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Towards removal of striped phase in matrix model description of fuzzy field theories

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The UV/IR-mixing phenomenon of non-commutative field theories is manifested by the existence of a non-local, striped phase in the scalar field theory, where the field does not oscillate around the same value in the whole space. We will consider modifications of the standard "kinetic term plus potential" actions for Hermitian matrix models which describe theories free of the UV/IR-mixing and discuss the expected receding of the striped phase in the phase diagram. We will present results for the case of the modified theory on the fuzzy sphere and for the truncated Heisenberg algebra formulation of the Grosse-Wulkenhaar model on the plane.

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1. Introduction

It is widely accepted that in the eventually successful combination of quantum theory and general relativity the notion of continuous space-time will be replaced by some kind of discrete, quantum structure [1, 2]. Non-commutative spaces provide a toy model for such spaces [3].

However when we try to formulate physics, namely quantum field theory, on such spaces, we quickly run into trouble. Due to the non-local nature of the interactions the field theories suffer from the UV/IR-mixing phenomenon. And most importantly, this issue persists in the commutative limit and the theories that were originally defined on non-commutative space are not well defined as quantum theories even when the non-commutativity is removed. This is manifested by the presence of a non-uniform order, or striped, phase in the phase diagram of the theory.

It is possible to modify the naive version of the non-commutative theory in such a way, that the problem of the UV/IR-mixing is removed and the resulting quantum theory is well behaved in the commutative limit. In this report we collect some results related to the analysis of the matrix models which aspire to describe such formulations of field theories free of the UV/IR-mixing.

It is structured as follows. We give some necessary introductory information – about the fuzzy physics and its description in terms of the matrix models – in section 2. We then describe the results for the fate of the striped phase for the case of the fuzzy sphere from [4] in section 3 and results for the curvature part of the Grosse-Wulkenhaar model from [5] in section 4. We conclude with some outlook for future research.

2. Preliminaries

We will use this section to give some necessary background for what follows in sections 3 and 4. We will describe the construction of the fuzzy sphere as the prototypical example of a fuzzy space, the scalar field theory with the most important features, and finally the matrix model description of scalar field theories.

2.1 Fuzzy spaces

First, let us illustrate the construction of fuzzy spaces and the emergence of a minimal length scale by the prototypical example – the fuzzy sphere [6–8].

Functions on the usual, commutative sphere are given by

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_{lm}(\theta,\phi)\ , \tag{1}$$

where Y_{lm} are the spherical harmonics, the eigenfunctions of the spherical Laplacian

$$\Delta Y_{lm}(\theta,\phi)=l(l+1)Y_{lm}(\theta,\phi)\ ,\ \Delta=\frac{1}{R^2}\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial\phi^2}\right) \tag{2}$$

and R radius of the sphere. To describe features at a small length scale we need Y's with a large l. If we truncate the possible values of l in the expansion (1) and define new objects

$$f(\theta,\phi)=\sum_{l=0}^{L}\sum_{m=-l}^{l}c_{lm}Y_{lm}(\theta,\phi)\ , \tag{3}$$

we will not be able to see any features of functions under certain length scales. Points on the sphere, defined as the limit of a series converging to a δ-function, cease to exist. However, expressions defined in this way are not closed under multiplication – it is easy to see that multiplying two spherical harmonics with $l = 1$ leads to a function that requires $l = 2$ in the expansion.

On the other hand, a quick calculation shows that the number of independent functions with $l \leq L = N - 1$ is N^2, the same as the dimension of the space of $N \times N$ Hermitian matrices. The idea to close the algebra of objects (3) is to map functions onto matrices and borrow their closed matrix product. In order to do so, we consider a $N \times N$ matrix as a product of two N-dimensional representations $\underline{N}$ of the algebra $SU(2)$. It is a reducible representation which reduces to a sum of irreducible representations according to

$$\begin{array}{ccccccccc} \underline{N} \otimes \underline{N} & = & \underline{1} & \oplus & \underline{3} & \oplus & \underline{5} & \oplus & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & = & \{Y_{0m}\} & \oplus & \{Y_{1m}\} & \oplus & \{Y_{2m}\} & \oplus & \ldots \end{array} \quad . \tag{4}$$

We thus have a map from spherical harmonics to matrices $\varphi : Y_{lm} \to M$ and we define the product

$$Y_{lm} \star Y_{l'm'} := \varphi^{-1} \left(\varphi \left(Y_{lm} \right) \varphi \left(Y_{l'm'} \right) \right) \ . \tag{5}$$

We multiply images of the two spherical harmonics as matrices and then find a function, which corresponds to this product. Clearly, this $\star$-product closes the algebra (3) but is not commutative by construction.

In objects (3) we have a short distance structure, but the price we had to pay was a non-commutative product $\star$ of functions. The space, for which these objects form an algebra of functions, is called the fuzzy sphere, which we will label S^2_N. And the functions themselves can be thought of as matrices. Opposing some lattice discretization this space still possesses a full rotational symmetry, since we can rotate the functions in (3) by any rotation and still obtain a well defined element of the algebra. And we can see that by construction we recover the original sphere in the limit N or $L \to \infty$.

There is a different way where the same idea – non-commutativity of the product and short distance structure – emerges. The regular sphere S^2 is given by the points in $\mathbb{R}^3$ satistifing the constraints

$$x_i x_i = R^2 \quad , \quad x_i x_j - x_j x_i = 0 \ , \ i, j = 1, 2, 3 \ . \tag{6}$$

And when viewed as coordinate functions, these generate the algebra of functions (1). For the fuzzy sphere S^2_N we define the set of operators

$$\hat{x}_i \hat{x}_i = r^2 \quad , \quad \hat{x}_i \hat{x}_j - \hat{x}_j \hat{x}_i = i\theta\varepsilon_{ijk}\hat{x}_k \ , \ i, j = 1, 2, 3 \ . \tag{7}$$

Such $\hat{x}_i$'s generate a different, non-commutative, algebra and the fuzzy sphere is an object, which has this algebra as an algebra of functions. The above conditions can be realized in terms of the generators L_i of the $N = 2s + 1$ dimensional representation of $SU(2)$

$$\hat{x}_i = \frac{2r}{\sqrt{N^2 - 1}} L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2 - 1}} \sim \frac{2}{N} \quad , \quad \rho^2 = \frac{4r^2}{N^2 - 1} s(s+1) = r^2 \ . \tag{8}$$

Since the group $SO(3)$ still acts on $\hat{x}_i$'s and rotates this triplet into a rotated set of coordinates $\hat{x}'_i$, this space enjoys a full rotational symmetry. And since $\theta \to 0$ in the limit $N \to \infty$, we recover the original sphere. But most importantly nonzero commutators (7) imply uncertainty relations for positions

$$\Delta x_i \Delta x_j \neq 0 \tag{9}$$

and we are no longer able to locate objects arbitrarily precisely on the fuzzy sphere. This means that the configuration space itself is analogous to the phase space of quantum mechanics.

In a similar fashion it is possible to construct an analogous deformation of the plane

$$\hat{x}_i \hat{x}_j - \hat{x}_j \hat{x}_i = i\theta \varepsilon_{ij} = i\theta_{ij} \ , \ i = 1, 2 \ , \tag{10}$$

with construction analogous to our first approach using the $\star$-product

$$f \star g = f \, e^{\frac{i}{2} \bar{\partial} \, \theta \, \vec{\partial}} g = fg + \frac{i\theta^{\mu\nu}}{2} \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\nu} + \cdots \ . \tag{11}$$

This algebra has no finite dimensional representations, which we will discuss in more detail in section 4.

Going back to the fuzzy sphere, we have divided it into N cells. Function on the fuzzy sphere is given by a matrix M and the eigenvalues of M represent the values of the function on these cells. However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.

Fuzzy, and more broadly non-commutative, spaces have been at first suggested as a regularization of infinities in the standard quantum field theory [10]. This problem was later successfully solved by the renormalization program, but non-commutative space have found use in other areas of theoretical physics. For regularization of field theories in numerical simulations [9], as an effective description of the open string dynamics in a magnetic background in the low energy limit [11, 12], as solutions of various matrix formulations of the string theory [13], in geometric unification of the particle physics and theory of gravity [14], as an effective description of various systems in a certain limit (eg. QHE) [15] and as toy models of spaces with discrete quantum structure, which is expected to arise in quantum theory of gravity. This last application is the most relevant in the context of the presented work.

2.2 Fuzzy field theory

Formulation

The standard commutative euclidean theory of a real scalar field is given by an action

$$S(\Phi) = \int d^2x \left[\frac{1}{2} \Phi \Delta \Phi + \frac{1}{2} m^2 \Phi^2 + V(\Phi) \right] \tag{12}$$

and any questions to be asked are answered by path integral correlation functions

$$\langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}} \ . \tag{13}$$

We construct the non-commutative theory as an analogue with replacing the commutative objects with their fuzzy counterparts. The field becomes a Hermitian matrix, the functional integral a matrix integral, spacetime integral a trace and derivatives become commutators with the matrices defining the fuzzy space – L_i's in the case of the fuzzy sphere. This way, the non-commutative action becomes

$$S(M) = \frac{4\pi R^2}{N}\mathrm{Tr}\left[\frac{1}{2}M\frac{1}{R^2}[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + V(M)\right], \tag{14}$$

with the path integral

$$\langle F\rangle = \frac{\int dM\, F(M)e^{-S(M)}}{\int dM\, e^{-S(M)}}\,. \tag{15}$$

We can now go ahead and calculate these matrix integrals using the usual Feynman diagram approach of the field theory and try to see what has changed. For great reviews of such consideration, see [16–18].

UV/IR mixing

The key property of the non-commutative field theories is the UV/IR mixing phenomenon, which arises as a result of the non-locality of the theory [19–21]. As the name suggests there is an interplay of UV and IR divergences in loop calculations, where very energetic fluctuations (UV physics) have consequences at large distances (IR physics).

In technical terms of the Feynman diagrams this problem exhibits itself in different properties of the ones that are planar and the ones that are non-planar. And the key issue is, that the (matrix) vertex is not invariant under permutation of incoming momenta. When we expand the matrix M in the base of polarization tensor and look at the standard ϕ^4 interaction term

$$M = \sum_{l=0}^{N-1}\sum_{m=-l}^{l} c_{lm}T_{lm}\ ,\ \mathrm{Tr}\left(M^4\right) = \sum_{l_{1\ldots 4}}\sum_{m_{1\ldots 4}} c_{l_1,m_1}c_{l_2,m_2}c_{l_3,m_3}c_{l_4,m_4}\mathrm{Tr}\left(T_{l_1,m_1}T_{l_2,m_2}T_{l_3,m_3}T_{l_4,m_4}\right) \tag{16}$$

we see that the contraction of neighboring T's yields a different result that the crossed contractions thanks to the fact that the trace is symmetric only under cyclic permutation of matrices.

Employing the standard machinery of QFT one arrives at the following difference between the non-planar and planar contributions to the one-loop effective action [20, 21] [1]

$$I^{NP} - I^P = \sum_{j=0}^{N-1}\frac{2j+1}{j(j+1)+m^2}\left[(-1)^{l+j+N-1}\begin{Bmatrix} l & s & s \\ j & s & s\end{Bmatrix} - 1\right]. \tag{17}$$

The crucial observation is that this difference is finite in $N\to\infty$ limit and in this limit the effective action is different from the standard S^2 effective action. In other words, regularization of the field theory by a non-commutative space is anomalous. In the planar limit $S^2 \to \mathbb{R}^2$ one recovers singularities and the standard UV/IR-mixing to be discussed shortly. The motto is thus that space

Figure 1: Planar and non-planar diagrams contributing to (18) and (19) respectively.

(geometry) forgets where it came from, but quantum field theory (physics) remembers its fuzzy origin.

When doing the analogous calculation for the non-commutative plane with the action

$$S = \int d^2x \left(\frac{1}{2}\partial_\mu \phi \star \partial_\mu \phi + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right) , \tag{18}$$

the contribution of the planar diagram is the same as in the commutative field theory

$$I_P = \frac{\lambda}{4!} \int \frac{d^2k}{(2\pi)^2} \frac{2}{k^2 + m^2} . \tag{19}$$

The non-planar diagrams are however different

$$I_{NP} = \frac{\lambda}{4!} \int \frac{d^2k}{(2\pi)^2} \frac{\exp(ik_\mu \theta^{\mu\nu} p_\nu)}{k^2 + m^2} = \frac{\lambda}{96\pi} \log \frac{\Lambda_{\text{eff}}^2}{m^2} + \cdots , \ \Lambda_{\text{eff}}^2 = \frac{1}{1/\Lambda^2 + |\theta^{\mu\nu} p_\nu|^2} . \tag{20}$$

Thanks to the oscillating factor they are effectively regulated at Λ_{eff} and finite when the momentum cutoff Λ is removed ... unless the external momentum vanished. In that case, the contribution diverges. This is a grave problem in the case of such non-planar loops on further loops. The UV and IR scales can not be separated, which is a crucial requirement of renormalizability of the theory and if we first define the field theory on a non-commutative space, then quantize it and then take the commutative limit, we end up with a theory that is different from a theory that was defined on a commutative space from the beginning. Moreover with a theory that is not renormalizable.

Spontaneous symmetry breaking patterns

One of the ways the UV/IR-mixing phenomenon exhibits itself are the possibilities of spontaneous symmetry breaking. Let us consider the standard ϕ^4 theory for simplicity. The commutative quantum field theory with action

$$S[\phi] = \int d^2x \left(\frac{1}{2}\partial_i \phi \partial_i \phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right) \tag{21}$$

Has two different phases [23, 24]. A disordered phase where the field oscillates around $\langle \phi \rangle = 0$, which for $m^2 < 0$ is a false vacuum. If the wells of the potential deep enough the quantum fluctuations of the field are not large enough to keep it out of the wells and the field transitions into an ordered phase, oscillating around a non-zero value $\langle \phi \rangle \neq 0$ which is the global minimum of the quartic potential. This breaks the $\phi \to -\phi$ symmetry of the model spontaneously.

[1]For a less standard machinery see [22].

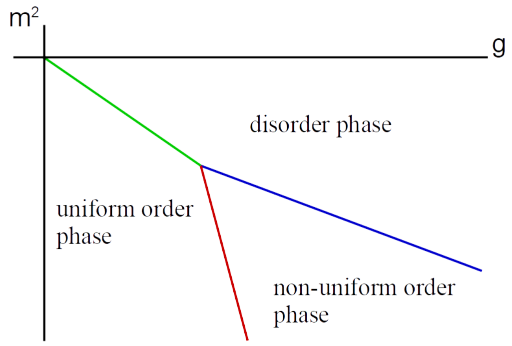

Figure 2: A typical phase diagram of the non-commutative field theory, even in the commutative limit. The green line is present also in the case of commutative field theories. The red and the blue lines are due to the UV/IR-mixing.

In the non-commutative case the theories have one more phase [25, 26]. It is a non-uniform order phase, or a striped phase. In this phase, the field does not oscillate around one given value in the whole space but rather forms stripes of oscillations around both the positive and negative minimum of the potential. The translational symmetry is broken spontaneously. This has been established in numerous numerical works for variety different spaces [27–39], see [9] for a review. What is important for our discussion is the fact that the existence of this phase is linked to the UV/IR-mixing.

Analytic description of this phenomenon relies on the matrix model description of fuzzy field theories.

2.3 Matrix model description of fuzzy field theories

We will be unapologetically brief in our discussion of the random matrix theory and refer the reader to excellent reviews [40–42] for more information. Random matrix model is simply an ensemble of random matrices with a particular probability distribution and questions to be answered in the form of expected values of functions of the random variable. An important example is the ensemble of $N \times N$ Hermitian matrices with probability

$$dM\, P(M) = dM\, e^{-N\mathrm{Tr}(V(M))}\ ,\ \text{usually}\ V(x)\ =\ \frac{1}{2}r\,x^2 + g\,x^4 \tag{22}$$

and the measure

$$dM = \left[\prod_{i=1}^{N} M_{ii}\right]\left[\prod_{i<j} Re\, M_{ij} Im\, M_{ij}\right]\ . \tag{23}$$

Both the measure and the probability distribution are invariant under $M \to UMU^\dagger$ with $U \in SU(N)$. Requirement of such invariance is however, quite restrictive. One is usually interested in the distribution of eigenvalues.

If we now recall the action of the fuzzy scalar field theory (14) and the functional integral formulation of the correlation functions (15) it is a rather trivial observation that the fuzzy field theory is a particular random matrix model. Namely one with a probability distribution[2]

$$P(M) = e^{-N^2 S(M)} \ , \ S(M) = \frac{1}{N}\mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]]\right) + \frac{1}{N}\mathrm{Tr}\left(\frac{1}{2}m^2M^2\right) + \frac{1}{N}\mathrm{Tr}\left(gM^4\right) \ . \quad (24)$$

Without the kinetic term the large N limit of the model is quite well understood. However with the kinetic term the situation is more complicated, the key issue being that diagonalization $M = U\,\mathrm{diag}(\lambda_1,\ldots,\lambda_N)U^\dagger$ is no longer straightforward. For the correlation functions one encounters integrals of the form

$$\langle F\rangle \sim \int\left(\prod_{i=1}^N d\lambda_i\right)\int dU\ F(\lambda_i,U)\ e^{-N^2\left[\frac{1}{2}m^2\frac{1}{N}\sum\lambda_i^2+g\frac{1}{N}\sum\lambda_i^4-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]} \\ \times\ e^{-\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)} \ . \quad (25)$$

If the function F does not depend on the angular degrees of freedom U, we need to calculate the dU integral of the form

$$\int dU\, e^{-\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)} =: e^{-N^2 S_{\mathrm{eff}}(\Lambda)} \ , \quad (26)$$

where we have introduced the kinetic term effective action S_{eff}. A perturbative expansion of this quantity has been obtained in [43, 44] using explicit integration over the unitary group. A very different approximation is possible as follows.

Second moment approximation

It has been shown, that for the free theory $g = 0$ the kinetic term in (24) just rescales the Wigner semicircle eigenvalue distribution [45, 46]. It has also been shown, that there is a unique parameter independent effective action that reconstructs this rescaling [47]

$$S_{eff}(\lambda_i) = \frac{1}{2}\log\left(\frac{c_2}{1-e^{-c_2}}\right) + \mathcal{R} \ , \ c_n = \frac{1}{N}\mathrm{Tr}\left(M^n\right) \ , \quad (27)$$

where the remainder term $\mathcal{R}$ vanishes when evaluated on the semicircle distribution. This idea can also be generalized to a more complicated kinetic term $\mathrm{Tr}\,(M\mathcal{K}M)$.

Introducing the asymmetry into the model by considering the symmetrized moments $c_2 \to c_2 - c_1^2$ we obtain a matrix model

$$S(M) = \frac{1}{2}F(c_2-c_1^2) + \frac{1}{2}r\,c_2 + g\,c_4 \ , \ F(t) = \log\left(\frac{t}{1-e^{-t}}\right) \ . \quad (28)$$

This matrix model has been analyzed and it has been shown that it qualitatively recovers phase structure of the field theory on the fuzzy sphere described above [48]. Most importantly it allows for all the three phases, which meet at the triple point.

[2]After necessary rescalings of field and parameters.

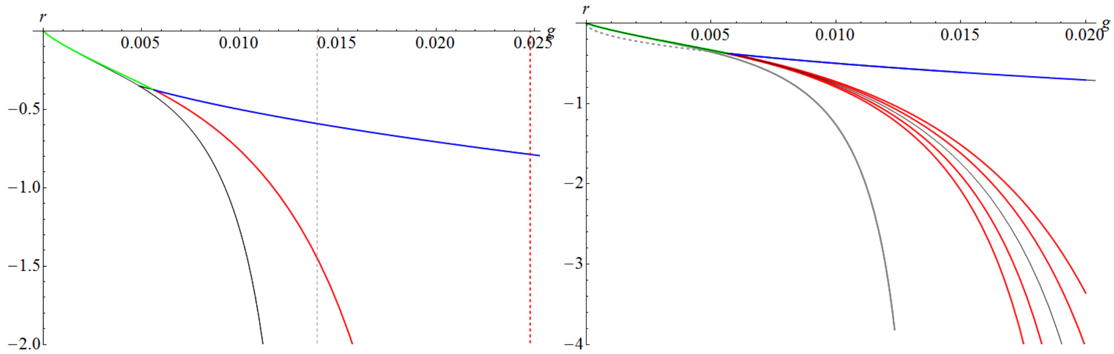

Figure 3: On the left phase diagram of the model (32) with $a = 3e^{3/2}$, $b = 0$, on the right phase diagram for the same value of a and values of $b = -4, -2, 0, 2, 4$.

3. Removal of striped phase in fuzzy sphere matrix model

We have finally reached the main part of our discussion. Since the persistence of the striped phase in the phase diagram of the commutative limit of non-commutative field theories is consequence of the UV/IR-mixing, we expect it not to be present in the modifications of the models free of this phenomenon.

One such modification for the theory on the fuzzy sphere has been given in [49]. As a matrix model it is given by the following action

$$S = \frac{1}{N}\mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + 12gMQM + \frac{1}{2}rM^2 + gM^4\right) , \tag{29}$$

where the operator Q acts on the polarization tensor as follows.

$$QT_{lm} = -\underbrace{\left(\sum_{j=0}^{N-1}\frac{2j+1}{j(j+1)+r}\left[(-1)^{l+j+N-1}\left\{\begin{array}{ccc} l & s & s \\ j & s & s\end{array}\right\} - 1\right]\right)}_{Q(l)} T_{lm} . \tag{30}$$

This removes the UV/IR mixing in the theory, essentially by removing the problematic part of the effective action by brute force.

Operator Q can be expressed as a power series in the Laplacian $C_2 = [L_i, [L_i, \cdot]]$

$$Q = q_1 C_2 + q_2 C_2^2 + \dots . \tag{31}$$

As a starting point, it is interesting to see the phase structure of such a simplified model [43]. This means that we will study the models with the kinetic terms

$$\mathcal{K} = (1 + ag)C_2 \quad \text{or} \quad \mathcal{K} = (1 + ag)C_2 + bg\, C_2^2 , \tag{32}$$

for paramteres a and b. This is expected not to remove the striped phase completely, but we expect the region of the existence of the striped phase to recede.

Such models have been analyzed in [4]. The important part of the phase diagram is given in the figure 3. As we can see, in both cases the phase transition between the uniform and non-uniform order phases shifts to righ as expected from the model towards removal of the UV/IR-mixing. Note that the transition line between the disorder and non-uniform order phases did not shift at all and the disorder to uniform order phase transition line shifted only slightly, together with the triple point.

4. Removal of striped phase in Grosse-Wulkenhaar matrix model

A modification of the naive formulation of non-commutative scalar field theory free of the UV/IR-mixing for the case of the non-commutative plane is also available. It is the celebrated Grosse-Wulkenhaar model [50, 51] with the action

$$S_{GW} = \int d^2x \left(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{1}{2}\Omega^2(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi) + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right) , \qquad (33)$$
$$\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu}x^\nu \ .$$

Introduction of the harmonic-oscillator-like-term introduces a symmetry between the UV and IR regime of the theory and the model is renormalizable again. Moreover, it can be described by a model of finite-sized matrices in terms of truncated Heisenberg algebra [52].

The coordinate "functions" on the standard non-commutative plane can be realized in terms of the following matrices

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} & +\sqrt{1} & & & \\ +\sqrt{1} & & +\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & \ddots \\ & & & \ddots & \end{pmatrix} , \ Y = \frac{i}{\sqrt{2}} \begin{pmatrix} & -\sqrt{1} & & & \\ +\sqrt{1} & & -\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & \ddots \\ & & & \ddots & \end{pmatrix} \qquad (34)$$

and straightforwardly $[X, Y] = i$. This algebra is then truncated to a finite dimension. We define finite matrices

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} & +\sqrt{1} & & & \\ +\sqrt{1} & & +\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & \sqrt{N-1} \\ & & & \sqrt{N-1} & \end{pmatrix} \qquad (35)$$

and similarly for Y, which gives

$$[X, Y] = i(1 - Z) \ , \ Z = \mathrm{diag}\,(0, \ldots, N) \ . \qquad (36)$$

Original algebra is recovered in the $N \to \infty$ limit or under the $Z = 0$ condition. The kinetic term becomes

$$\frac{1}{2}\int d^2x\, \partial_\mu\phi \star \partial_\mu\phi \to \mathrm{Tr}\left([X, M][X, M] + [Y, M][Y, M] \right) \ . \qquad (37)$$

The harmonic potential becomes

$$\frac{1}{2}\int d^2x\,\Omega^2(\tilde{x}_\mu\phi)\star(\tilde{x}_\mu\phi)\to \mathrm{Tr}\left(RM^2\right)\ , \tag{38}$$

where R is a fixed external matrix

$$R=\frac{15}{2}-4Z-8\left(X^2+Y^2\right)=\frac{31}{2}-16\,\mathrm{diag}\,(1,2,\ldots,N-1,N/2)\ , \tag{39}$$

and the second expression assumes the $Z=0$ condition. It has an interpretation of coupling to the curvature of the space. At the end of the day, we are left with a matrix model with action

$$S=\mathrm{Tr}\left(M[X,[X,M]]+M[Y,[Y,M]]\right)-g_r\mathrm{Tr}\left(RM^2\right)-g_2\mathrm{Tr}\left(M^2\right)+g_4\mathrm{Tr}\left(M^4\right)\ , \tag{40}$$

where we have introduced a little different notation for the constants than in (24). This model has been analyzed numerically in [53, 54] and features consistent with the expectation of removal of the striped phase due to the lack of UV/IR-mixing has been observed.

We have analyzed the model also analytically in [5]. As the first step, we concentrate on the effect of the curvature term and discard the kinetic term

$$S(M)=-\mathrm{Tr}\left(g_rRM^2\right)-g_2\,\mathrm{Tr}\left(M^2\right)+g_4\,\mathrm{Tr}\left(M^4\right)\ . \tag{41}$$

This leads to the angular integral in (26)

$$\int dU\, e^{g_r\mathrm{Tr}\left(URU^\dagger\Lambda^2\right)}\ , \tag{42}$$

which gives up to g_r^4

$$S(\Lambda)\ =N\,\mathrm{Tr}\Big(-g_2\Lambda^2+8g_r\Lambda^2+g_4\Lambda^4-\frac{32}{3}g_r^2\Lambda^4\Big)+\frac{1024}{45}g_r^4\Lambda^8+ \tag{43}$$

$$+\frac{32}{3}g_r^2\left(\mathrm{Tr}\left(\Lambda^2\right)\right)^2+\frac{1024}{15}g_r^4\left(\mathrm{Tr}\left(\Lambda^4\right)\right)^2-\frac{4096}{45}g_r^4\mathrm{Tr}\left(\Lambda^6\right)\mathrm{Tr}\left(\Lambda^2\right)\ . \tag{44}$$

This formula has been obtained both by expansion of the HCIZ formula and by direct integration along the lines of [43]. We ended up with a multitrace matrix model which can be analyzed using the standard matrix model techniques.

The phase diagrams from this analysis are shown in the figure 4. The left image shows the phase transition lines at the order $O(g_r^3)$ between the disorder and the non-uniform order phases – i.e. modification of the blue line from the figure 2. As we can see, the introduction of the curvature term shifts the line away from the origin, again shrinking the region where the problematic phase exists. The orange part of the line is not to be trusted, as in that region the approximation we are using is no longer reasonable. In the right image the blue solid line represents $O(g_r)$ correction to the pure potential transition line. Orange dashed line represents perturbative $O(g_r^4)$. Green dotted line represents a numerically calculated transition line for the multitrace model (44). All three plots are for $g_r=0.1$. This confirms that the transition line between the two symmetric phases of the model (44) is well behaved and is shifted away from the origin – confirming the expectation that a matrix model towards physics free of the UV/IR-mixing has a smaller region, where the problematic phase of the theory is realized.

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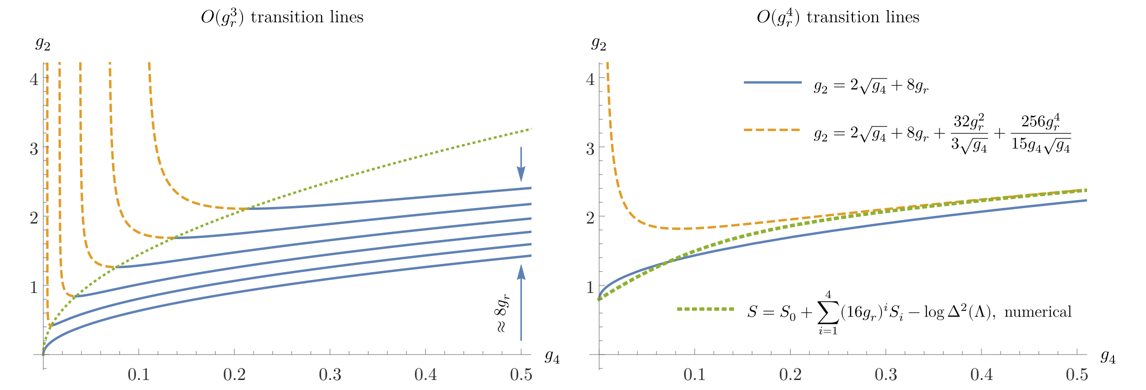

Figure 4: Different phase diagrams of the multi-trace model (44), with a different orientation of the vertical axis thanks to different definitions of parameters g_2 and r. See text for the description.

5. Conclusion and outlook

As we have seen, the non-commutative field theories are naturally described in terms of Hermitian random matrix models. The kinetic term of the field theory introduces technical complications and some approximations to these models can be cast in terms of particular multi-trace models. The UV/IR-mixing is exhibited as a non-uniform order, or striped, phase of the model, which can be seen both numerically and analytically.

We have summarized results of some previous works, which suggest that when one considers models describing theories free of the UV/IR-mixing this non-commutative phase recedes from the phase diagram as expected.

However, there is still plenty of work to be done before this project can be considered finished. Most importantly we need to find a way to incorporate the kinetic term in the analytic treatment in the Grosse-Wulkenhaar case. This part is crucial in describing the asymmetric regime of the theory. It would also be interesting to consider the matrix model for the GW-inspired $U(1)$ gauge field theory [55], where the UV/IR-mixing seems to persist, and see what happens to the striped phase there. Finally, we have presented only the perturbative treatment of the curvature term in the GW model and we would like to entertain the possibility of treating it in a non-perturbative way.

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