).

The field space $\mathcal{M}$ can be thought of as the ordered medium itself, whereas functional singularities correspond to topological defects. The functional order parameter ψ defines the map

$$\psi : \mathcal{M} \to \mathbb{S}^1,$$

from the field space to the unit circle, the latter playing the role of the order parameter space. The singularities can be characterized by the fundamental group (first homotopy group).

Since $\pi_1(\mathbb{S}^1) = \mathbb{Z}$, the homotopy classes are labeled by the winding number $\mathcal{W}$.

A functional singularity exists whenever $\mathcal{W} \neq 0$.

Let us consider as an example a flat Friedmann universe filled with a massless scalar field, minimally coupled to gravity. There is the singularity of the Big Bang - Big Crunch type. This singularity can be eliminated by a field reparametrization. Direct (while tricky) calculation shows that in this case the winding number is equal to zero. Let us discuss it in more detail. The action here is

$$\tilde{\Gamma} = \int_\Omega d^4x\,\sqrt{-g}\left(\frac{R}{16\,\pi\,G_N} - \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi\right),$$

where G_N denotes the Newton's constant. The simplest cosmological spacetime is given by the spatially-flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric:

$$ds^2 = -N^2\,dt^2 + a^2\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right],$$

where $N = N(t)$ denotes the lapse function and $a = a(t)$ is the scale factor. For a homogeneous scalar field $\phi = \phi(t)$, we find

$$a_{\pm}^{3}(t) = \pm 3\sqrt{\kappa}\, p_{\phi}\, t \tag{1}$$

$$\phi_{\pm}(t) = \pm\frac{1}{\sqrt{\kappa}}\,\log\left(\pm\frac{t}{t_0}\right) , \tag{2}$$

where t_0 is an integration constant, $p_\phi = a^3\,\dot{\phi}$ is a constant of motion that follows from the equation for ϕ and we have set $N = 1$ in the final expressions. The different signs above correspond to different regimes of evolution of the universe. Expansion takes place for the positive sign, with $0 < t < \infty$, and contraction for the negative sign, with $-\infty < t < 0$. We have also adjusted the integration constants accordingly in order to obtain $a_{\pm}(0) = 0$. With such a choice we can join the two regimes of evolution at $t = 0$ to form a "bouncing" configuration, which shall be denoted by $\varphi_{\rm s}^{i} = (a_{\rm s}(t), \phi_{\rm s}(t))$.

It is not difficult to show that the Ricci scalar for the solution (1) diverges for $t \to 0$, which indicates the existence of a spacetime singularity at the bounce. We also note that the determinant of the spacetime metric vanishes at the bounce, which could suggest the presence of a covariant singularity. Thus, one can suspect that the spacetime singularity at $t = 0$ corresponds to a functional singularity, which would prevent us from defining observables for the bouncing solution $\varphi_{\rm s}^{i}$. However, the calculation of the functional winding number $\mathcal{W}$ shows otherwise. Following the formalism described in [17], we encircle the potentially singular configuration φ_{s}^{i} with a curve γ_1 parameterised as

$$\gamma^{I}(t;\theta) = (a_{\rm s}(t) + A\,\cos\theta, \phi_{\rm s}(t) + A\,\sin\theta, 1) , \tag{3}$$

for all values of t for which $a = a_{\rm s}(t)$ and $\phi = \phi_{\rm s}(t)$ are defined, and A is a positive constant. Since the effective action diverges when computed along (3), the calculation of the winding number for such a parameterization is quite tricky. Nothing forbids us from parameterizing γ_1 differently, but it is easier to exploit the freedom to add total derivatives to the effective action. By including the total derivative

$$F = \frac{A^3}{3}\,\cos^3\theta\,\dot{\phi} + \frac{A^2}{2}\,\cos^2\theta\,\dot{a} , \tag{4}$$

we can cancel out the divergence in the time integral over the configurations (3). This results in $\Gamma[\varphi_{\rm s}] = 0$, already suggesting that the apparent singularity is removable. Indeed, the effective action evaluated along (3) vanishes identically, namely $\Gamma(\theta) = 0$, which yields

$$\mathcal{W} = 0 . \tag{5}$$

This implies that the apparent singularity at $\varphi_{\rm s}$ is indeed removable by local alterations of the effective action in the vicinity of $\varphi_{\rm s}$. In fact, by imposing a cutoff $T > 0$ in the lower limit of the time integral in Eq. (5) and taking $T \to 0$ in the end, one finds

$$\lim_{T\to 0}\Gamma_T[\varphi_{\rm s}] = 0 , \tag{6}$$

where $\Gamma_T[\varphi_{\rm s}]$ denotes the regularized effective action. Therefore, the spacetime singularity at $t = 0$ does not correspond to a functional singularity and physical observables can be defined normally.

This shows that configuration-space coordinates must exist in which the spacetime singularity vanishes completely. Indeed, the corresponding reparameterizations were found not only for the Friedmann universe [14] but also for the Bianchi - I universe [24] and for the Kantowski-Sachs model [25].

6. Conclusions

Let us recapitulate briefly the content of our works. We have proposed to investigate singularities in the field space rather than in spacetime. Existing examples show that certain singularities in spacetime can be removed by field redefinitions albeit being non-removable under change of coordinates. Finding field redefinitions that can eliminate singularities is not always feasible in practice. The promising approach is to calculate curvature invariants in field space. We showed that the Kretschmann scalar of the DeWitt functional metric turns out to be free of singularities. We should note that removing a singularity from a field configuration certainly makes its description more complete, but the fact that a singularity is removable in field space does not imply that there is no interesting physics occurring around it. Horizons as removable spacetime singularities clearly teach us that investigating the physics likely requires a case by case study. In fact, studying specific models of self-gravitating systems is one of the natural developments of the present work, as it is the formal analysis of field-space invariants for more general theories than pure gravity. We have discussed also the relation between the structure of the quantum effective action and the presence of the singularities in field space. We have treated in some detail the case of the flat Friedmann cosmological model with a massless scalar field and have shown that the singularity in the field space is absent.

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