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Quantum $N = 2$ Minkowski Superspace

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We give the $N = 2$ SUSY quantum complex Minkowski superspace as big cell in the quantum super Grassmannian of $2|0$ spaces in $\mathbb{C}^{4|2}$, presented through its super Plücker embedding. Both super Minkowski and super Grassmannian are quantized as homogeneous superspaces for the quantum Poincarè and quantum conformal supergroups.

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1. Introduction

Non commutative geometry provides us with a natural language to encode symmetries in the quantum setting: quantum homogeneous spaces are deformations of the coordinate algebra of an homogeneous space together with a coaction of a quantum group, that is an Hopf algebra [13]. This framework is then ready to be generalized to the supersymmetric (SUSY) context.

In [17, 18], the $N = 1$ (complex) conformal superspace is realized as the superflag $\mathrm{Fl}(2|0, 2|1; 4|1)$, of $2|0$ dimensional subspaces into the superspace $\mathbb{C}^{4|1}$ ($N = 1$ SUSY refers to the odd dimension of such vector space). Since the superflag is a quotient of the special linear supergroup, we have that such supergroup, called the complex conformal supergroup $\mathrm{SL}(4|1)$ acts naturally on $\mathrm{Fl}(2|0, 2|1; 4|1)$. The space $\mathbb{C}^{4|1}$, underlying the defining representation of $\mathrm{SL}(4|1)$, is the space of supertwistors (see [2, 3] and [18] for a more mathematical description). The superflag $\mathrm{Fl}(2|0, 2|1, 4|1)$ can be embedded in the product

$$\mathrm{Fl}(2|0, 2|1, 4|1) \subset \mathrm{Gr}(2|0, 4|1) \times \mathrm{Gr}(2|1, 4|1),$$

and using the super Segre embedding [19] the superflag is embedded into the projective superspace $\mathbb{P}^{80|64}$ [13, 20].

For $N = 2$, we can try to reproduce the construction by looking at the embedding:

$$\mathrm{Fl}(2|0, 2|2, 4|2) \subset \mathrm{Gr}(2|0, 4|2) \times \mathrm{Gr}(2|2, 4|2)$$

However this superflag is too big to be physically interesting. The scalar superfields associated to it have too many field components to be useful in the formulation of supersymmetric field theories. Still, the antichiral $\mathrm{Gr}(2|0, 4|2)$ and chiral $\mathrm{Gr}(2|2, 4|2)$ superspaces do have physical applications, so it is useful to study them. They are both embedded in $\mathbb{P}^{8|8}$.

The present work is organized as follows.

In Sec. 2 we describe the super Grassmannian $\mathrm{Gr}(2|0, 4|2)$ of $2|0$ subspaces in the superspace $\mathbb{C}^{4|2}$, that is for $N = 2$ supersymmetry (SUSY) and its superprojective embedding via the super Plücker coordinates.

In Sec. 3, we described a quantum deformation of the super Grassmannian $\mathrm{Gr}(2|0, 4|2)$.

In Sec. 4, we define $N = 2$ Minkowski superspace and we give a quantization of it. We also introduce a quantum principal superbundle on it.

In this note, we are just introducing our approach and outlining our main results, for all the details we refer the reader to the article [14].

2. Super Grassmannain $\mathrm{Gr}(2|0, 4|2)$

We start with a brief review of Grassmannians in the ordinary setting and then we go to their generalization to SUSY. For the notation and main definitions of SUSY refer to [7], [13].

In classical geometry, we can view Grassmannians as algebraic projective varieties via the *Plücker embedding*. The image of a Grassmannian in a projective space under this embedding is characterized by the well-known *Plücker relations*. In the case of the Grassmannian Gr(2, 4) of 2 dimensional spaces into $\mathbb{C}^4$, the *Plücker embedding* is given explicitly by:

$$Gr(2,4) \longrightarrow \mathbb{P}(\bigwedge^2(\mathbb{C}^4)) \cong \mathbb{P}(\mathbb{C}^6)$$

$$(a,b) \mapsto [a \wedge b] \equiv [y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}]$$

where (a, b) denotes a basis of the corresponding point, i.e. a 2-subspace in the Grassmannian Gr(2, 4) and $y_{ij} : a_i b_j - b_i a_j$ are called *Plücker coordinates* of (a, b). In this case, the *Plücker relation* reads:

$$y_{12}y_{34} - y_{13}y_{24} + y_{14}y_{23} = 0.$$

We define *big cell* in Gr(2, 4) as $U_{12} := \{(a, b) : y_{12} \neq 0\}$. One can easily see that $U_{12} \cong \mathbb{C}^4$, which can then be identified with the complex Minkowski space (For more details, see [13], Chap. 2).

We now extend these ideas to the super setting to construct Grassmannian $Gr(2|0, 4|2)$ and then we give its quantization.

Let $\{e_1, e_2, e_3, e_4, \epsilon_5, \epsilon_6\}$ be a basis for $\mathbb{C}^{4|2}$ and $E := \bigwedge^2(\mathbb{C}^{4|2})$. As usual we use latin letters for even and greek letters for odd elements. We then have following set as a basis for E:

$$\begin{array}{llllll}(\text{Even}): & e_i \wedge e_j & 1 \leq i < j \leq 4, & \epsilon_5 \wedge \epsilon_5, & \epsilon_6 \wedge \epsilon_6, & \epsilon_5 \wedge \epsilon_6, \\ (\text{Odd}): & & e_k \wedge \epsilon_l & 1 \leq k \leq 4,\ 5 \leq l \leq 6. & & \end{array}$$

A general element Q of E can be written as:

$$Q = q + \lambda_5 \wedge \epsilon_5 + \lambda_6 \wedge \epsilon_6 + a_{55}\epsilon_5 \wedge \epsilon_5 + a_{66}\epsilon_6 \wedge \epsilon_6 + a_{56}\epsilon_5 \wedge \epsilon_6$$

. with

$$q = q_{ij} e_i \wedge e_j, \quad \lambda_m = \lambda_{mi} e_i, \qquad i, j = 1, \ldots, 4, \quad m = 5, 6\,.$$

The element Q is decomposable if $Q = a \wedge b$, where

$$a = r + \xi_5\epsilon_5 + \xi_6\epsilon_6, \qquad b = s + \eta_5\epsilon_5 + \eta_6\epsilon_6,$$

with $r = r_i e_i$, $s = s_i e_i$. One obtains the following equalities:

$$\begin{array}{lll} q = r \wedge s & & \\ \lambda_5 = \xi_5 s - \eta_5 r, & \lambda_6 = \xi_6 s - \eta_6 r, & \\ a_{55} = \xi_5\eta_5, & a_{66} = \xi_6\eta_6, & a_{56} = \xi_5\eta_6 + \xi_6\eta_5 \end{array} \tag{1}$$

which imply,

$$\begin{array}{lll} q \wedge q = 0, & & \\ q \wedge \lambda_5 = 0, & q \wedge \lambda_6 = 0, & \\ \lambda_5 \wedge \lambda_5 = -2a_{55}q, & \lambda_6 \wedge \lambda_6 = -2a_{66}q, & \lambda_5 \wedge \lambda_6 = -a_{56}q, \\ \lambda_5 a_{55} = 0, & \lambda_6 a_{66} = 0, & \\ \lambda_5 a_{66} = -\lambda_6 a_{56}, & \lambda_6 a_{55} = -\lambda_5 a_{56}, & \\ a_{55}^2 = 0, & a_{66}^2 = 0, & a_{56}a_{56} = -2a_{55}a_{66} \\ a_{55}a_{56} = 0, & a_{66}a_{56} = 0. & \end{array} \tag{2}$$

More explicitly, we can write them in coordinates in the following way (always $1 \leq i < j < k \leq 4$ and $5 \leq n \leq 6$):

$$\begin{array}{l} q_{12}q_{34} - q_{13}q_{24} + q_{14}q_{23} = 0, \\ q_{ij}\lambda_{kn} - q_{ik}\lambda_{jn} + q_{jk}\lambda_{in} = 0, \\ \lambda_{in}\lambda_{jn} = a_{nn}q_{ij}, \\ \lambda_{i5}\lambda_{j6} + \lambda_{i6}\lambda_{j5} = a_{56}q_{ij}, \\ \lambda_{in}a_{nn} = 0, \qquad \lambda_{i5}a_{66} = -\lambda_{i6}a_{56}, \qquad \lambda_{i6}a_{55} = -\lambda_{i5}a_{56} \\ a_{56}a_{56} = -2a_{55}a_{66}, \qquad a_{55}a_{56} = 0, \qquad a_{66}a_{56} = 0, \\ a_{nn}^2 = 0. \end{array} \tag{3}$$

We call relations (3) the *super plücker relations*. These relations completely characterize the super Grassmannian $\mathrm{Gr}(2|0,4|2)$, since decomposable Q are evidently corresponding to subspaces; moreover the $q_{ij}, a_{nm}, \lambda_{nk}$ give the superprojective coordinates in complete analogy to the ordinary setting described above (see also [13], [21]). We then have the following result.

Theorem 2.1. *The graded superring associated to the image of* $\mathrm{Gr}(2|0,4|2)$ *under the super Plücker embedding is* $\mathbb{C}[\mathrm{Gr}] \cong \mathbb{C}[q_{ij}, a_{nm}, \lambda_{nk}]/\mathbb{I}_P$ *where* $\mathbb{I}_P$ *the ideal generated by relations (3).*

For a proof, one can see [21]. Though our notations in here and [14] are different, however, the relations in [21] and ours coincide.

Another observation, that is helpful for quantization is: one can also view $\mathbb{C}[\mathrm{Gr}(2|0,4|2)]$ as a subalgebra sitting inside $\mathbb{C}[\mathrm{SL}(4|2)]$. Let us display the generators of the algebra $\mathbb{C}[\mathrm{GL}(4|2)]$ in the matrix form

$$\left(\begin{array}{cccc|cc} g_{11} & g_{12} & g_{13} & g_{14} & \gamma_{15} & \gamma_{16} \\ g_{21} & g_{22} & g_{23} & g_{24} & \gamma_{25} & \gamma_{26} \\ g_{31} & g_{32} & g_{33} & g_{34} & \gamma_{35} & \gamma_{36} \\ g_{41} & g_{42} & g_{43} & g_{44} & \gamma_{45} & \gamma_{46} \\ \hline \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & g_{55} & g_{56} \\ \gamma_{61} & \gamma_{62} & \gamma_{63} & \gamma_{64} & g_{65} & g_{66} \end{array}\right)$$

then

$$\mathbb{C}[\mathrm{SL}(4|2)] \cong \mathbb{C}[g_{ij}, g_{mn}, \gamma_{im}, \gamma_{nj}]/(\mathrm{Ber} - 1)\,,$$

where Ber is the *Berezinian* of the matrix and $1 \leq i, j \leq 4$ and $5 \leq m, n \leq 6$. Now, one can identify $\mathbb{C}[Gr]$ as subalgebra of $\mathbb{C}[\mathrm{SL}(4|2)]$ generated by:

$$y_{ij} = g_{i1}g_{j2} - g_{i2}g_{j1}, \qquad \eta_{kn} = g_{i1}\gamma_{n2} - g_{i2}\gamma_{n1}$$
$$x_{55} = \gamma_{51}\gamma_{52}, \qquad x_{66} = \gamma_{61}\gamma_{62} \qquad x_{56} = \gamma_{51}\gamma_{62} + \gamma_{61}\gamma_{52}. \tag{4}$$

3. The Quantum Super Grassmannian $\mathrm{Gr}_q(2|0, 4|2)$

The key idea for quantization is to replace the geometric objects with a deformation of algebra of functions on these objects and replace actions with coactions. With the inclusion $\mathbb{C}[\mathrm{Gr}] \subset \mathbb{C}[\mathrm{SL}(4|2)]$ as described in the last section, the idea for quantization is to consider $\mathbb{C}_q[\mathrm{SL}(4|2)]$ (we follow [13]) and see how the corresponding subalgebra looks like. This approach was also used in the classical setting for a quantization of grassmannians and flag varities (see [9], [11], [12]).

Firstly, let us recollect briefly, some basic facts about quantum supermatrices. We refer the reader to [17] and [13] for notation and details.

Definition 3.1. The *quantum matrix superalgebra* $\mathrm{M}_q(r|s)$ is defined as

$$\mathrm{M}_q(r|s) := \mathbb{C}_q\langle z_{ij}, \xi_{kl}\rangle / \mathbb{I}_M$$

where $\mathbb{C}_q\langle z_{ij}, \xi_{kl}\rangle$ denotes the free superalgebra over $\mathbb{C}_q = \mathbb{C}[q, q^{-1}]$ generated by the even variables

$$z_{ij}, \qquad \text{for} \quad 1 \leq i, j \leq r \quad \text{or} \quad r+1 \leq i, j \leq r+s.$$

and by the odd variables

$$\xi_{kl} \qquad \begin{array}{l} \text{for} \quad 1 \leq k \leq r, \quad r+1 \leq l \leq r+s \\ \text{or } \ r+1 \leq k \leq r+s, \quad 1 \leq l \leq r, \end{array}$$

satisfying the relations $\xi_{kl}^2 = 0$ and $\mathbb{I}_M$ is an ideal that we describe below by relations 5. We can visualize the generators as a matrix $\begin{bmatrix} z_{m\times m} & \xi_{m\times n} \\ \xi_{n\times m} & z_{n\times n} \end{bmatrix}$.

It is convenient sometimes to have a common notation for even and odd variables.

$$a_{ij} = \begin{cases} z_{ij} & 1 \leq i, j \leq r, \ \text{or} \quad r+1 \leq i, j \leq r+s, \\ & \\ \xi_{ij} & 1 \leq i \leq r, \quad r+1 \leq j \leq r+s, \ \text{or} \\ & r+1 \leq i \leq r+s, \quad 1 \leq j \leq r\,. \end{cases}$$

We assign a parity to the indices: $p(i) = 0$ if $1 \leq i \leq r$ and $p(i) = 1$ if $r+1 \leq i \leq r+s$. The parity of a_{ij} is $\pi(a_{ij}) = p(i) + p(j)$ mod 2. Then, the ideal $\mathbb{I}_M$ is generated by the relations:

$$a_{ij}a_{il} = (-1)^{\pi(a_{ij})\pi(a_{il})} q^{(-1)^{p(i)+1}} a_{il}a_{ij}, \qquad \text{for } j < l$$

$$a_{ij}a_{kj} = (-1)^{\pi(a_{ij})\pi(a_{kj})} q^{(-1)^{p(j)+1}} a_{kj}a_{ij}, \qquad \text{for } i < k$$

$$a_{ij}a_{kl} = (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij}, \qquad \text{for } i < k,\ j > l \ \text{ or } \ i > k,\ j < l$$

$$a_{ij}a_{kl} - (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij} = (-1)^{\pi(a_{ij})\pi(a_{kl})} (q^{-1} - q) a_{kj}a_{il}, \qquad \text{for } \ i < k,\ j < l \tag{5}$$

Theorem 3.2. $\mathrm{M}_q(r|s)$ *a super bialgebra with comultiplication and counit defined as:*

$$\Delta(a_{ij}) := \sum_k a_{ik} \otimes a_{kj} \qquad \epsilon(a_{ij}) := \delta_{ij}. \tag{6}$$

Definition 3.3. Define the *quantum general linear supergroup* $\mathrm{GL}_q(r|s)$ as:

$$\mathrm{GL}_q(r|s) := \mathrm{M}_q(r|s)[D_1^{-1}, D_2^{-1}]$$

where D_1 and D_2 denotes the quantum determinants of the upper and lower even blocks respectively:

$$\begin{aligned} D_1 &:= \sum_{\sigma \in S_r} (-q)^{-l(\sigma)} a_{1\sigma(1)} \ldots a_{r\sigma(r)} \\ D_2 &:= \sum_{\sigma \in S_s} (-q)^{-l(\sigma)} a_{r+1,r+\sigma(1)} \ldots a_{r+s,r+\sigma(s)}. \end{aligned}$$

Remark 3.4. It turns out that $\mathrm{GL}_q(r|s)$ is a Hopf superalgebra where the comultiplication and the counit is induced from $\mathrm{M}_q(r|s)$ as in 6. However, the antipode S is a bit more involved. Moreover, one also have a notion of quantum Berezinian Ber_q. We do not need an explicit description of these for this note; we refer interested readers to [13, 15, 16].

Definition 3.5. Define the *quantum special linear supergroup* $\mathrm{SL}_q(r|s)$ as the quotient

$$\mathrm{SL}_q(r|s) := \mathrm{GL}_q(r|s)/(\mathrm{Ber}_q - 1)$$

where Ber_q denotes the quantum Berezianian [13].

We can now define the quantum Grassmannian $\mathrm{Gr}_q(2|0, 4|2)$ in analogy to our euristic classical derivation (4), see also [9–11] for a theoretical motivation in the quantum ordinary i.e. non super case, holding also here.

Definition 3.6. The *quantum super Grassmannian* $\mathrm{Gr}_q := \mathrm{Gr}_q(2|0,4|2)$ is the subalgebra of $\mathrm{SL}_q(4|2)$ generated by the elements (quantum determinants):

$$\begin{aligned} &D_{ij} := a_{i1}a_{j2} - q^{-1}a_{i2}a_{j1} && D_{in} := a_{i1}a_{n2} - q^{-1}a_{i2}a_{n1} \\ &D_{55} := a_{51}a_{52} && D_{66} := a_{61}a_{62} \\ &D_{56} = a_{51}a_{62} - q^{-1}a_{52}a_{61} && \end{aligned} \tag{7}$$

with $1 \leq i < j \leq 4$ and $n = 5, 6$.

To give a presentation of the graded ring Gr_q in terms of generators and relations, we need to find a quantum version of the commutation relations and super Plücker relations. After some tedious calculations we arrive at the following *quantum commutation relations*:

- Let $1 \leq i, j, k, l \leq 6$ be not all distinct, and D_{ij}, D_{kl} not both odd,

$$D_{ij}D_{kl} = q^{-1}D_{kl}D_{ij}, \tag{8}$$

  for $(i, j) < (k, l),\ i < j, k < l$ where the ordering '$<$' of pairs is the lexicographical ordering.

- Let $1 \leq i, j, k, l \leq 6$ be all distinct, and D_{ij}, D_{kl} not both odd and $D_{ij}, D_{kl} \neq D_{56}$. Then

$$\begin{aligned} &D_{ij}D_{kl} = q^{-2}D_{kl}D_{ij}, && 1 \leq i < j < k < l \leq 6, \\ &D_{ij}D_{kl} = q^{-2}D_{kl}D_{ij} - (q^{-1} - q)D_{ik}D_{jl} && 1 \leq i < k < j < l \leq 6, \\ &D_{ij}D_{kl} = D_{kl}D_{ij} && 1 \leq i < k < l < j \leq 6, \end{aligned} \tag{9}$$

- Let $1 \leq i < j \leq 4$, $5 \leq n \leq m \leq 6$. Then

$$\begin{aligned} &D_{in}D_{jn} = -q^{-1}D_{jn}D_{in} - (q^{-1} - q)D_{ij}D_{nn} = -qD_{jn}D_{in}, \\ &D_{ij}D_{nm} = q^{-2}D_{nm}D_{ij}, \\ &D_{i5}D_{j6} = -q^{-2}D_{j6}D_{i5} + (q^{-1} - q)D_{ij}D_{56}, \\ &D_{i6}D_{j5} = -D_{j5}D_{i6}, \\ &D_{i5}D_{i6} = -q^{-1}D_{i6}D_{i5}, \\ &D_{i5}D_{i6} = -q^{-1}D_{i6}D_{i5}, \\ &D_{55}D_{66} = -q^{-2}D_{66}D_{55}\,, \\ &D_{55}D_{56} = 0\,. \end{aligned} \tag{10}$$

Moreover, the *super Plücker relations* get quantized as follows: One has for $1 \leq i < j \leq 4$ and

$n = 5, 6$:

$$\begin{aligned}
&D_{12}D_{34} - q^{-1}D_{13}D_{24} + q^{-2}D_{14}D_{23} = 0,\\
&D_{ij}D_{kn} - q^{-1}D_{ik}D_{jn} + q^{-2}D_{in}D_{jk} = 0,\\
&D_{i5}D_{j6} + q^{-1}D_{i6}D_{j5} = qD_{ij}D_{56},\\
&D_{in}D_{jn} = qD_{ij}D_{nn},\\
&D_{in}D_{nn} = 0,\\
&D_{i5}D_{66} = -q^{-1}D_{i6}D_{56},\\
&D_{i6}D_{55} = -q^{2}D_{i5}D_{56},\\
&D_{nn}^2 = 0, \qquad D_{55}D_{56} = 0, \qquad D_{66}D_{56} = 0,\\
&D_{56}D_{56} = (q^{-1} - 3q)D_{55}D_{66}.
\end{aligned} \tag{11}$$

We end this section with the observation that we can also view $\mathrm{Gr}_q(2|0,4|2)$ as *quantum homogeneous space* (by which we mean, it admits a coaction of a quantum group $\mathrm{SL}_q(4|2)$).

Theorem 3.7. *The restriction of the comultiplication in* $\mathrm{SL}_q(4|2)$ *to* $\mathrm{Gr}_q(2|0,4|2)$ *is of the form:*

$$\mathrm{Gr}_q(2|0,4|2) \longrightarrow \mathrm{SL}_q(4|2) \otimes \mathrm{Gr}_q(2|0,4|2).$$

The proof is just a calculation, see [14].

4. $N = 2$ Minkowski Superspace and its Quantization

In this section, we mimic the construction of Minkowski space as the big cell inside $\mathrm{Gr}(2,4)$ in this super and quantum setting. A similar treatment for a quantization of the classical (complex) Minkowski space is described in [12, 13].

Consider the set S of $(4|2) \times 2$ matrices:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ \alpha_{51} & \alpha_{52} \\ \alpha_{61} & \alpha_{62} \end{bmatrix}$$

with $a_{11}a_{22} - a_{12}a_{21}$ invertible. There is a natural right action of $\mathrm{GL}_2(\mathbb{C})$ on S. Under this action, every element of S can be written uniquely as:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ u_{31} & u_{32} \\ u_{41} & u_{42} \\ \nu_{51} & \nu_{52} \\ \nu_{61} & \nu_{62} \end{bmatrix}$$

In other words, the quotient of S under the action of $\mathrm{GL}_2(\mathbb{C})$ is an affine superspace $\mathbb{M} \cong \mathbb{C}^{4|4}$. One can easily compute u_{ij} and v_{kl} for an arbitrary element of S:

$$\begin{aligned} u_{i1} &= -d_{2i}d_{12}^{-1} \quad \text{and} \quad u_{i2} = d_{1i}d_{12}^{-1} \\ v_{k1} &= -d_{2k}d_{12}^{-1} \quad \text{and} \quad u_{k2} = d_{1k}d_{12}^{-1} \end{aligned} \tag{12}$$

where $i = 3, 4$ and $k = 5, 6$ and $d_{rs} := a_{r1}a_{s2} - a_{r2}a_{s1}$.

Definition 4.1. We call $\mathbb{M}$ the quotient $S/\mathrm{GL}_2(\mathbb{C}) \cong \mathbb{C}^{4|4}$ as the $N = 2$ *Minkowski superspace*.

We also viewed it as a *trivial quantum principal bundle*. We follow [4] section 2, in giving the definition of quantum principal bundle in the super category.

Definition 4.2.

- Let (H, Δ, ϵ, S) be a Hopf superalgebra and A be an H-comodule superalgebra with coaction $\delta : A \longrightarrow A \otimes H$. Let

$$B := A^{coinv(H)} := \{a \in A \mid \delta(a) = a \otimes 1\}$$

  The extension A of the superalgebra B is called *H-Hopf-Galois* (or simply *Hopf-Galois*) if the map

$$\chi : A \otimes_B A \longrightarrow A \otimes H, \qquad \chi = (m_A \otimes \mathrm{id})(\mathrm{id} \otimes_B \delta)$$

  called the *canonical map*, is bijective (m_A denotes the multiplication in A).

- We define quantum principal superbundle as a pair (A, B), where A is an H-Hopf Galois extension and A is H-equivariantly projective as a left B-supermodule.

- Let H be a Hopf superalgebra and A an H-comodule superalgebra. The algebra extension $A^{coinv\ H} \subset A$ is called a *cleft extension* if there is a right H-comodule map $j : H \longrightarrow A$, called *cleaving map*, that is convolution invertible, , i.e. there exists a map $h : H \longrightarrow A$ such that the convolution product $j \star h$ satisfies:

$$(j \star h)(f) := (m_A \circ (j \otimes h) \circ \Delta)(f) = \epsilon(f).1$$

  for all $f \in H$.

- An extension $A^{coinv\ H} \subset A$ is called a *trivial extension* if there is an H-comodule algebra map $j : H \to A$. In this case, the convolution inverse is just $h = j \circ S$.

With these definitions we prove the following result (see [14]).

Theorem 4.3. *The natural projection $p : S \longrightarrow S/\mathrm{GL}_2(\mathbb{C}) \cong \mathbb{M}$ is a trivial principal bundle.*

Finally, we move towards the quantization. Let $\mathbb{C}_q[S]$ be the quantization of S obtained by taking the Manin relations among the entries with D_{12} invertible. By getting a quantum version of (12) we give following definition.

Definition 4.4. The $N = 2$ *quantum chiral Minkowski superspace* $\mathbb{C}_q[\mathbb{M}]$ is the superalgebra in $\mathbb{C}_q[\mathrm{GL}(4|2)]$ generated by the elements:

$$\begin{aligned} \tilde{u}_{i1} &:= -q^{-1}D_{2i}D_{12}^{-1} \qquad \tilde{u}_{i2} := D_{1i}D_{12}^{-1} \qquad i = 3,4 \\ \tilde{v}_{k1} &:= -q^{-1}D_{2k}D_{12}^{-1} \qquad \tilde{u}_{k2} := D_{1k}D_{12}^{-1} \qquad k = 5,6. \end{aligned} \tag{13}$$

By computing the commutation relations, we arrive at the following proposition, which is tedious, yet straightforward.

Theorem 4.5. $\mathbb{C}_q[\mathbb{M}]$ *is isomorphic as a superalgebra to the superalgebra of matrices* $\mathrm{M}_q(2|2)$ *(as defined in 3).*

Moreover, we also viewed it as quantum principal bundle, as we prove in [14] (see also [4, 5, 13] for more details).

Theorem 4.6. *The quantum superalgebra* $\mathbb{C}_q[S]$ *is a trivial quantum principal bundle over* $N = 2$ *quantum chiral Minkowski superspace* $\mathbb{C}_q[\mathbb{M}]$*. Moreover it carries a natural action of the quantum Poincaré supergroup.*

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References

[1] P. Aschieri, P. Bieliavsky, C. Pagani, A. Schenkel, *Noncommutative Principal Bundles Through Twist Deformation*, Comm. Math. Phys. 352 (2017) 287-344.

[2] R. Penrose, *Twistor algebra*, J. Math. Phys. 8 345–66, (1967).

[3] A. Ferber, *Supertwistors and conformal supersymmetry*, Nucl. Phys. B 132 55–64, (1978).

[4] P. Aschieri, R. Fioresi, E. Latini, *Quantum Principal Bundles on Projective Bases*, Comm. Math. Phys. 382 (2021) 1691–1724.

[5] P. Aschieri, R. Fioresi, E. Latini, T. Weber, *Differential Calculi on Quantum Principal Bundles over Projective Bases*, arXiv:2110.03481, 2021.

[6] E. J. Beggs, S. Majid, *Quantum Riemannian Geometry*, Springer, 2020.

[7] C. Carmeli, L. Caston, R. Fioresi, *Mathematical Foundation of Supersymmetry*, Ser. Lect. Math., European Math. Soc., Zurich, 2011.

[8] N. Ciccoli, R. Fioresi, F. Gavarini, *Quantization of Projective Homogeneous Spaces and Duality Principle*, J. Noncommut. Geom. 2 (2008) 449-496.

[9] R. Fioresi, *Quantum deformation of the Grassmannian manifold*, J. of Alg. 214 (1999) 418-447.

[10] R. Fioresi, *Commutation relations among generic quantum minors in* $O_q(M_n(k))$, J. of Alg. 280 (2004) 655-682.

[11] R. Fioresi, *Quantum deformation of the flag variety*, Commun. Algebra 27, 11 (1999).

[12] R. Fioresi, *Quantizations Of Flag Manifolds and Conformal Spacetime*, Rev. Math. Phys. (1997) 453-465.

[13] R. Fioresi, M. A. Lledó, *The Minkowski and conformal superspaces: the classical and the quantum pictures*, World Scientific Publishing, Singapore, (2015).

[14] R. Fioresi, M. A. Lledó, J. Razzaq, *N = 2 quantum chiral superfields and quantum super bundles*, J. Phys. A: Math. Theor. 55 384012 (2022).

[15] H. H. Phung, *On the structure of the quantum supergroups* $\mathrm{GL}_q(m|n)$, J. of Alg. (1999).

[16] H. Zhang, R. B. Zhang, *Dual canonical bases for the quantum general linear supergroup*, J. of Algebra 304, 1026–1058, (2006).

[17] Y. I. Manin, *Topics in noncommutative geometry*, M. B. Porter Lectures. Princeton University Press, Princeton, NJ, (1991).

[18] Y. I. Manin, *Gauge field theory and complex geometry*; translated by N. Koblitz and J.R. King, Springer-Verlag, Berlin-New York, 1988.

[19] R. Fioresi, E. Latini, M. A. Lledo, F. A. Nadal *The Segre embedding of the quantum conformal superspace*, Adv. Theor. Math. Phys. 22 1939–2000, (2018).

[20] D. Cervantes, R. Fioresi, M. A. Lledo *The quantum chiral Minkowski and conformal superspaces*, Adv. Theor. Math. Phys. 15 565–620, (2011).

[21] E. Shemyakova, T. Voronov, *On super Plücker embedding and cluster algebras*, Selecta Mathematica, Springer, (2022).