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Cosmology under the fractional calculus approach: a possible H_0 tension resolution?

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Fractional cosmology has recently emerged, based on the formalism of fractional calculus, which modifies the integer order derivative by a fractional derivative generating changes in the Friedmann equations. The standard evolution of the cosmic species densities is modified, depending on the μ fractional parameter and the age of the Universe t_0. The modified Friedmann equations provide a late cosmic acceleration at the background level without introducing a dark energy component. This radical approach could be a new path to tackle problems not resolved until now in cosmology. We estimate stringent constraints on the fractional and cosmological parameters using observational Hubble data, Type Ia supernovae and joint analysis to elucidate that. According to our results, the Universe would be older than the standard estimations. Finally, we analyze whether fractional cosmology can alleviate H_0 tension.

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1. Introduction

Fractional calculus is a subject that has been introduced previously with two works by Niels Henrik Abel in 1823 and 1826. It is a natural extension of the calculus discovered independently in the late 17th century by Isaac Newton and Gottfried Leibniz, where differentiation and integration are extended to noninteger or complex orders. Denoting by μ the fractional orders of the operations, the new operations coincide with the results of classical calculus when $\mu \in \mathbb{Z}$, and if it is a positive integer (differentiation) or negative integer (integration). Fractional calculus has drawn increasing attention to studying physical behaviours. Research in fractional differentiation is inherently multi-disciplinary and has its application across various disciplines, for example, fractional quantum mechanics and gravity for fractional spacetime, fractional quantum field theory and cosmology [1–20].

In references [18] and [19], the fractional calculus free parameters are estimated using cosmological data. The analysis from the type Ia supernovae (SNe Ia) data, the observational Hubble parameter data (OHD), and the joint analysis lead to best-fit values for the free parameters. Moreover, these best-fit values are used to calculate the age of the Universe, a current deceleration parameter, and a current matter density parameter. In [20], dynamical systems were used to analyze fractional cosmology for different matter contents, obtaining a late-time accelerating cosmology. Despite the discrepancy between the age of the Universe predicted by the fractional calculus approach and that of globular clusters, it is essential to highlight that fractional cosmology would contribute to the solutions to other problems associated with the ΛCDM model. For example, the late-time acceleration without dark energy can alleviate the so-called Cosmological Constant problem, in which the observational value of the Λ differs between 60 and 120 orders of magnitude compared with the value anticipated by particle physics [21]. Another problem related to the Dark Energy (DE) is the coincidence problem, which stipulates that, currently, Dark Matter (DM) and DE densities are of the same order of magnitude, with a fine-tuning problem associated with the context of ΛCDM model [22]. Another issue that fractional cosmology can alleviate is the Hubble tension. Measurements of the Hubble parameter at the current time, H_0, exhibit a discrepancy of 5σ between the observational value obtained from cepheids and SNe Ia from the Hubble Space Telescope (HST) [23], and the one inferred from Planck CMB [24]. The first corresponds to model-independent measurements, while the second depends on the ΛCDM model. According to [25], observational issues like the H_0 tension are strong evidence that physics beyond ΛCDM model is required, being the fractional cosmology one of these options. Therefore, following this line, a possible alternative to solve this tension is considering extensions beyond ΛCDM (see [26, 27] for a review). In [18], some results related to the H_0 tension were discussed in the context of fractional cosmology. In the present paper, we discuss the main results of [18–20].

2. Modified Friedmann equations

In Fractional cosmology, the Friedmann equation for the Friedmann-Lemaître-Robertson-Walker (FLRW) metric in a flat Universe is modified according to [8]

$$3H^2 + 3(1-\mu)Ht^{-1} = \sum_i \rho_i, \tag{1}$$

where μ is the fractional constant parameter, $H = \dot{a}/a$ is the Hubble parameter, the dot means derivative with respect the cosmic time, and we use units where $8\pi G = 1$.

Assuming separated conservation equations, we have [18]

$$\dot{\rho}_i + 3\left(H + (1-\mu)/(3t)\right)(\rho_i + p_i) = 0. \tag{2}$$

If $\mu = 1$, the standard cosmology without Λ is recovered. We are assuming the equation of state $p_i = w_i \rho_i$, with $w_i \neq -1$ constants, setting $a(t_0) = 1$, where t_0 is the current age of the Universe, and denoting by ρ_{0i} the current value of energy density of the i-th species, we obtain $\rho_i(t) = \rho_{0i} a(t)^{-3(1+w_i)} (t/t_0)^{(\mu-1)(1+w_i)}$. For $\mu \neq 1$, we have

$$\sum_i p_i = 6(\mu-3)Ht^{-1} + 3H^2 - 3(\mu-2)(\mu-1)t^{-2}. \tag{3}$$

Eliminating $\sum_i p_i$ and $\sum_i \rho_i$, we obtain

$$\dot{H} = -2(\mu-4)Ht^{-1} - 3H^2 + (\mu-2)(\mu-1)t^{-2}, \tag{4}$$

Defining the dimensionless time variable $\xi = t/t_0$, we obtain by integrating (4), the following:

$$E(\xi) = \frac{H(\xi)}{H_0} = \frac{1}{3\alpha_0 \xi}\left[\frac{9-2\mu+r}{2} - \frac{cr}{c+\xi^r}\right], \tag{5}$$

$$z(\xi) = -1 + \xi^{\frac{1}{6}(2\mu+r-9)} \sqrt[3]{\frac{(c+1)}{c+\xi^r}}, \tag{6}$$

$$p(\xi) = \frac{H_0^2\left(2(4\mu-9)r\left(\xi^{2r}-c^2\right) + r^2\left(\xi^r - c\right)^2 - 7(4\mu(2\mu-9)+45)\left(c+\xi^r\right)^2\right)}{12\alpha_0^2\xi^2\left(c+\xi^r\right)^2}, \tag{7}$$

$$\rho(\xi) = \frac{H_0^2\left(-2(5\mu-12)r\left(\xi^{2r}-c^2\right) + r^2\left(\xi^r - c\right)^2 + (2\mu-9)(8\mu-15)\left(c+\xi^r\right)^2\right)}{12\alpha_0^2\xi^2\left(c+\xi^r\right)^2}, \tag{8}$$

$$q(\xi) = -\frac{c^2(2\mu+r-9)(2\mu+r-3) + 2c\left(4\mu^2-24\mu+5r^2+27\right)\xi^r + (-2\mu+r+3)(-2\mu+r+9)\xi^{2r}}{\left((-2\mu+r+9)\xi^r - c(2\mu+r-9)\right)^2}, \tag{9}$$

$$w_{\text{eff}}(\xi) = \frac{2(4\mu-9)r\left(\xi^{2r}-c^2\right) + r^2\left(\xi^r-c\right)^2 - 7(4\mu(2\mu-9)+45)\left(c+\xi^r\right)^2}{\left((-2\mu+r+9)\xi^r - c(2\mu+r-9)\right)\left((-8\mu+r+15)\xi^r - c(8\mu+r-15)\right)}, \tag{10}$$

$$\Omega_{\text{m}}(\xi) = \frac{(-8\mu+r+15)\xi^r - c(8\mu+r-15)}{(-2\mu+r+9)\xi^r - c(2\mu+r-9)}, \tag{11}$$

where $\alpha(t) = tH$ is the age parameter, $\alpha_0 = H_0 t_0$, $c = \frac{-2\mu+r-6\alpha_0+9}{2\mu+r+6\alpha_0-9}$, and $r = \sqrt{8\mu(2\mu-9)+105}$. Taking the limit as $\xi \to \infty$, we have

$$\lim_{\xi\to\infty} z(\xi) = -1, \quad \lim_{\xi\to\infty} E(\xi) = 0, \quad \lim_{\xi\to\infty} p(\xi) = 0, \quad \lim_{\xi\to\infty} \rho(\xi) = 0,$$

$$\lim_{\xi\to\infty} q(\xi) = \frac{-13-2(\mu-4)\mu + \sqrt{8\mu(2\mu-9)+105}}{2(\mu-2)(\mu-1)}, \lim_{\xi\to\infty} \alpha(\xi) = \frac{1}{6}\left(9-2\mu+\sqrt{8\mu(2\mu-9)+105}\right) \geq 0,$$

$$\lim_{t\to\infty} w_{\text{eff}}(t) = \frac{-7+\sqrt{8\mu(2\mu-9)+105}}{4(\mu-1)}, \quad \lim_{t\to\infty} \Omega_{\text{m}}(t) = \frac{5-\sqrt{8\mu(2\mu-9)+105}}{2(\mu-2)}. \tag{12}$$

The main difficulty of this approach is the need to invert (6) to obtain ξ as a function of z because data is in terms of redshift, which is impossible using analytical tools. After all, the equation is a rational one. By introducing the logarithmic independent variable $s = -\ln(1+z)$, with $s \to -\infty$ as $z \to \infty$, $s \to 0$ as $z \to 0$, and $s \to \infty$ as $z \to -1$, and given the initial conditions $\alpha(0) = t_0 H_0, t(0) = t_0$, we obtain the initial value problem

$$\alpha'(s) = 9 - 2\mu - 3\alpha(s) + \frac{(\mu-2)(\mu-1)}{\alpha(s)}, \tag{13}$$

$$t'(s) = t(s)/\alpha(s). \tag{14}$$

The equation (13) gives a one-dimensional dynamical system. The equilibrium points are $T_1 : \alpha = \frac{1}{6}\left(9 - 2\mu - \sqrt{8\mu(2\mu-9)+105}\right)$, satisfying $\alpha > 0$ for $1 < \mu < 2$, and $T_2 : \alpha = \frac{1}{6}\left(9 - 2\mu + \sqrt{8\mu(2\mu-9)+105}\right)$, that satisfies $\alpha > 0$ for $\mu \in \mathbb{R}$. T_1 is a source whenever it exists and T_2 is always a sink. That is the asymptotic behaviour for large t, which is consistent with [20]. We introduce the parameter ϵ_0 such that

$$\epsilon_0 = \frac{1}{2}\lim_{t\to\infty}\left(\frac{t_0 H_0 - tH}{tH}\right), \quad \alpha_0 = \frac{1}{6}\left(9 - 2\mu + \sqrt{8\mu(2\mu-9)+105}\right)(1+2\epsilon_0). \tag{15}$$

ϵ_0 is a measure of the limiting value of the relative error in the age parameter tH when it is approximated by $t_0 H_0$.

3. Methodology and dataset

A Bayesian Markov Chain Monte Carlo (MCMC) analysis is performed to constrain the phase-space parameter $\mathbf{\Theta} = \{h, \Omega_{0m}, \mu\}$ of the fractional cosmology using observational Hubble data OHD, SNe Ia dataset and joint analysis. Under the `emcee` Python package environment [28], after the auto-correlation time criterion warranty the convergence of the chains, a set of 4000 chains with 250 steps each is performed to establish the parameter bounds. Additionally, the configuration for the priors are Uniform distributions allowing vary the parameters in the range $h \in [0.2, 1]$, $\Omega_{0m} \in [0, 1]$ and $\mu \in [1, 3]$. Hence the figure-of-merit for the joint analysis is built through the Gaussian log-likelihood given as $-2\ln(\mathcal{L}_{\rm data}) \propto \chi^2_{\rm data}$ and $\chi^2_{\rm Joint} = \chi^2_{\rm CC} + \chi^2_{\rm SNeIa}$, where each term refers to the χ^2-function for each dataset. Now, each piece of data is described.

3.1 Cosmic chronometers

Up to now, a set of 31 points obtained by differential age tools, namely cosmic chronometers (CC), represents the measurements of the Hubble parameter, which is cosmological independent [29]. In this sense, this sample is useful to bound alternative models to ΛCDM. Thus, the figure-of-merit function to minimize is given by

$$\chi^2_{\rm CC} = \sum_{i=1}^{31}\left(\frac{H_{th}(z_i) - H_{obs}(z_i)}{\sigma^i_{obs}}\right)^2, \tag{16}$$

where the sum runs over the whole sample, and $H_{th} - H_{obs}$ is the difference between the theoretical and observational Hubble parameter at the redshift z_i and σ_{obs} is the uncertainty of H_{obs}.

3.2 Type Ia Supernovae

Ref. [30] provides 1048 luminosity modulus measurements, known as Pantheon sample, from Type Ia Supernovae, which cover a region $0.01 < z < 2.3$. Due to this sample, the measurements are correlated, and it is convenient to build the chi-square function as

$$\chi^2_{\rm SNeIa} = a + \log\left(\frac{e}{2\pi}\right) - \frac{b^2}{e}, \tag{17}$$

where

$$a = \Delta\tilde{\boldsymbol{\mu}}^T \cdot \mathbf{Cov_P^{-1}} \cdot \Delta\tilde{\boldsymbol{\mu}}, \quad b = \Delta\tilde{\boldsymbol{\mu}}^T \cdot \mathbf{Cov_P^{-1}} \cdot \Delta\mathbf{1}, \quad e = \Delta\mathbf{1}^T \cdot \mathbf{Cov_P^{-1}} \cdot \Delta\mathbf{1}. \tag{18}$$

Furthermore, $\Delta\tilde{\boldsymbol{\mu}}$ is the vector of residuals between the theoretical distance modulus and the observed one, $\Delta\mathbf{1} = (1, 1, \dots, 1)^T$, $\mathbf{Cov_P}$ is the covariance matrix formed by adding the systematic and statistic uncertainties, i.e. $\mathbf{Cov_P} = \mathbf{Cov_{P,sys}} + \mathbf{Cov_{P,stat}}$. The super-index T on the above expressions denotes the transpose of the vectors.

The theoretical distance modulus is estimated by

$$m_{th} = \mathcal{M} + 5\log_{10}\left[\frac{d_L(z)}{10\,pc}\right], \tag{19}$$

where $\mathcal{M}$ is a nuisance parameter which has been marginalized by Eq. (17).

The luminosity distance, denoted as $d_L(z)$, is computed through

$$d_L(z) = (1+z)c\int_0^z \frac{dz'}{H(z')}, \tag{20}$$

being c the speed of light.

In [18], the theoretical $H(z)$ is obtained by solving numerically the system

$$E(z) = -fF(z) + F(z)^{-\mu}\left\{f^2F(z)^{2(\mu+1)} + \Omega_{0m}(z+1)^3F(z)^{\mu+1} + \Omega_{0r}(z+1)^4F(z)^{\frac{2(\mu+2)}{3}}\right\}^{1/2}, \tag{21}$$

$$F'(z) = \frac{2fF(z)^{\mu+2}}{(\mu-1)(z+1)}\left\{fF(z)^{\mu+1} - \left[f^2F(z)^{2\mu+2} + \Omega_{0m}(z+1)^3F(z)^{\mu+1} + \Omega_{0r}(z+1)^4F(z)^{\frac{2(\mu+2)}{3}}\right]^{1/2}\right\}^{-1}, \tag{22}$$

where $f \equiv (1-\mu)/(2t_0H_0)$ is going to be the *fractional constant* that will act as the cosmological constant. For $\mu > 1$, $\Omega_{0m} + \Omega_{0r} < 1$, and notice that we choose the positive branch in order to have $E(z) > 0$ and where $\Omega_{0r} = 2.469 \times 10^{-5}h^{-2}(1 + 0.2271N_{\rm eff})$, where $N_{\rm eff} = 2.99 \pm 0.17$ [24].

On the other hand, in reference [19] was constrained the free parameters with the SNe Ia data and OHD using more data points. In particular, the first one has used the same sample as in [18]. The second one is considered the OHD sample compiled by Magaña *et al.* [32], which consists of 51 data points from cosmic chronometers and BAO estimations in the redshift range $0.07 \leq z \leq 2.36$. Hence, for the constraint, we numerically integrate the system given by Eqs. (13) and (14), which represent a system for the variables (α, t) as a function of $s = -\ln(1+z)$, and for which we consider the initial conditions $\alpha(s=0) \equiv \alpha_0 = t_0H_0$ and $t(s=0) \equiv t_0 = \alpha_0/H_0$. Then, the Hubble parameter is obtained numerically by $H_{th}(z) = \alpha(z)/t(z)$. For this integration, we consider the NumbaLSODA code, a python wrapper of the LSODA method in ODEPACK to C++ [1]. For further comparison, were also constrained the free parameters of the ΛCDM model by excluding radiation, whose respective Hubble parameter as a function of the redshift is given by $H(z) = H_0\sqrt{\Omega_{m,0}(1+z)^3 + 1 - \Omega_{m,0}}$.

Finally, the free parameter α_0 was considered through the parameterization given by Eq. (15). Therefore, the free parameters of the Fractional cosmological model are $\theta = \{h, \mu, \epsilon_0\}$, and the free parameters of the ΛCDM model are $\theta = \{h, \Omega_{m,0}\}$. For the free parameters μ, ϵ_0, and $\Omega_{m,0}$ were considered the following flat priors: $\mu \in F(1, 4)$, $\epsilon_0 \in F(-0.1, 0.1)$, and $\Omega_{m,0} \in F(0, 1)$. On the other hand, the prior chosen for ϵ_0 is because ϵ_0 is a measure of the limiting value of the relative error in the age parameter tH when it is approximated by $t_0 H_0$ as given by Eq. (15). For the mean value $\epsilon_0 = 0$, we acquire $\alpha_0 = \frac{1}{6}(-2\mu + r + 9)$, and then, we have the leading term for $E(z) \simeq (1+z)^{\frac{6}{(9-2\mu+r)}}$. The lower prior of μ is because the Hubble parameter $H \simeq (\mu - 1)/t$ becomes negative when $\mu < 1$, in the absence of matter.

3.3 Results and discussion

The constraints obtained in [18] through cosmic chronometers, Type Ia Supernovae, and joint analysis are summarized in Fig. 1 and Table 1 (middle rows). The fractional parameter prefers $\mu = 2.839^{+0.117}_{-0.193}$ for a joint analysis which suggests a solid presence of fractional calculus in the dynamical equations of cosmology; however, it generates crucial differences as it is possible to observe from Figs 2a, 2b, and 2b. From one side, the term $(1-\mu)H/t$ acts like an extra source of mass, closing the Universe and not allowing the observed dynamics, in particular, the Universe acceleration at late times if $\mu < 2$, but, for $\mu > 2$, we can have an accelerated power-law solution. Furthermore, from Figs. 2a, 2b, and 2b it is possible to notice that the fractional constant f can act like the object that causes the Universe acceleration. It is possible to observe from $H(z)$ and $q(z)$ essential differences when we compare them with the standard model, mainly at high redshifts. In addition, the Jerk parameter also shows that the source of the Universe acceleration is not a cosmological constant because, at $z = 0$, the fractional parameter does not converge to $j = 1$; this is in agreement with recent studies that suggest that Λ is not the cause of the Universe acceleration [33].

Moreover, the Universe's age obtained under this scenario is $t_0 = 33.617^{+3.411}_{-4.511}$ Gyrs based on our Joint analysis, around 2.4 times larger than the age of the Universe expected under the standard paradigm. However, this value does not contradict the minimum bound expected for the universe age imposed by globular clusters. As far as we know, the maximum bound does not exist and is model-dependent. Fig. 2d displays the reconstruction of the $\mathbb{H}0(z)$ diagnostic [34] for the fractional cosmology and its error band at 3σ of confidence level (CL).

In reference [19], the best-fit values of the free parameters space for the ΛCDM model and the Fractional cosmological model, obtained from the SNe Ia data, OHD, and in their joint analysis, with their corresponding χ^2_{min} criteria, are presented in Table 1 (last rows). The uncertainties correspond to 1σ, 2σ, and 3σ CL. In Figures 3a and 3b, we depict the posterior distribution and joint admissible regions of the free parameters space of the ΛCDM model and the Fractional cosmological model, respectively. The joint admissible regions correspond to 1σ, 2σ, and 3σ CL. Due to the degeneracy between H_0 and $\mathcal{M}$, the distribution of h for the SNe Ia data was not represented in their full parameter space. The analysis from the SNe Ia data leads to $h = 0.696^{+0.302}_{-0.295}$, $\mu = 1.340^{+2.651}_{-0.339}$ and

[1]Available online in the GitHub repository https://github.com/Nicholaswogan/numbalsoda

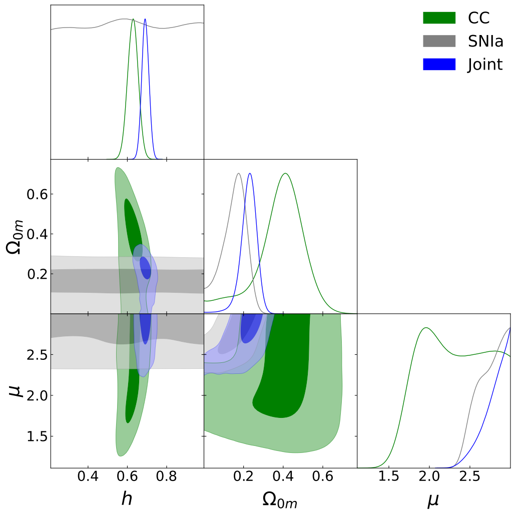
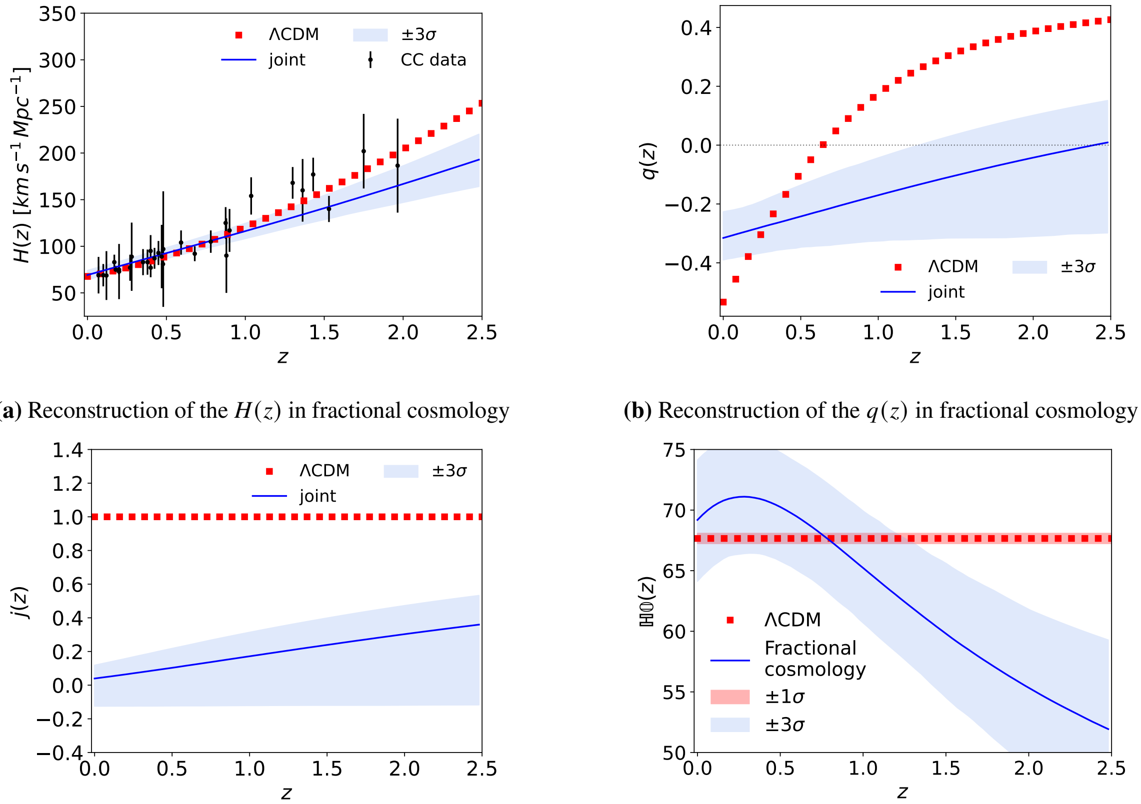


Figure 1: 2D likelihood contours at 68% and 99.7% CL, alongside the corresponding 1D posterior distribution of the free parameters (see Ref. [18])

Table 1: Best-fit values and χ^2_{min} criteria for the ΛCDM model with free parameters h and $\Omega_{m,0}$; for the Fractional cosmological model (dust + radiation) [18]; and the Fractional cosmological model with free parameters h, μ, and ϵ_0 [19]. The MCMC analysis obtained the values from the SNe Ia data, OHD (or CC), and their joint analysis. The ΛCDM model is used as a reference model.

	Best-fit values				
Data	h	$\Omega_{m,0}$	μ	$\epsilon_0 \times 10^2$	χ^2_{min}
		ΛCDM model			
SNe Ia	$0.692^{+0.209\ +0.296\ +0.307}_{-0.120\ -0.278\ -0.292}$	$0.299^{+0.022\ +0.046\ +0.068}_{-0.021\ -0.042\ -0.059}$	$\cdots$	$\cdots$	1026.9
OHD	$0.706^{+0.012\ +0.024\ +0.035}_{-0.012\ -0.024\ -0.036}$	$0.259^{+0.018\ +0.038\ +0.059}_{-0.017\ -0.033\ -0.047}$	$\cdots$	$\cdots$	27.5
SNe Ia+OHD	$0.696^{+0.010\ +0.020\ +0.029}_{-0.010\ -0.020\ -0.029}$	$0.276^{+0.014\ +0.030\ +0.043}_{-0.014\ -0.027\ -0.040}$	$\cdots$	$\cdots$	1056.3
	Fractional cosmological model (dust + radiation) [18] (The uncertainties presented correspond to 1σ(68.3%) CL)				
SNe Ia	$0.599^{+0.275}_{-0.269}$	$0.160^{+0.050}_{-0.072}$	$2.771^{+0.161}_{-0.214}$	$\cdots$	54.83
CC	$0.629^{+0.027}_{-0.027}$	$0.399^{+0.093}_{-0.122}$	$2.281^{+0.492}_{-0.433}$	$\cdots$	16.14
SNe Ia+CC	$0.692^{+0.019}_{-0.018}$	$0.228^{+0.035}_{-0.040}$	$2.839^{+0.117}_{-0.193}$	$\cdots$	78.69
	Fractional cosmological model [19] (The uncertainties presented correspond to 1σ(68.3%), 2σ(95.5%), and 3σ(99.7%) CL)				
SNe Ia	$0.696^{+0.215\ +0.293\ +0.302}_{-0.204\ -0.284\ -0.295}$	$\cdots$	$1.340^{+0.492\ +2.447\ +2.651}_{-0.245\ -0.328\ -0.339}$	$1.976^{+0.599\ +1.133\ +1.709}_{-0.905\ -1.848\ -2.067}$	1028.1
OHD	$0.675^{+0.013\ +0.029\ +0.041}_{-0.008\ -0.015\ -0.021}$	$\cdots$	$2.239^{+0.449\ +0.908\ +1.386}_{-0.457\ -0.960\ -1.190}$	$0.865^{+0.395\ +0.650\ +0.793}_{-0.407\ -0.657\ -0.773}$	29.7
SNe Ia+OHD	$0.684^{+0.011\ +0.021\ +0.031}_{-0.010\ -0.020\ -0.027}$	$\cdots$	$1.840^{+0.343\ +1.030\ +1.446}_{-0.298\ -0.586\ -0.773}$	$1.213^{+0.216\ +0.383\ +0.482}_{-0.310\ -0.880\ -1.057}$	1061.1

$\epsilon_0 = \left(1.976^{+1.709}_{-2.067}\right) \times 10^{-2}$, which are the best-fit values at 3σ CL. In this case, the value obtained for h cannot be considered as a best fit due to the degeneracy between H_0 and $\mathcal{M}$. On the other hand,



(a) Reconstruction of the $H(z)$ in fractional cosmology

(b) Reconstruction of the $q(z)$ in fractional cosmology

(c) Reconstruction of the $j(z)$ in fractional cosmology

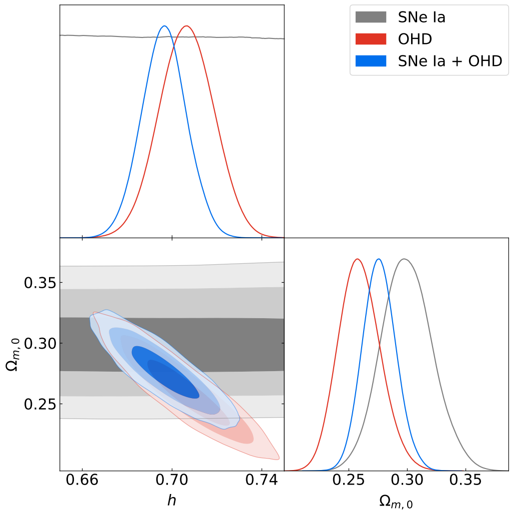
(d) $\mathbb{H}0(z)$ diagnostic for fractional cosmology

Figure 2: Reconstruction of Hubble Factor $H(z)$, deceleration parameter $q(z)$, Jerk $j(z)$, and $\mathbb{H}0(z)$ diagnostic and its comparison against ΛCDM model (red dashed lines) with $h = 0.6766$ and $\Omega_{m0} = 0.3111$ [24] (see Ref. [18])

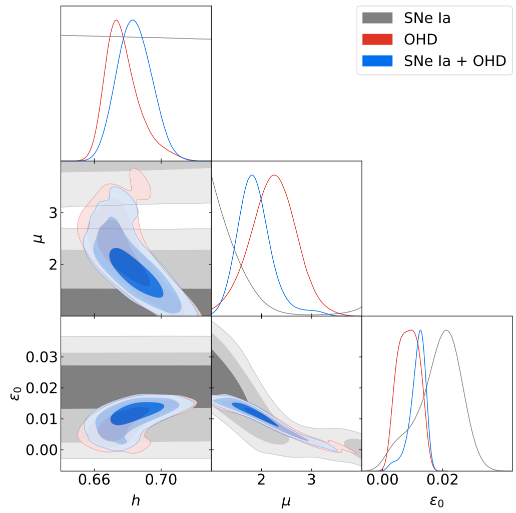
the lower limit of the best fit for μ is very close to 1. That is because the posterior distribution for this parameter is close to this value, as seen from Figure 3b. That indicates that a value of the SNe Ia data prefers $\mu < 1$, but, as a reminder, this value leads to a negative Hubble parameter in the absence of matter. However, as can be seen from the same Figure 3b, the posterior distribution for these parameters is multi-modal. Therefore, it is possible to obtain a best-fit value that satisfies $\mu > 1$.

It is important to mention that the OHD and the joint analysis do not experience this issue, which allows us to maintain the validity of the prior used for μ. The analysis from OHD leads to $h = 0.675^{+0.041}_{-0.021}$, $\mu = 2.239^{+1.386}_{-1.190}$ and $\epsilon_0 = \left(0.865^{+0.793}_{-0.773}\right) \times 10^{-2}$, which are the best-fit values at 3σ CL. In this case, note how the OHD can properly constrain the free parameters h, μ and ϵ_0, i.e., we obtain the best fit for the priors considered in our MCMC analysis. Also, note how the posterior distribution of μ includes the value of 1, as seen from Figure 3b, but greater than 3σ CL.

Finally, the joint analysis with data from SNe Ia + OHD leads to $h = 0.684^{+0.031}_{-0.027}$, $\mu = 1.840^{+1.446}_{-0.773}$ and $\epsilon_0 = \left(1.213^{+0.482}_{-1.057}\right) \times 10^{-2}$, which are the best-fit values at 3σ CL. Focusing our analysis on these results, we can conclude that the region in which $\mu > 2$ is not ruled out by observations. On the other hand, these best-fit values lead to an age of the Universe with a value of $t_0 = \alpha_0/H_0 = 25.62^{+6.89}_{-4.46}$ Gyrs at 3σ CL. This fact to find a universe roughly twice older as one of the ΛCDM models, which is also



(a) Posterior distribution and joint admissible regions of the free parameters h and $\Omega_{m,0}$ for the ΛCDM model, obtained in the MCMC analysis.



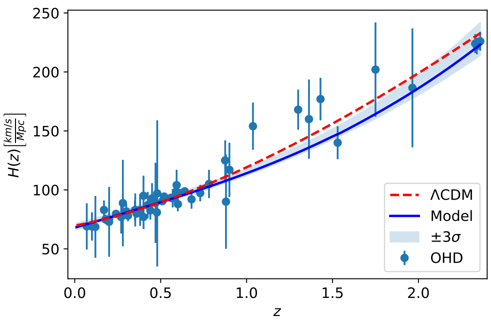
(b) Posterior distribution and joint admissible regions of the free parameters h, μ, and ϵ_0 for the Fractional cosmological model, obtained in the MCMC analysis.

Figure 3: Posterior distribution and joint admissible regions of the free parameters obtained in the MCMC analysis for each model. The admissible joint regions correspond to $1\sigma(68.3\%)$, $2\sigma(95.5\%)$, and $3\sigma(99.7\%)$ CL, respectively. The best-fit values for each model free parameter are shown in Table 1 (see Ref. [19]).

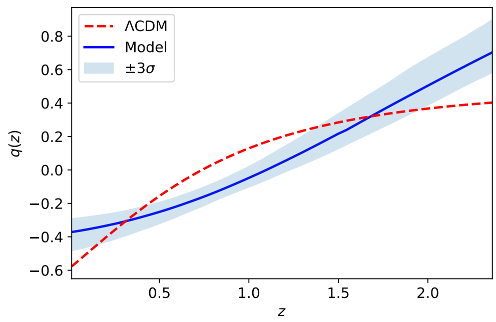
in disagreement with the value obtained with globular clusters, with a value of $t_0 = 13.5^{+0.16}_{-0.14} \pm 0.23$ [35], is a distinction of the Fractional Cosmology. This result also agrees with the analysis made in [18], section 8, where the best-fit μ-value is obtained from the reconstruction of $H(z)$ for different priors of μ. In [18] was considered a set of 31 points obtained by differential age tools, namely cosmic chronometers (CC), represents the measurements of the Hubble parameter, which is cosmological independent [29] (in [19] was considered the datasets from [32], which consists of 51 data points in the redshift range $0.07 \leq z \leq 2.36$, 20 more points as compared with [29]; nevertheless, the additional points are model-dependent). The 1048 luminosity modulus measurements, known as the Pantheon sample, from Type Ia Supernovae cover a region $0.01 < z < 2.3$ [31]. In [18], results depend on the priors used for μ. Say, for the prior $0 < \mu < 1$, $\mu = 0.50$ and $t_0 = 41.30$ Gyrs; for $1 < \mu < 3$, $\mu = 1.71$ and $t_0 = 27.89$ Gyrs; and for $0 < \mu < 3$, $\mu = 1.15$ and $t_0 = 33.66$ Gyrs.

From the values for the χ^2_{min} criteria presented in Table 1 (first rows), it is possible to see that the ΛCDM model is the best model to fit the SNe Ia, OHD, and joint data. Nevertheless, the Fractional cosmological model studied in [19] exhibits values of the χ^2_{min} criteria close to the values of the ΛCDM model, with differences of 1.2 for the SNe Ia data, 2.2 for the OHD data, and 4.8 for their joint analysis. So, this Fractional cosmological model is suitable for describing the SNe Ia and OHD data, as seen from Figure 4a and Figure 5, showing the transition from a deceleration expansion phase to an accelerated one. Therefore, Fractional Cosmology can be considered an alternative valid cosmological model to describe the late-time Universe.

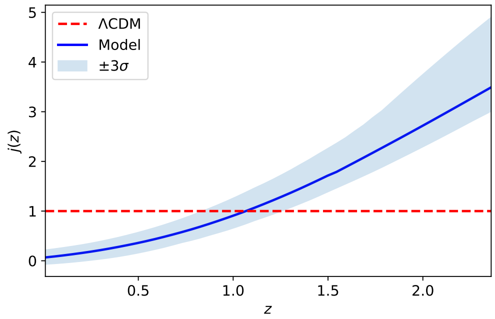
On the other hand, Fig. 4a shows the theoretical Hubble parameter for the ΛCDM model (red dashed line) and the Fractional cosmological model (solid blue line) as a function of the redshift z, contrasted with the OHD sample. The shaded curve represents the confidence region of the



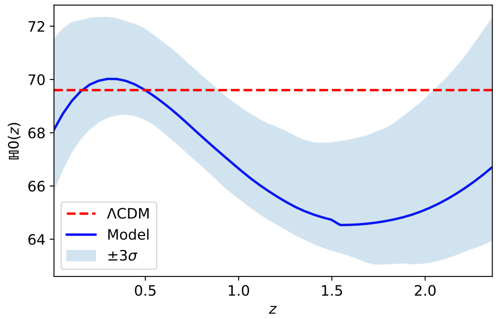
(a) Reconstruction of the $H(z)$ in fractional cosmology



(b) Reconstruction of the $q(z)$ in fractional cosmology



(c) Reconstruction of the $j(z)$ in fractional cosmology



(d) $\mathbb{H}0(z)$ diagnostic for fractional cosmology

Figure 4: Theoretical Hubble parameter, Deceleration parameter, Jerk, and $\mathbb{H}0$ diagnostic (solid blue line) as a function of the redshift z for the Fractional cosmological model. The shaded curve represents the confidence region at $3\sigma(99.7\%)$ CL. Each model is compared with the ΛCDM model (red dashed line). Fig. 4a is contrasted with the OHD sample. The rest of the figure is obtained using the best-fit values from the joint analysis in Table 1 (see Ref. [19])

Hubble parameter for the Fractional cosmological model at $3\sigma(99.7\%)$ CL. The figure is obtained using the best-fit values from the joint analysis in Table 1 (last rows). Additionally, in order to establish that this Fractional cosmological model can describe a universe that experiences a transition from a decelerated expansion phase to an accelerated one, we compute the deceleration parameter $q = -1 - \dot{H}/H^2$, which using the Riccati Equation (4), leads to

$$q(\alpha(s)) = 2 + \frac{2(\mu-4)}{\alpha(s)} - \frac{(\mu-2)(\mu-1)}{\alpha^2(s)}. \tag{23}$$

Following this line, in Figure 4b, we depict the deceleration parameter for the Fractional cosmological model as a function of the redshift z, obtained for the best-fit values from the joint analysis in the Table 1 (last rows), with an error band at 3σ CL. We also depict the deceleration parameter for the ΛCDM model as a reference model. From this figure, we can conclude that the Fractional cosmological model effectively experiences this transition at $z_t \gtrsim 1$, with the characteristic that $z_t > z_{t,\Lambda CDM}$, being $z_{t,\Lambda CDM}$ the transition redshift of the ΛCDM model. Even more, the current deceleration parameter of the Fractional cosmological model is $q_0 = -0.37^{+0.08}_{-0.11}$ at 3σ CL. Moreover, we compute the cosmographic parameter known as the Jerk, which quantifies if the Fractional

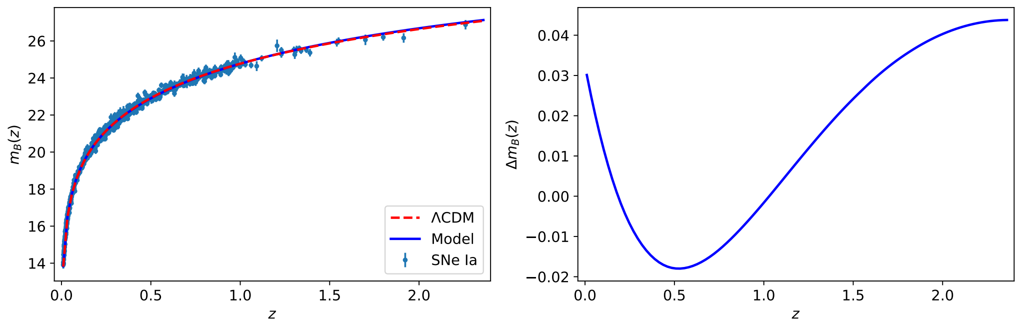


Figure 5: (*left panel*) Theoretical apparent B-band magnitude for the ΛCDM model (red dashed line) and the Fractional cosmological model (solid blue line) as a function of the redshift z, contrasted with the Pantheon data set. (*right panel*) Variation of the theoretical apparent B-band magnitude of the Fractional cosmological model concerning the ΛCDM model as a function of the redshift z. The curve is obtained through the expression $\Delta m_B = m_{B,Model} - m_{B,\Lambda CDM}$. The figures are obtained using the best-fit values from the joint analysis in Table 1 (see Ref. [19])

cosmological model tends to Λ or its another kind of DE, which can be written as

$$j(s) = q(s)(2q(s) + 1) - \frac{dq(s)}{ds}, \tag{24}$$

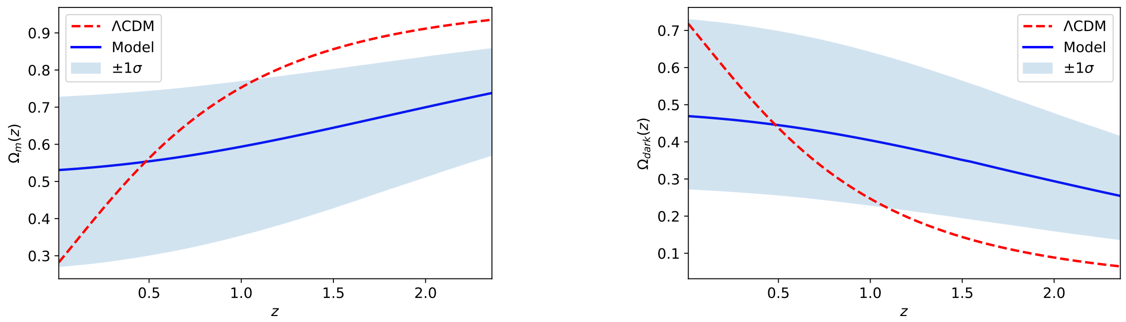
where q is given by Eq. (23). Hence,

$$j(\alpha(s)) = \frac{12(\mu - 4)}{\alpha(s)} + \frac{(\mu - 21)\mu + 50}{\alpha(s)^2} - \frac{2(\mu - 3)(\mu - 2)(\mu - 1)}{\alpha(s)^3} + 10. \tag{25}$$

Figure 4c shows the Jerk for the ΛCDM model (red dashed line) and the Fractional cosmological model (solid blue line) as a function of the redshift z. The figure is obtained using the best-fit values from the joint analysis in Table 1 (last rows) with an error band at 3σ CL, represented by a shaded region. A departure of more than 3σ of CL for the current value for ΛCDM shows an alternative cosmology with an effective dynamical equation of state for the Universe for late times in contrast to ΛCDM.

On the other hand, in Figure 4d, we depict $\mathbb{H}0$ diagnostic for the ΛCDM model (red dashed line) and the Fractional cosmological model (solid blue line) as a function of the redshift z. The figure is obtained using the best-fit values from the joint analysis in Table 1 (last rows), with an error band at 3σ CL, represented by a shaded region. As a reminder, in both Figures 4c and 4d, we also depict the Jerk and the $\mathbb{H}0$ diagnostic for the ΛCDM model as a reference model.

In Figures 6a and 6b, we depict the matter density and fractional density parameters for the Fractional cosmological model (the last one interpreted as dark energy), respectively, as a function of the redshift z, for the best-fit values from the joint analysis in the Table 1, with an error band at 1σ CL. We depict the matter density and dark energy density parameters in both figures for the ΛCDM model. From Figure 6a, we can see that the matter density parameter for the Fractional cosmological model, obtained from Eq. (11), presents significant uncertainties, which can be a consequence of their reconstruction from a Hubble parameter that does not take into account any EoS. In this sense, the current value of this matter density parameter at 1σ CL is $\Omega_{m,0} = 0.531^{+0.195}_{-0.260}$, a value that is in



(a) Matter density parameter for the ΛCDM model (red dashed line) and the Fractional cosmological model (solid blue line) as a function of the redshift z.

(b) Dark energy density parameter for the ΛCDM model (red dashed line) and the Fractional cosmological model (solid blue line) as a function of the redshift z.

Figure 6: Dark energy and Dark Matter density parameters for the ΛCDM model and the Fractional cosmological model as a function of the redshift z. The shaded curve represents the confidence region of the matter density parameter for the Fractional cosmological model at 1σ(68.3%) CL. The figure is obtained using the best-fit values from the joint analysis in Table 1 (see Ref. [19])

agreement with the asymptotic value obtained from Eq. (12) of $\Omega_{m,t\to\infty} = 0.519^{+0.199}_{-0.262}$, computed at 1σ CL for the best-fit values from the joint analysis in the Table 1 (last rows). Therefore, this greater value of $\Omega_{m,0}$ for the Fractional cosmological model can, in principle, explain the lower value of the current deceleration parameter q_0 and the excess of matter in the effective term $\rho_{\text{frac}} = 3(\mu-1)t^{-1}H$ with $\Omega_{\text{frac}}(\alpha(s)) = (\mu-1)/\alpha(s)$. Note that the current value $\Omega_{\text{frac},0}$ can be interpreted as the dark energy density parameter for the Fractional cosmological model as $\Omega_{\text{frac},0} = 0.469^{+0.260}_{-0.195}$, which satisfies the condition $\Omega_{m,0} + \Omega_{\text{frac},0} = 1$. Observe that the energy densities of DE and DM are of the same order of magnitude today, alleviating the Coincidence Problem.

Finally, we estimate the free parameters (α_0, μ) using cosmological data. Using the re-parameterization $H_0 = 100\frac{\text{km/s}}{\text{Mpc}}h$, $\alpha_0 = \frac{1}{6}\left(9 - 2\mu + \sqrt{8\mu(2\mu-9)+105}\right)(1+2\epsilon_0)$.

The analysis from the SNe Ia data, OHD and the joint analysis with data from SNe Ia + OHD leads respectively to $h = 0.696^{+0.302}_{-0.295}$, $\mu = 1.340^{+2.651}_{-0.339}$ and $\epsilon_0 = \left(1.976^{+1.709}_{-2.067}\right)\times 10^{-2}$, $h = 0.675^{+0.041}_{-0.021}$, $\mu = 2.239^{+1.386}_{-1.190}$ and $\epsilon_0 = \left(0.865^{+0.793}_{-0.773}\right)\times 10^{-2}$, and $h = 0.684^{+0.031}_{-0.027}$, $\mu = 1.840^{+1.446}_{-0.773}$ and $\epsilon_0 = \left(1.213^{+0.482}_{-1.057}\right)\times 10^{-2}$, where the best-fit values are calculated at 3σ CL. On the other hand, these best-fit values lead to an age of the Universe with a value of $t_0 = \alpha_0/H_0 = 25.62^{+6.89}_{-4.46}$ Gyrs, a current deceleration parameter of $q_0 = -0.37^{+0.08}_{-0.11}$, both at 3σ CL, and a current matter density parameter of $\Omega_{m,0} = 0.531^{+0.195}_{-0.260}$ at 1σ CL. Finding a Universe roughly twice older as the one of ΛCDM is a distinction of Fractional Cosmology. Focusing our analysis on these results, we can conclude that the region in which $\mu > 2$ is not ruled out by observations. This parameter region is relevant because, in the absence of matter, fractional cosmology gives a power-law solution $a(t) = (t/t_0)^{\mu-1}$, which is accelerated for $\mu > 2$. In summary, we presented a fractional origin model that leads to an accelerated state without appealing to Λ or Dark Energy.

4. Conclusions

This paper discusses the formalism of fractional calculus, which modifies the integer order derivative by a fractional derivative of order μ. It generates changes in the Friedmann equations, where the standard evolution of the cosmic species densities depends on the fractional parameter and the Universe's current age t_0. The additional term in the new cosmic dynamics equation can support the late-time accelerated expansion without a dark energy component. We estimated stringent constraints on the fractional and cosmological parameters using observational Hubble data, Type Ia supernovae and joint analysis to elucidate that. According to our results, the Universe would be older than the standard estimations. We have obtained modified Friedmann equations at the background level under fractional calculus, which provides a late cosmic acceleration without introducing a dark energy component. This radical approach could be a new path to tackle problems not resolved until now in cosmology. Finally, we analyzed whether fractional cosmology can alleviate H_0 tension. We observe a trend of H_0 to the value obtained by the Supernova H_0 for the Equation of State [25] at current times, and in agreement with Planck's value [24] for $z \lesssim 1.5$. However, a discrepancy between both values in the region $1.5 < z < 2.5$ holds, so H_0 tension is not fully resolved.

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