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Spin $\frac{1}{2}$ from Gluons

A. P. Balachandran

Syracuse University, NY 13244-1130

E-mail: apbal1938@gmail.com

The theta vacuum in QCD is the standard vacuum, twisted by the exponential of the Chern-Simons term. But what is the quantum operator $U(g)$ for winding number 1?
We construct $U(g)$ in this note. The Poincaré rotation generators commute with it only if they are augmented by the spin $\frac{1}{2}$ representation of the Lorentz group coming from large gauge transformations. This result is analogous to the 'spin-isopin' mixing result due to Jackiw and Rebbi [1], and Hasenfratz and 't Hooft [2] and a similar result in fuzzy physics [3].
Hence states can drastically affect repreentations of observables. This fact is further shown by charged states dressed by infrared clouds. Following Mund, Rehren and Schroer [5], we find that Lorentz invariance is spontaneously broken in these sectors. This result has been extended earlier to QCD (references [6] given in the Final Remarks) where even the global QCD group is shown to be broken.

It is argued that the escort fields of [5] are the Higgs fields for Lorentz and colour breaking. They are string-localised fields where the strings live in a union of de Sitter spaces. Their oscillations and those of the infrared clouds generate the associated Goldstone modes.

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1. Introduction

In quantum physics, there are always two related aspects. The first is the algebra $\mathcal{A}$ of observables which represent elements subject to experimental measurements. The second is the state ω which represents the quantum ensemble which will be subject to measurements. ω is a positive linear functional on $\mathcal{A}$, so that if $a \in \mathcal{A}$, $\omega(a)$ is a complex number. Also $\omega(a^*a) \geq 0$ and $\omega(\mathbb{I}) = 1$, the two properties needed for a probability measure.

In this view, the Hilbert space H and the representation of $\mathcal{A}$ on H are emergent concepts which can be found using the GNS construction. The abstract algebra $\mathcal{A}$ is always the same, but the representations of $\mathcal{A}$ depend on ω.

It can happen that two ω's give equivalent representations, but matrix elements of observables between vectors in these representations vanish: this vanishing theorem may require taking the direct sum of these representations.

It can also happen that the emergent representations are inequivalent. Here too, no observable can excite a vector in one to a vector in the other.

In either case, we say that the representations are superselected. If a Lagrangian symmetry changes the superselection sector, it is said to be spontaneously broken.

In an infinite ferromagnet, the vector states in an irreducible representation can be all those with the same direction of asymptotic spins. Observables can be those which affect the local spins without changing the asymptotic value. That defines an irreducible representation of observables.

Another Hilbert space will have vectors with asymptotic direction of spins being in a different direction, but still observables causing only local disturbances of spins. These two irreducible representations are equivalent, but no observable has a non-zero matrix element between vectors of the two representations.

In the case of a charged Higgs field ϕ breaking say $U(1)$ gauged symmetry, it can happen that we have two families of states defining their Hilbert spaces, the expectation values of $\phi(x)$ as $|\vec{x}|$ goes to infinity differing in magnitude. This difference can be caused by the Higgs potential. In this case, the $U(1)$ gauge field has different masses in the two cases so that the representations are inequivalent. But still the local observables define the same algebra .

These remarks illustrate that we need both the abstract algebra $\mathcal{A}$ and a state on it to realise $\mathcal{A}$ as operators on a Hilbert space.In this paper, we elaborate on this idea for $SU(N)$ theta vacua in non-abelian gauge theories. These vacua are based on the fact that the homotopy group $\pi_3(SU(N)) = Z$, for $N \geq 2$. The quantum states are classified by representations of this group. If g_n is a winding number n transformation, then n has the image $\exp(in\theta)$ on the theta states. They define a representation of observables on these states. If $U(g_n)$ is the quantum operator implementing the winding number n transformation on theta states, and g denotes g_1, then $U(g)$ acting on a theta state must have eigenvalue $e^{i\theta}$. We will find $U(g)$ explicitly. It is a 'large' gauge transformation so that all observables must commute with it. The g in question is the configuration that occurs for Skyrme solitons.

For clarification, we add that observables are all the operators commuting with the complete commuting set (CCS) of large gauge transformations whose eigenvalues label the superselection sector. .They include all small gauge transformations (generated by Gauss law) and all local obsrvables. The small gauge transformations vanish on all the quantum vector states and commute also with the local observables.

2. Remarks on Gauge Transformations

We will work with an $SU(N)$ gauge theory with the Gell-Mann matrices λ_α as its Lie algebra generators in its defining N-dimensional representation. We also fix an $SU(2)$ subgroup with Pauli matrices τ_i as its generators.

On a spatial slice $\mathbb{R}^3$, the gauge group $\mathcal{G}$ is the group of smooth maps

$$g : \mathbb{R}^3 \to SU(N)$$

with $g(x)$ having a definite limit as the spatial coordinate goes to infinity, that is as $|\vec{x}| \to \infty$. It has been called the Sky group by Balachandran and Vaidya [7]. It is the analogue of the Spi group for asymptotically flat spaces introduced by Ashtekar and Hanson [8].

Let λ_α be the $SU(N)$ Gell-Mann matrices. Then if Ξ is a Lie algebra valued test function,

$$\Xi(\vec{x}) = \Xi^\alpha(\vec{x})\lambda_\alpha,$$

with $\Xi^\alpha(\vec{x})$ approaching definite limits as $|\vec{x}| \to \infty$,the Lie algebra generators of the sky group are

$$Q(\Xi) = \int d^3x \; tr \; (D_i\Xi(\vec{x}) \; E_i(\vec{x}) + \Xi(\vec{x}) \; J_0(\vec{x}))$$

where D_i is the covariant derivative, E_i is the (Lie algebra valued) electric field, J_0 is the $SU(N)$ charge density from matter sources and the trace is in the Lie algebra representation.

If the test functions are compactly supported or vanish fast at infinity, $Q(\Xi)$ represents the smeared Gauss law as one can see by partial integration. So all observables are required to commute with it. In addition, $Q(\Xi)$ is required to vanish on quantum states. These are called ‘small gauge transformations’.

If Ξ^α do not all vanish at infinity, considerations based on locality show that observables still commute with them [11]. But $Q(\Xi)$ need no longer vanish on quantum states. For example , in QED, if Ξ goes to a constant at infinity and does not vanish on quantum states, then it means that we are working in a charged sector.

So an isometry operator which does not commute with such $Q(\Xi)$ is not an observable , it changes the superselection sector. It is an intertwiner between two representations of the observables. We will see that generic elements of the Lorentz or $SU(N)$ groups do precisely that.Hence they are spontaneously broken.

3. The Theta Vacua

The theta vacua are quantum vector states which respond to $U(g)$ with eigenvalue $exp(i\theta)$ and which are invariant under small gauge transformations.They can be inferred from instanton physics and are given by

$$|\theta\rangle = exp(i\theta \int K(A))|0\rangle$$

where $K(A)$ is the $SU(N)$ Chern-Simons term ,

$$K(A) = \frac{1}{8\pi^2} tr\ (A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

and $|0\rangle$ is the Poincaré-invariant vacuum. It is invariant under large and small gauge transformations.

The theta state above is also $SU(N)$ invariant. We restrict it to the irreducible representation of $SU(2)$ chosen earlier for later convenience.

Under a winding number 1 transformation g of A, $A \rightarrow gDg^{-1}$,
$\int K(A)$ acquires the additional term

$$\frac{1}{24\pi^2} \int tr\ (dg\ g^{-1})^3 = 1$$

so that the above Chern-Simons twisted vacuum is indeed the theta vacuum vector.

Note that $\int K(A)$ is invariant under small gauge transformations.

Gauge transformations of Sky act on wave functions (ψ_1, ψ_2, ψ_3) where ψ_j belongs to $L^2(S^2)$. So if g is an element of the Sky group, and $U(g)$ is the operator implementing it, $U(g)\psi_j(\vec{x}) = \psi_k(\vec{x})U(g(x))_{kj}$ and the scalar product for two wave functions is $\int d\mu(\hat{x}) \sum \psi_i(\hat{x})^* \psi_j(\hat{x})$. where $d\mu(\hat{x}) = d(cos\theta)d\phi$ in the usual notation.

4. The Operator Implementing Winding Number Transformations

We can guess that it is a large gauge transformation. We propose to show that it is the finite gauge transforrmation generated by $Q(h)$ where $h(\vec{x}) = (\vec{\tau} \cdot \hat{\vec{x}})\tilde{h}(r)$ with

$$\tilde{h}(0) = 0, \qquad \tilde{h}(\infty) = -\pi.$$

This test function is well-defined at r equals 0 as $\tilde{h}$ vanishes there. But as $\tilde{h}$ is not zero as r becomes ∞, it generates a large gauge transformation. It will be recognised that h is the winding number 1 Skyrmion configuration. (See for example [9]). An important feature of h is that it is invariant only under the simultaneous rotation of $\hat{\vec{x}}$ and $\vec{\tau}$. This plays a crucial role in describing spin $\frac{1}{2}$ nucleons using the chiral model of pions.

5. More on Superselection Sectors : Many Theta Vacua

Let us note a striking feature of theta vacua .When one writes $\vec{\tau}.\hat{\vec{x}}$, there is an identification of directions such as the third direction in $\vec{\tau}$ and $\vec{x}$ spaces, or an identification of angular momentum generators in the two spaces. We can also write h' equals $(\vec{\tau}'.\hat{\vec{x}})\tilde{h}$ where τ_i's are any rotated Pauli matrices. That too will give a theta sector from its h. But $(h' - h)$ does *not* vanish at infinity and so $Q(h' - h)$ is a large gauge transformation. Hence $Q(h')$ and $Q(h)$ define different superselection sectors even though their eigenvalues on the Chern-Simons-twisted vacua are the same ! *This is like the situation in a ferromagnet when the spins located at the points at infinity are in different directions.The algebras of observables in the two cases are isomorphic : the isometry is provided by the rotation of spins from one direction to the other.*

Superposition of such theta vacua produces a mixed state for the observables. If CP violation from instantons is found, we can ask which mixed state is responsibe for it.

6. Spin 1/2 from Gluons

There is a 1980 paper by Friedman and Sorkin [4] with a similar title and we have adapted our title from theirs. There are also papers with similar results by Jackiw and Rebbi [1] and Hasenfratz and 't Hooft [2] in the theory of non-abelian monopoles.

As emphasised in the introduction, quantum theory requires both an algebra of observables and a state .(That is the case also in classical theory.) In functional integral approaches, the latter is defined by the Lagrangian. It can happen that the latter is defined entirely by bosonic variables, but still quantum theory contains spinorial states. There are plenty of examples. The books [9] and [10] describe many instances, both from soliton physics(eg.Skyrmions) and from molecular physics (such as the ethylene molecule). The theta states are other examples. A vector state in this case is defined by the vacuum twisted by a Chern-Simons term. The algebra of observables is gauge invariant.

A superselection sector contains a large gauge transformation $U(g)$. We claimed above that this $U(g)$ for us generates a winding number 1 transformation. We also claimed that this $U(g)$ is given by the Skyrmion configuration for g. Let us prove this result.

Let Ψ be a coloured field in the N-dimensional $SU(N)$ representation. A finite transformation on Ψ is then given by

$$\begin{aligned} e^{iQ(h)}\Psi(x)e^{-iQ(h)} &= \sum_n \frac{i^n}{n!}[Q(h),[Q(h),\cdots[Q(h),\Psi]\cdots]] \\ &= \sum_n \frac{i^n}{n!}((\vec{\tau}\cdot\hat{x})\tilde{h}(r))^n\Psi((x) = exp(i(\tau\cdot\hat{x})\tilde{h}(r))\Psi(x) \\ &\equiv g(h)\Psi(x) \end{aligned} \tag{1}$$

Here g is a Skyrmion configuration which is well-defined :

$$g(h) = \cos h(r) + i(\tau\cdot\hat{\vec{x}})\sin h(r).$$

A Remark

Let h' be defined using τ'. The $g(h)$ above and a $g(h')$ are both $-\mathbb{I}$ at $r \to \infty$ although $Q(h-h')$ does not come from the Gauss law and need not vanish on quantum states. Thus when restricted to the sphere at ∞, the map from the Lie algebra to the Lie group level is not injective. This result has played a role in the above analysis.

Back to the main theme. The expression (1) is valid also in the pure gluon sector when the state is given by the Chern-Simons-twisted vacuum. The latter involves the connection $A = A^\alpha\lambda_\alpha$ and $U(g)$ gauge transforms it with $g(h)$ as in (1).

Now the gluons rotate only with tensorial angular momentum (2π rotation = $+\mathbb{I}$). This operator rotates just $\hat{\vec{x}}$ in $\hat{h}$. But that changes $Q(h)$, changing also the superselection sector. We can conclude that the canonical angular momentum L_i for the gluon sector is spontaneously broken.

But consider adding the gauge rotation $Q(\mathbb{I}\,\tilde{h}(r)\tau_i/2)$ to L_i. where $\mathbb{I}$ is the constant function with value 1 on $\mathbb{R}^3$ and let us choose the vector state $|\theta > \otimes(a,b,0), |a|^2+|b|^2 = 1$. The added term rotates τ_i as well in $Q(h)$ so that $L_i + Q(\mathbb{I}\,\tilde{h}(r)\,\tau_i/2)$ commutes with $Q(h)$: it does not change the supeselection sector. That is, the total angular momentum $J_i = L_i + Q(\mathbb{I}\ \tilde{h}(r)\tau_i/2)$ does not change the superselection sector.

The 2π rotation from J_i acting on the above twisted vacuum state changes its sign : the $SU(2)$ Chern-Simons twisted vacuum is spinorial.(Recall that we restricted the Chern-Simons term to an $SU(2)$ irreducible representation). In this way we get spinorial states in the gluon sector.

If we had considered the vector state $|\theta > \otimes(a, b, c)$ *where* $|a|^2 + |b|^2 + |c|^2 = 1$*, the subspace spanned by* $(a, b, 0)$ *will become fermionic and the subspace spanned by* $(0, 0, c)$ *will stay bosonic. This also means that if* $\mathbb{C}^3$ *is associated with the quarks, two of the quarks become bosonic and the third fermionic. This has many phenomenological consequences which can be used to test for theta vacua, We will return to this issue in a later work.*

7. The Lorentz Group

Let K_i be the canonical boost associated to L_i. Then $K_i + Q(i\mathbb{I}\ \tau_i/2)$ and $L_i + Q(\mathbb{I}\ \tau_i/2)$ fulfil the $SL(2, C)$ algebra and are appropriate generators for a Majorana field.

(We can also consider $L_i + Q(-i\tau_i/2)$). A Majorana field transforming unitarily by these operators can also be constructed using Weinberg's methods [12].)

Unfortunately this choice of boosts does not seem to preserve the superselection sector. For example , in $Q(h)$, τ_i will transform by the non-unitary $(1/2, 0)$ representation of $SL(2, C)$ and that does not seem to be compensated by the transformation of x. So the Lorentz group is spontaneously broken, a result known from other papers. But the spinorial cover of the Euclidean group with $L_i + Q(\mathbb{I}\ \tau_i/2)$ and spacetime translations seem implementable in the theta sectors.

In a subsequent paper, we show that infrared effects canonically induce fields on the two-sphere at 'infinity' with covariant $SL(2, C)$. Acting on the vacuum, they create states on the local algebra which under $SL(2, C)$ intertwine inequivalent irreducible representations.

8. The Chern-Simons term for $SO(3) \subset SU(N)$

When $N \geq 3$, there is an $SO(3)$ subgroup in $SU(N)$ acting say on the first three components of the N-dimensional vector space. This group had a prominent role in our work on dibaryons [13] as solitons.

Now we can construct the Chern-Simons term $K(A)$ and the associated twisted vacuum with A valued in the $SO(3)$ algebra. The image of $\tau_i/2$ are the 3×3 angular matrices l_i. (These are conventionally called θ_i as in our group theory book [14] , but we will use l_i instead to avoid confusion with the theta of theta vacua.) Accordingly, the Skyrmion configuration is changed to

$$h'(\vec{x}) = (\vec{\mathbf{l}} \cdot \hat{\vec{x}})\tilde{h}(r).$$

Its finite transformation equals

$$\hat{g}(\hat{\vec{x}}) = e^{i\vec{l}\cdot\hat{\vec{x}}}h(r)$$

with winding number 4 and so the eigenvalue of winding number transformation on the Chern-Simons twisted vacuum is $e^{4i\theta}$. The periodicity in theta now is accordingly $\frac{2\pi}{4}$.

The angular momentum $J_i = L_i + l_i$ is now tensorial. The boost generators are $K_i + Q(il_i)$, but the associated Lorentz group changes the superselection sector.

9. Brief Remarks on Escort Fields

We add this brief para to draw attention to the remarkable developments in the theory of string-localised quantum fields and their escort fields, They have a bearning on this paper too.

In the abstract, we remarked that the Goldstone modes of Lorentz symmetry breaking are incorporated in the escort fields of [5].That is the case : these fields incorporate a 'string' from the direction of the Wilson line, and it can locally fluctuate creating quantised Goldstone modes. But to keep this paper focused, we will discuss such points in later work[15].

10. Final Remarks

There is more to be said on superselection sectors and their relation for example to Wilson lines and the Rindler space. They will be discussed in later work.

An older result discussed in [6] concerns QCD :As it is non-abelian, its generators do not generically commute with $Q(\Xi)$: only the stability group of $Q(\Xi)$ does so.

A particular result among others with direct application is the calculation of the Landau-Yang process, strctly forbidden by Lorentz invariance and allowed by its breaking. This calculation with Asorey, Balachandran and Momen[16] is complete and will soon be reported.

It is striking that theta vacua can convert the gluon sector to spinorial states and that the theta states are infinitely degenerate. These results will have an impact on axion phenomenology, which is yet to be explored.

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