PROCEEDINGS OF SCIENCE

Is M_B really a constant?

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M_B is the Supernova absolute magnitude. Ref. [1] isolates the distance ladder $d_L(z)$ from the Baryon Acoustic Oscillations (BAO) and from the Type Ia supernova data to determine M_B. The degeneracy between the absolute magnitude and the Hubble constant H_0 for the supernova data is replaced by a degeneracy between M_B and the sound horizon r_d. Since the data is for certain redshifts, the Gaussian processes (GP) regression extrapolates the values for the whole redshift range. By canceling the distance ladder, the GP gives $M_B = -19.25 \pm 0.39$ as a mean for the full distribution when using the sound horizon from late time measurements. The estimations provide up to $1\,\sigma$ possibility of a nuisance parameter presence $\delta M_B(z)$ at higher redshifts.

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PoS(CORFU2022)259

1. Introduction

The ΛCDM model is a six parameter model that has historically been the most successful in explaining cosmological observations. In the most recent reporting, ΛCDM has shown to be in an excellent agreement with measurements of the cosmic microwave background (CMB) radiation, the abundances of elements in the early Universe, the large scale structure of the Universe and other major astronomical measurable [2–5]. The revisiting of the foundations of ΛCDM has come about due to the growing observational crisis primarily centered on the value of the Hubble constant H_0, which appears to be the most serious of the emerging cosmological tensions [6–9]. The discrepancy between early and late time measurements of the value of H_0 came to the fore with the reporting of local values of H_0 from the SH0ES [10, 11], the latest of which gives $H_0^{\rm S21} = 73.04 \pm 1.04\,{\rm km\,s^{-1}\,Mpc^{-1}}$ [10], which are brought down by strong lensing measurements [12] and measurements from the tip of the red giant branch [13, 14]. On the other end of the spectrum, early Universe measurements give a drastically lower value with the latest Planck Collaboration value being $H_0^{\rm Pl18} = 67.4 \pm 0.5\,{\rm km\,s^{-1}\,Mpc^{-1}}$ [2]. Other early time measurements are largely consistent with this value such as the extended Baryon Oscillation Spectroscopic Survey (eBOSS) [3], the Dark Energy Survey [15], and results from the Atacama Cosmology Telescope (ACT) [16]. The 5σ disagreement between the early time estimation and the late time estimation of H_0 is so dramatic that it represents one of the biggest challenges in modern cosmology.

Another approach to viewing this tension is through the sound horizon at last scattering, r_d, which depends on the matter content of the pre-recombination Universe and which can have an enormous impact on the way we interpret expansion data [17]. In this work, we will utilize both the Planck Collaboration fiducial value of the sound horizon, $r_d^{\rm Pl18} = 147.09 \pm 0.26\,{\rm Mpc}$ [2, 18], and the late time value given by $r_d^{\rm HW+SN+BAO+SH0ES} = 136.1 \pm 2.7\,{\rm Mpc}$ [19]. The reaction to the cosmological tension has been varied with numerous proposals for solutions [20, 21] with a large portion of the community still seeking some systematics source to the cosmological tensions problem. However, these efforts are diminished with every new data release. By and large, the efforts to resolve the tension fall into the following possible modifications: departures from GR on cosmic scales [22–27]; adding new species to the ΛCDM model such as those described in Refs. [20, 28]; radical new physics at recombination that alters the sound horizon [29–32]; or alterations to the expansion history at late times [33–36].

In [1] we test if the M_B is really a constant. This may provide another avenue by which the tension in the Hubble diagram may be interpreted. Our approach involves taking several parametric models of $M_B = M_B(z)$, together with two non-parametric approaches. In the parametric case, we take a number of literature models for the possible variation in absolute magnitude [37–41]. On the other hand, for the non-parametric approaches we test for possible dependence on redshift through two independent machine learning probes, namely Gaussian processes (GPs). A full resolution to the tension remains largely an open problem. In this work, we attempt to probe the nuanced features of the Hubble tension using late time expansion data by exploring the constancy of the absolute magnitude of SNIa. We start with the luminosity distance which is fixed by the evolution of the Hubble parameter through

$$d_L(z) = (1+z)\int_0^z \frac{c\,dz'}{H(z')}\,, \tag{1}$$

PoS(CORFU2022)259

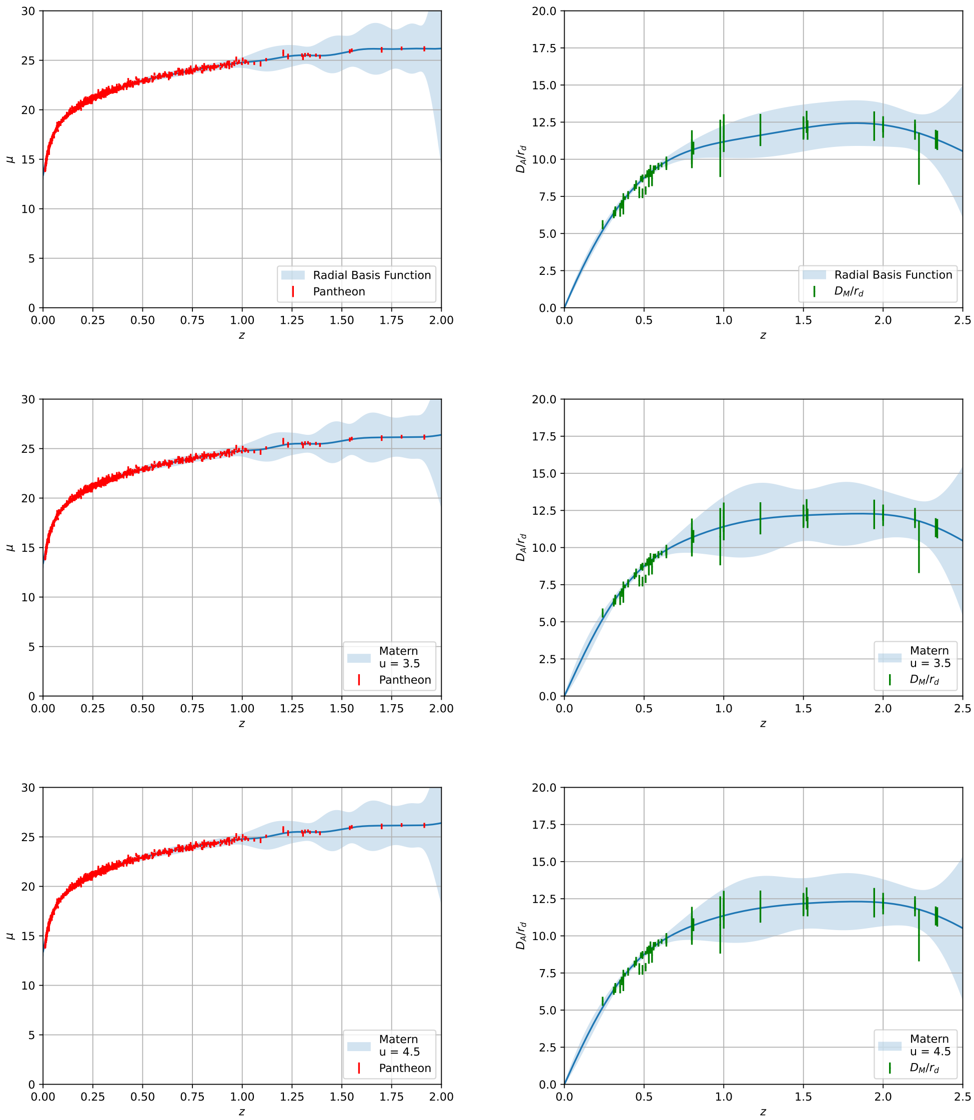

Figure 1: *The non parametric reconstruction of the Type Ia supernova (left) and the Baryon Acoustic Oscillations (right) with different Kernels.*

where c is the speed of light, and which naturally leads to the angular diameter distance $D_A(z) = d_L(z)/(1+z)^2$. Thus, irrespective of the source of the corrections needed to resolve the Hubble tension, these measurements can constrain the evolution of $H(z)$. We also note that it is not possible to include CMB data without assuming a cosmological model and so we only consider late time data to avoid adding this further constraint on our hypothesis. We can thus retain independence from a cosmological model throughout the GPs.

For the SNIa standard candles, the distance modulus $\mu_{Ia}(z)$ (defined as the difference between absolute and relative magnitudes) is related to the luminosity distance through

$$\mu_{Ia}(z) = 5\log_{10}[d_L(z)] + 25 + M_B(z)\,, \tag{2}$$

where $d_L(z)$ is measured in units of Mpc, and where $M_B(z)$ represents the possible redshift dependence of the intrinsic magnitude, which is regularly assumed to be a constant. In the present work, we explore the possibility that this may inherit some dynamics due to the cosmic evolution across redshift. We do not speculate on the potential sources, whether astrophysical or cosmological, for such a redshift dependence to arise in this work. The emergence of any possible redshift dependence would be sourced from the absolute magnitude, while $M_B(z)$ is largely assumed to be a constant in most of the literature.

The possibility of a redshift variation in the absolute magnitude was suggested in Refs. [42, 43] where they claim that the Hubble tension can also be reinterpreted through these variations in $M_B(z)$. Indeed, Refs. [44–49] suggest that a transition in the dark energy equation of state or in the absolute magnitude as a possible solution to the Hubble tension. In addition to the SNIa observations, we also make use of BAO measurements [50, 50, 50–61, 61–65] in order to take a fuller account of the possible evolution of $M_B(z)$. We incorporate BAO measurements into Eq. (2) by substituting in for the angular diameter distance giving

$$M_B = \mu_{Ia} - 5\log_{10}\left[(1+z)^2\left(\frac{D_A}{r_d}\right)_{\text{BAO}}\cdot r_d\right] - 25\,, \tag{3a}$$

where the reciprocal theorem was used, and which gives uncertainties

$$\Delta M_B = \Delta\mu_{Ia} + \frac{5}{\ln 10}\left[\frac{\Delta r_d}{r_d} + \frac{\Delta\left(D_A/r_d\right)_{BAO}}{\left(D_A/r_d\right)_{BAO}}\right]. \tag{3b}$$

Since Eq. (3b) is an analytical derivation of the error for Eq. (3a), it does not make an assumption on the Gaussianiaty of the data. Furthermore, here r_d enters as a measurement with its mean and error and not as a prior. We reconstruct the μ_{Ia} from the Type Ia supernova and the $(D_A/r_d)_{\text{BAO}}$ from the BAO measurements. Through these reconstructions, we are able to produce intermediary points for both types of data, so that be used to assess to what level the M_B nuisance parameter adheres to being a constant. As a consequence of this approach, the degeneracy between the M_B and H_0, is replaced by M_B and r_d.

2. Gaussian Process Reconstruction

GPs have been applied in a number of cosmological scenarios. The GPs will be defined via a mean function $\mu(z)$, and a kernel, function $k(z,\tilde{z})$ which together describe the continuous realisation of the GPs reconstruction $\xi(z) \sim \mathcal{GP}\left(\mu(z), k(z,\tilde{z})\right)$, where $\tilde{z}$ represents the elements of the input data sets. GPs utilizes a Bayesian approach to optimizing its kernel hyperparameters leaving open the functional choice of the kernel, so that they are fit using Bayesian optimizer approach where the difference between the reconstructed and observed behaviors is minimized at the redshift points where observations exists. For any two redshift points z and $\tilde{z}$, the kernel incorporates the strength of the correlation for the reconstructed parameter [66]. There exist a number of literature kernel choices [67, 68] which were designed to be general purpose, but which generally agree with each other to within an amount of uncertainty. In this work we consider the Radial Basis (RB) function

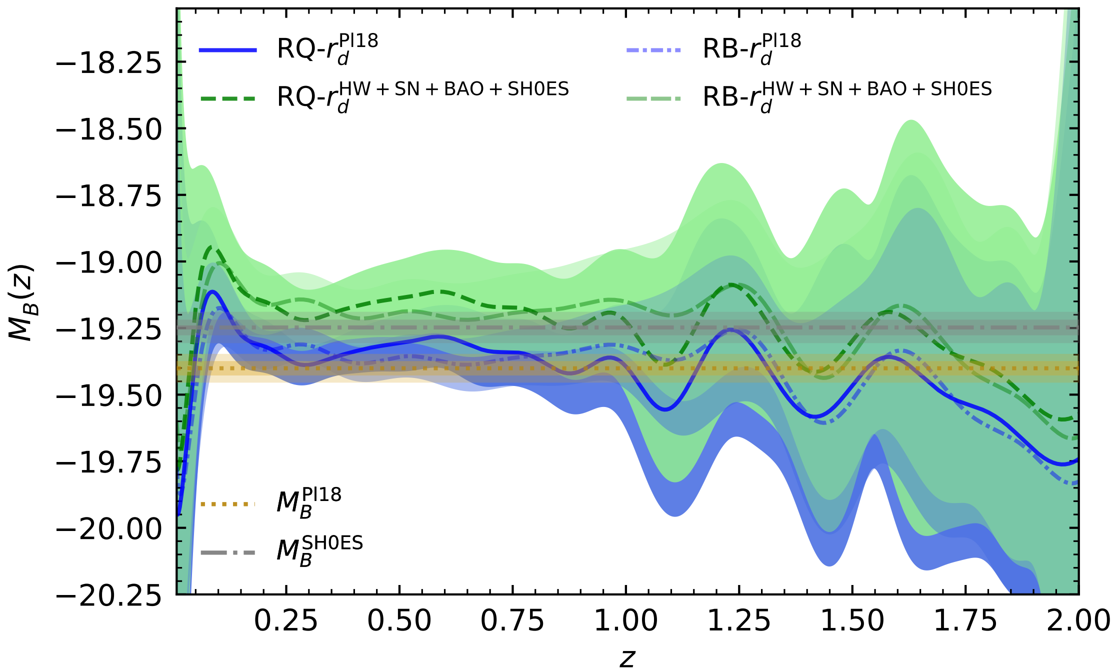

Figure 2: *Nonparametric Reconstruction for the Absolute Magnitude from GPs with a 68% confidence interval. The kernels are Rational Quadratic (RQ) and Radial Basis (RB) function. The reconstructions of $M_B(z)$ when assuming the sound horizon r_d from the Planck (blue) and the H0LiCOW+SN+BAO+SH0ES (green) are illustrated in both panels.*

kernel:

$$k(z,\tilde{z}) = \sigma_f^2 \exp\left(-\frac{(z-\tilde{z})^2}{2l^2}\right), \tag{4}$$

and the Rational Quadratic (RQ) kernel:

$$k(z,\tilde{z}) = \frac{\sigma_f^2}{\left(1+|z-z'|^2/2\alpha l^2\right)^{\alpha}}. \tag{5}$$

We have also tested the Matern kernel, but we exclude these results from the figures since they give identical results to those obtained with the RQ kernel. These kernels are infinitely differentiable and represent different expressions of the hyperparameters σ_f and l which characterize the smoothness and overall profile of the reconstructed profile. The length-scale l signifies the distance to which pairs of elements in the data set can influence each other to a significant enough extent, while σ_f controls the uncertainties across the redshift range of the reconstruction.

Fig 1 shows the reconstruction of the supernova data and the BAO data and Fig. 2 shows the combined reconstruction of the M_B assuming different values of sound horizon. Both the RQ and the RB kernels show the same behavior of δM_B at higher redshifts. For the low redshift, the GPs predicts a singularity for $z = 0$ which comes from a numerical artifact, since we take the quantity $\sim \log_{10}(D_A(z))$ and for $z = 0$ the D_A is zero. The fluctuations for $z > 1$ may be caused by the the BAO data set possessing only a small number of measurements in this redshift range. However, the possibility for $\delta M_B(z) < 0$ remains.

Technique	$r_{d,fit}^{\rm Pl18}$	$r_{d,fit}^{\rm HW+SN+BAO+SH0ES}$	$r_{d,full}^{\rm Pl18}$	$r_{d,full}^{\rm HW+SN+BAO+SH0ES}$
GP-RQ	-19.35 ± 0.03	-19.18 ± 0.03	-19.42 ± 0.35	-19.25 ± 0.39
GP-RB	-19.35 ± 0.07	-19.18 ± 0.07	-19.42 ± 0.29	-19.25 ± 0.33

Table 1: *The inferred GPs mean values of $M_B(z)$ are shown here when adopting $r_d^{\rm Pl18}$ and $r_d^{\rm H0LiCOW+SN+BAO+SH0ES}$ fiducial values. The values for GP are fitted for a Gaussian without a tail.*

The new degeneracy $M_B - r_d$ from the combination Type Ia + BAO can yield the value of $M_B(z)$ assuming a particular r_d. The mean of $M_B(z)$ (over the two kernels) gives $M_B = -19.42 \pm 0.35$ for the sound horizon from $r_d^{\rm Pl18}$, and $M_B = -19.25 \pm 0.39$ for the sound horizon using H0LiCOW+SN+BAO+SH0ES. Different kernels give only a different error, see Table 1. The results are consistent with latest measurement of M_B for the sound horizon by H0LiCOW+SN+BAO+SH0ES.

3. Discussion

Supernovae used as distance indicators that lead to the discovery of cosmic acceleration. Cosmic acceleration is usually attributed to a form of DE. However, in recent years, several studies have raised the question of whether or not intrinsic SNIa luminosities might evolve with redshift [37–40, 69]. Here we test the soundness of inferred absolute magnitude combining SNIa data with BAO measurements. We isolate the luminosity distance parameter $d_L(z)$ from the BAO dataset and cancel the expansion part from the observed distant modulus $\mu_{Ia}(z)$. Consequently, the combination with the BAO replaces the known degeneracies between M_B and H_0 for the Type Ia supernova data and between H_0 and the r_d for the BAO data with a new degeneracy between r_d and M_B. By choosing two different values for r_d, one specifies the M_B from NRT. The sound horizon from Planck $r_d^{\rm Pl18} = 147.09 \pm 0.26\,$Mpc [2] gives $M_B = -19.42 \pm 0.35$ from GPs. The sound horizon from H0LiCOW+SN+BAO+SH0ES $r_d^{\rm HW+SN+BAO+SH0ES} = 136.1 \pm 2.7\,$Mpc [19] $M_B = -19.25 \pm 0.39$ from GPs. The latter value of M_B matches the recently measured value by Ref. [10]. These mean values have been obtained assuming one Gaussian distribution for the whole reconstruction. The GP can be fit with a Gaussian with a tail, which when accounted for makes the mean values a bit lower than the cited above. From the GP approaches, we see a 1σ possibility of a nuisance parameter presence ($\delta M_B(z) \neq 0$) in higher redshifts. Over-all, the two different methods show very similar behaviour in the range $z \in [0.25, 2]$ – fluctuations and a decreasing trend with the redshift with an increasing error due to the low number of datapoints for higher redshifts.

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PoS(CORFU2022)259

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