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Gauge Theory, Sigma Models and Generalised Geometry

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When the target space of a supersymmetric sigma model is a generalised Kähler manifold, there are two topological twists, generalising the A-model and B-model on a Kähler manifold. Kapustin and Witten considered a reduction of a 4-dimensional $N = 4$ gauge theory to such a sigma model and explained geometric Langlands programme by electric-magnetic duality. The target space is Hitchin's moduli space, which is hyper-Kähler, and the sigma model at low energies is either a B-model or a B-field transform of an A-model, all of which are anomaly-free. In this paper, we consider the reduction of the $N = 4$ gauge theory on an orientable 4-manifold containing embedded non-orientable surfaces. The resulting theory is a sigma model on a worldsheet whose boundary lives on branes from Hitchin's moduli space for non-orientable surfaces. We show that these branes are supported on submanifolds preserved by the generalised complex structures and that the low energy theory remains anomaly-free at the quantum level. We match the topological sectors and discrete symmetries of the high and low energy theories in a way that is manifestly covariant on the worldsheet.

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1. Introduction

The classical Maxwell equations in the vacuum display an apparent symmetry between electricity and magnetism. The equations are invariant if the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ interchanges by $(\boldsymbol{B},\boldsymbol{E}) \mapsto (\boldsymbol{E},-\boldsymbol{B})$. The symmetry is lost in the real world due to the presence of electric charges and the absence of magnetic charges. Attempt to restore it can be made by considering magnetic monopole solutions that are singular at some spatial points. More fundamentally, a complete description of electromagnetic configurations requires not only the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, but also phases coming from integrating the vector potential along large loops. Such phases are holonomies of a connection on a U(1)-bundle. The Maxwell equations then split into two halves: one is the Bianchi identity, of geometric origin; the other is dynamical, from variation of an action. So even in the vacuum, the symmetry of electricity and magnetism is lost in our formulation of the theory.

In quantum theory, we can do perturbations near the trivial background and obtain a Hilbert space which decomposes into subspaces according to the electric charges of the states. We can also perturb around a non-trivial, solitonic background, like a magnetic monopole, and obtain new sectors which are labelled by the magnetic charges. In this way, the symmetry between electricity and magnetism, lost at the classical level, can be recovered in the quantum theory, and this is exactly what was conjectured by Montonen and Olive [30] for non-Abelian gauge theories. The electric-magnetic duality, or S-duality, exchanges strong and weak coupling, and the gauge group G and its Langlands dual ${}^{L}G$ [15]. Over the past decades, understanding S-duality in supersymmetric gauge theories has led to not only conceptual breakthrough of the non-perturbative aspects of quantum gauge theory [32, 33], but also highly non-trivial predictions in mathematics, whose proofs supported the validity of the duality. For example, in the $N = 2$ and $N = 4$ gauge theories, S-duality provides new insights on the geometry of 4-manifolds [40, 44]. In [26], Kapustin and Witten considered another version of twisted $N = 4$ gauge theory and show that upon dimensional reduction, duality in four dimensions reduces to T-duality or mirror symmetry in two dimensions, and this explains the geometric Langlands programme. For some of the subsequent and recent developments, see [46] and references therein.

In this paper, we revisit a variant [49] of the Kapustin-Witten theory, when the 4-manifold is not a product of two orientable surfaces but contains embedded non-orientable surfaces. The resulting low energy theory is a sigma model on a worldsheet with boundary. The target space is Hichin's moduli space $\mathcal{M}_{\mathrm{H}}(C,G)$ [20] from an orientable surface C like in [26], equipped with a generalised Kähler structure from its hyper-Kähler structure. But the boundary lives on branes constructed by the moduli space $\mathcal{M}_{\mathrm{H}}(C',G)$ [23] from a non-orientable surface C' whose orientation double cover is C. More precisely, the map $p\colon \mathcal{M}_{\mathrm{H}}(C',G) \to \mathcal{M}_{\mathrm{H}}(C,G)$ by pulling back connections and fields from C' to C is a regular cover (on the smooth part) over the support of the branes and can be used to construct Chan-Paton line bundles of the branes.

The rest of the paper is organised as follows. In §2, we explain the mathematical techniques for computing the topological sectors of 4-dimensional gauge theories and 2-dimensional sigma models. We recall supersymmetric gauge theories, especially the twisted $N = 4$ theory in [26], as well as the bi-Hermitian [13] or generalised Kähler [16] geometry of the target space of the 2-dimensional sigma model with an $N = (2,2)$ supersymmetry. In §3, we consider the dimensional reduction of the twisted $N = 4$ theory when the 4-manifold is a product of two orientable surfaces

[26] or contains embedded non-orientable surfaces [49]. In both cases, we match the topological sectors and discrete symmetries of the theories in high and low energies in a way that is manifestly covariant on the worldsheet. The matchings are consistent with the S-duality in four dimensions and the mirror symmetry in two dimensions. We conclude in §4 with a summary and outlook.

2. The topology of gauge theory and sigma models

2.1 Topological sectors of a gauge theory

Classical gauge theory is about the geometry of principal fibre bundles. The simplest and earliest example is electromagnetism. A complete description of classical configurations requires not only the electromagnetic field, but also a gauge potential modulo gauge transformations. This extra information has global ramifications that can be confirmed by experiments such as the Aharonov-Bohm effect and the quantisation of magnetic fluxes in superconductivity. The gauge potential is a connection on a principal $\mathrm{U}(1)$-bundle over the spacetime and the electromagnetic field is its curvature. Generalisation to non-Abelian gauge groups was proposed in the physics literature: [54] for $\mathrm{SU}(2)$ and [39] for arbitrary Lie groups, and subsequently non-Abelian gauge potentials were identified [27, 52] with connections on principal fibre bundles. Though the global properties of fibre bundles were extensively studied much earlier [36], they did not play a role in physics until the discovery of instantons [2] and discrete fluxes [38].

We consider gauge theory with a compact connected gauge group G on a space or a spacetime X. Mathematically, we have a principal G-bundle P over the manifold X. The bundle is topologically trivial if and only if it admits a global section. The obstructions to the existence of such a section are in the cohomology groups $H^{k+1}(X,\pi_k(G))$ for $k \geq 1$ [36]. When $\dim X = 2$ or 3, the only obstruction is a class $\xi(P) \in H^2(X,\pi_1(G))$. For example, if $X = T^3$ and $G = \mathrm{PU}(N)$, then $\xi(P) \in H^2(T^3,\mathbb{Z}_N)$ is the discrete magnetic flux discovered in physics by 't Hooft [38]. If P is an $\mathrm{SO}(n)$-bundle over X of any dimension, then $\xi(P) = w_2(P)$ is the second Stiefel-Whitney class of P. If X is an oriented Riemannian manifold of dimension n and P is the bundle of positively oriented orthonormal frames on X, then $\xi(P)$ is the obstruction to the existence of a spin structure on X. We will mostly be concerned with gauge theory on a four dimensional spacetime manifold. When $\dim X = 4$, there is an additional obstruction $k(P) \in H^4(X,\pi_3(G))$. If X is closed orientable and G is simple, then $\pi_3(G) \cong \mathbb{Z}$ and $k(P) \in H^4(X,\mathbb{Z}) \cong \mathbb{Z}$ is the instanton number.

We now discuss discrete symmetries in gauge theories. In the pure gauge theory, if we modify the holonomy of a connection along a non-contractible loop in X by a group element in the centre $Z(G)$, the curvature remains unchanged and therefore this is a symmetry of the action. Since holonomies multiply when loops are joined, the symmetry group is $\mathrm{Hom}(\pi_1(X), Z(G)) \cong H^1(X,Z(G))$. In fact, this is an example of 1-form symmetry [12] (cf. [51]). There are at least two ways to reduce the discrete symmetry of the theory to a subgroup of $H^1(X,Z(G))$. First, suppose the theory has bosonic and fermionic matter fields that are sections of the associated bundles of the principal bundle. If the centre $Z(G)$ does not act trivially on the fibres but a subgroup Z of it does, then the discrete symmetry is reduced to $H^1(X,Z)$. Second, an element $g \in H^1(X,Z(G))$ may change the topology of the principal bundle. In fact it modifies $\xi(P)$ to $\xi(P) + \beta^1_X(g)$, where $\beta^1_X\colon H^1(X,Z(G)) \to H^2(X,\pi_1(G))$ is the Bockstein map [49, §2.1]. So the symmetry is spontaneously broken to $\ker(\beta^1_X)$ if $\xi(P)$ is to be fixed. When $G = \mathrm{U}(1)$, this is related to the uncertainty of fluxes [9].

Like any global symmetry, the 1-form symmetry can be gauged, at least classically. In a gauge theory, if we take the quotient of the bundle by $Z(G)$, the gauge group G reduces to $G_{\mathrm{ad}} = G/Z(G)$, but the field strength (or the curvature of the connection) remains unchanged. What distinguishes G from other groups of the same Lie algebra is that the holonomy associated to each loop on the spacetime X is in G (modulo conjugation). Since the 1-form symmetry modifies the holonomies by elements in the centre $Z(G)$, the identification of two connections by this action means that the gauge group in the gauged theory is G_{ad}. Conversely, if we start from a gauge theory with gauge group G_{ad}, then the class $\xi(P_{\mathrm{ad}})$ of the G_{ad}-bundle P_{ad} is $H^2(X, \pi_1(G_{\mathrm{ad}}))$. Gauge theory with P_{ad} is equivalent to a theory of connection and curvature on a somewhat generalised G-bundle twisted by a flat $Z(G)$-gerbe $\zeta \in H^2(X, Z(G))$, which is the image of $\xi(P_{\mathrm{ad}})$ under the change of coefficients $\pi_1(G_{\mathrm{ad}}) \to Z(G)$ [28, 48]. We allow ζ to vary and weight each of them by a phase $\varepsilon(\zeta)$, where $\varepsilon \in H^2(X, Z(G))^\vee$, the Pontryagin dual of $H^2(X, Z(G))$. Summing over all ε, we recover the gauge group G.

2.2 Supersymmetric gauge theories and S-duality

For the pure $\mathrm{U}(1)$ gauge theory in four dimensions, the partition function is a Gaussian integral. A continuous and infinite dimensional version of the Poisson summation technique shows that the quantum theory with the complex coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi}{e^2}\sqrt{-1}$, where $e > 0$ is the real coupling constant and θ is the theta-angle, is equivalent to the same theory with the complex coupling $-\frac{1}{\tau}$ (see for example [45, §8.5]). When the gauge group G is non-Abelian, Goddard, Nuyts and Olive [15] proposed a dual, magnetic group, which turns out to be the Langlands dual ${}^L G$ and has the curious property that characters of G are the cocharacters of ${}^L G$, i.e., the electric charges in the G-theory are the magnetic charges in the ${}^L G$-theory, and vice versa. Montonen and Olive [30] further conjectured that the two theories with gauge groups G and ${}^L G$ are isomorphic at the quantum level by a duality now known as the S-duality that exchanges electric with magnetic fields and fundamental particles with collective excitations like monopoles.

It was found that S-duality for non-Abelian gauge theory is more plausible when there is supersymmetry. The $N = 1$ pure gauge theory has a vector multiplet: a gauge field and a Majorana or Weyl fermion in the adjoint representation of G. $N = 1$ supersymmetry remains present with the inclusion of $N = 1$ matter, i.e., a chiral multiplet consisting of a scalar boson and a fermion in the same representation. If the latter is the adjoint representation, the theory with a suitable superpotential is the $N = 2$ pure gauge theory. By having four Majorana or Weyl fermions and six real scalar bosons in the adjoint representation, we can have the $N = 4$ pure gauge theory. Some $N = 1$ and $N = 2$ gauge theories are strongly coupled at low energies. But there are dual theories which are weakly coupled and equivalent to the original theories at low energies. These Seiberg [32] and Seiberg-Witten [33] dualities are approximate dualities since they are valid for low energy descriptions only. On the contrary, the $N=4$ gauge theory is believed to exhibit exact S-duality [34].

The Euclidean versions of the $N = 2$ and $N = 4$ pure gauge theories can be twisted so that at least one supersymmetry survives on an arbitrary 4-manifold. The twisted $N = 2$ theory calculates Donaldson's invariants at high energies [41], and the duality of the low energy descriptions yields a relation between Donaldson and Seiberg-Witten invariants [44]. The $N = 4$ gauge theory has three inequivalent twists [29, 53]. One of them leads to the Vafa-Witten theory [40], in which S-duality predicts that the generating function of the Euler numbers of the instanton moduli spaces is a modular form; see [47] when the gauge group is not simply laced. Another twist leads to the Kapustin-Witten theory [26]. In this case, there are two supersymmetries, δ_l and δ_r, that can be defined on arbitrary 4-manifolds. Consequently there is a family of topological field theories parametrised by $t \in \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$, each with a BRST operator $\delta_t = \delta_l + t\delta_r$. The parameters τ and t combine to form a canonical parameter that determines the theory [26, §3.5],

$$\Psi = \frac{\theta}{2\pi} + \frac{4\pi\sqrt{-1}}{e^2} \cdot \frac{t - t^{-1}}{t + t^{-1}} = \frac{\tau t + \bar{\tau} t^{-1}}{t + t^{-1}},$$

taking values in $\mathbb{C}P^1$. It is real (i.e., in the $\mathbb{R}P^1$ inside $\mathbb{C}P^1$) if and only if $|t| = 1$; in particular, $\Psi = \infty$ if $t = \pm\sqrt{-1}$. The theory depends only on Ψ because the t-dependence of δ_t can be eliminated by the redefinition $\delta'_t := (1 + t^2)^{-1/2}\delta_t$ when $t \neq \pm\sqrt{-1}$, satisfying $(\delta'_t)^2 = -\sqrt{-1}£_\sigma$, σ being an adjoint valued bosonic field.

When G is simple, the duality transformation on the complex coupling is $S\colon \tau \mapsto -\frac{1}{n_\mathfrak{g}\tau}$, where $n_\mathfrak{g}$ is the ratio of the length-square of the long and short roots of the Lie algebra $\mathfrak{g}$. Let $T\colon \tau \mapsto \tau + 1$ be the shift of θ by 2π. Note that if G is not simply connected, then the instanton number can be fractional [40, 47] and T brings about a non-trivial phase in the partition function. The elements S, T generate the modular group $\mathrm{SL}(2, \mathbb{Z})$ if $\mathfrak{g}$ is simply laced ($n_\mathfrak{g} = 1$) and the Hecke group if $\mathfrak{g}$ is non-simply laced ($n_\mathfrak{g} = 2$ or 3). For the twisted $N = 4$ gauge theory parametrised by $t \in \mathbb{C}P^1$, the transformations on the parameters are $S\colon (\tau, t, \Psi) \mapsto \big(-\frac{1}{n_\mathfrak{g}\tau}, -\frac{\tau}{|\tau|}t, -\frac{1}{n_\mathfrak{g}\Psi}\big)$ and $T\colon (\tau, t, \Psi) \mapsto (\tau + 1, t, \Psi + 1)$ [26, §3.5]. For example, if $\Psi = \infty$, $t = \sqrt{-1}$ and τ is purely imaginary, the dual theory has ${}^L\Psi = 0$, ${}^Lt = 1$ and ${}^L\tau$ is also purely imaginary.

2.3 Topological sectors of a sigma model

Sigma model, named after a spinless scalar field in it, was introduced in the study of strong interaction [14]. The fields in the theory include a bosonic map u from a worldvolume Σ to a target space M, which is usually a Riemannian manifold and is often a Lie group or a symmetric space, as well as other matter fields (bosonic or fermionic) which are sections of the pull-backs u^*E, where E are bundles over M. Sigma models on a two dimensional worldsheet display many features shared by gauge theories in four dimensions such as having solitonic classical solutions and non-perturbative quantum effects: mass gap, confinement, etc. Moreover, the renormalisation group equation at one-loop for the Riemannian metric on the target is the Ricci flow [11], which was introduced into mathematics by [17].

If the worldsheet Σ has a boundary $\partial\Sigma$ which is a disjoint union of circles, we can impose a D-brane boundary condition. That is, the map $u\colon \Sigma \to M$ is required to send $\partial\Sigma$ to a submanifold N of M on which there is a Chan-Paton vector bundle (typically a line bundle) $L \to N$ with a connection. The dynamics of the bulk theory is affected by inserting the holonomies of L around the circles in $\partial\Sigma$ in the partition function. We will need a slight variant. Suppose there is a (finite) covering $p\colon M' \to N$ and, in addition to u, there is a map $u'\colon \partial\Sigma \to M'$ satisfying $p \circ u' = u|_{\partial\Sigma}$. We say that the brane M' wraps around its support N. It clearly includes a special case when M' is a finite number of disjoint copies of N. If the covering is regular, then any character ε of the group of deck transformation defines a flat line bundle $M' \times_\varepsilon \mathbb{C}$ over N. In this way, we have the usual notion of a D-brane consisting of a line bundle L over N.

For simplicity, suppose that other than the map u, the matter fields as sections of u^*E form a single homotopy class; this is certainly true when each E is a vector bundle. If Σ is a closed surface, the topological sectors of the sigma model are labelled by $[\Sigma, M]$, the set of homotopy classes of maps from Σ to M. If base points are chosen in Σ and M, let $[\Sigma, M]_0$ be the set of the (based) homotopy classes of based maps. If $\partial\Sigma \neq \emptyset$ and the boundary condition is given by a brane M' wrapping around $N \subset M$, then the topological sectors are labelled by the set $[(\Sigma, \partial\Sigma), (M, M')]$ of homotopy classes of the pair (u, u'). Let $[(\Sigma, \partial\Sigma), (M, M')]_0$ be the corresponding set of based map pairs if base points (in $\partial\Sigma$ and M') are specified.

We use the Puppe sequence (cf. [35, §7.1]) to compute the sets of homotopy classes of maps. If $f\colon X \to Y$ is a based map, the Puppe sequence is a long exact sequence of pointed sets

$$\cdots \to [\mathrm{s}(Y), M]_0 \to [\mathrm{s}(X), M]_0 \to [C(f), M]_0 \to [Y, M]_0 \to [X, M]_0,$$

where $C(f)$ is the mapping cone of f and $\mathrm{s}(X)$ is the (reduced) suspension of X. Let Σ be a closed orientable surface of genus $g(\Sigma)$. It has a cellular structure whose 0-skeleton $\Sigma^{(0)}$ is a single point and whose 1-skeleton $\Sigma^{(1)} = \bigvee_{i=1}^{2g(\Sigma)} S^1$ is a bouquet of $2g(\Sigma)$ circles; Σ itself is obtained by attaching a 2-cell via a map $f\colon (S^1, s_0) \to (\Sigma^{(1)}, \Sigma^{(0)})$. Then $\Sigma = C(f)$ and the Puppe sequence reduces to the short exact sequence [49, §A.7]

$$0 \to \pi_2(M) \to [\Sigma, M]_0 \to \mathrm{Hom}(\pi_1(\Sigma), \pi_1(M)) \to 0.$$

If $\pi_1(M)$ is Abelian and acts trivially on $\pi_2(M)$, then $[\Sigma, M]_0$ can be replaced by the set $[\Sigma, M]$ of homotopy classes of unbased maps and $\mathrm{Hom}(\pi_1(\Sigma), \pi_1(M))$ simplifies to $H^1(\Sigma, \pi_1(M))$.

If $(f, f')\colon (X, X') \to (Y, Y')$ is a based map of based spaces, then there is a relative Puppe sequence of homotopy classes of maps to another pair (M, N) [35, §7.1]. This can be generalised so that M' is not in M but map to M by $p\colon M' \to N \subset M$ [49, §A.7]. A corollary is the long exact sequence of relative homotopy groups, also in this more general situation. The relative Puppe sequence can be used to compute $[(\Sigma, \partial\Sigma), (M, M')]_0$, where Σ is an orientable surface whose boundary $\partial\Sigma$ is a single circle. Let $\hat\Sigma$ be the closed surface after attaching a disc to the boundary. As above, the 0-skeleton $\hat\Sigma^{(0)}$ is a single point and the 1-skeleton $\hat\Sigma^{(1)}$ is a bouquet of $2g(\hat\Sigma)$ circles. Suppose by attaching a disc to $\hat\Sigma^{(1)}$ via the map $\hat f\colon (S^1, s_0) \to (\hat\Sigma^{(1)}, \hat\Sigma^{(0)})$ we get $\hat\Sigma$. Then the mapping cone of the composite pair $(f, f')\colon (I, \partial I) \to (S^1, s_0) \to (\hat\Sigma^{(1)}, \hat\Sigma^{(0)})$, where I is an interval, is $(\Sigma, \partial\Sigma)$, and the relative Puppe sequence yields the short exact sequence [49, §A.7]

$$0 \to \pi_2(Z, Z') \to [(\Sigma, \partial\Sigma), (M, M')]_0 \to H^1(\hat\Sigma, \pi_1(M)) \to 0.$$

We obtain $[(\Sigma, \partial\Sigma), (M, M')]$ from $[(\Sigma, \partial\Sigma), (M, M')]_0$ by adding another graded component $\pi_0(M')$.

2.4 Target space geometry, generalised geometry, and T-duality

Consider a sigma model whose target space is a circle and the only field on the worldsheet Σ is a map to the circle. The theory is equivalent to one with the dual circle as the target. This is the bosonic Abelian duality in two dimensions (see for example [45, §8.1]) analogous to the S-duality of the pure U(1) gauge theory in four dimensions. More generally, when the target space has a circle factor and a possible B-field, we have T-duality given by the Buscher rule [6] or its global version [4] in the case of circle fibrations.

Duality between more general target spaces again requires supersymmetry. If the target is a Riemannian manifold M, we have an $N=(1,1)$ supersymmetric sigma model containing a bosonic map $u\colon \Sigma \to M$ and a fermionic field $\psi \in \Gamma(S_\Sigma \otimes u^*TM)$, where S_Σ is the spinor bundle over Σ. If the target space is Kähler and hyper-Kähler, the same theory has $N=(2,2)$ and $N=(4,4)$ sypersymmetry, respectively. The $N=(2,2)$ theory has two inequivalent twists: the A- and B-twists, leading to the A- and B-models, which are topological sigma models whose BRST operator is the surviving supersymmetry on a curved Σ [43]. The A-model depends only on the symplectic structure on M and can be defined when M is symplectic [42] whereas the B-model depends only on the complex structure on M and is anomaly free when M is Calabi-Yau. Mathematically, the A-model is about the Gromov-Witten invariants counting pseudo-holomorphic curves whereas the B-model is about periods and variations of complex structures. Two spaces M and $M^\vee$ are mirrors of each other if the A-model on M is equivalent to the B-model on $M^\vee$, and vice versa.

In the presence of a B-field, $N=(2,2)$ supersymmetry requires bi-Hermitian geometry on M [13]. There is a pair of complex structures J_+, J_- on M that are parallel under the respective connections ∇^+, ∇^-. Here $\nabla^\pm$ preserves an Hermitian metric g (with respect to both $J_\pm$) and their torsions are proportional to $\pm H = \pm dB$. These conditions are equivalent to having a pair of commuting (twisted) generalised complex structures

$$\mathcal{J}_\pm := \frac{1}{2}\begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^t \pm J_-^t) \end{pmatrix},$$

where $\omega_\pm = gJ_\pm$, which define a generalised Kähler metric [16, §6.4]. The theory can then be twisted in two ways, each leading to a topological sigma model that depends only on one of $\mathcal{J}_\pm$ (cf. [24, 31]). We choose one, say $\mathcal{J} := \mathcal{J}_+$. If $J_+ = J_-$, the twisted theory is a B-model in the complex structure J_+, whereas if $J_+ = -J_-$, it is an A-model in a symplectic form proportional to ω_+. The quantum theory is anomaly-free only if $c_1(T_+^{1,0}M) + c_1(T_-^{1,0}M) = 0$, where $T_\pm^{1,0}M$ are the holomorphic tangent bundles of M in the complex structures $J_\pm$, respectively [24]. Equivalently, the first Chern class of the $\sqrt{-1}$-eigenbundle of $\mathcal{J}$ must vanish. In the absence of a B-field, if $J_+ = J_-$ (B-model), this condition reduces $c_1(T^{1,0}M) = 0$ whereas if $J_+ = -J_-$ (A-model), it is always satisfied. When the B-field is closed, the condition reduces to the definition of generalised Calabi-Yau manifolds [22], of which the usual Calabi-Yau and symplectic manifolds are examples.

If the target space M is a hyper-Kähler manifold with a Riemannian metric g, three complex structures I, J, K and the corresponding Kähler forms $\omega_I, \omega_J, \omega_K$, there is a family of complex structures [26, §5.1]

$$J_w := \frac{1-\bar{w}w}{1+\bar{w}w}\, I + \sqrt{-1}\,\frac{w-\bar{w}}{1+\bar{w}w}\, J + \frac{w+\bar{w}}{1+\bar{w}w}\, K$$

parametrised by $w \in \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$. For any pair (J_+, J_-) of complex structures given by $(w_+, w_-) \in \mathbb{C}P^1 \times \mathbb{C}P^1$ and for any closed B-field, the conditions for $N=(2,2)$ supersymmetry are automatically satisfied. The theory is a B-model if $w_+ = w_-$. If $w_+ \neq w_-$, the target space has a Kähler structure (g, ω', J') given by

$$\omega' := \frac{(1+|w_+|^2)(1+|w_-|^2)}{2|w_+ - w_-|^2}(\omega_+ - \omega_-), \qquad J' := \frac{(1+|w_+|^2)^{1/2}(1+|w_-|^2)^{1/2}}{2|w_+ - w_-|}(J_+ - J_-).$$

Upon twisting, the sigma model is an A-model with the symplectic form ω' by a B-field transform of $B' = (\omega_+ + \omega_-)(J_+ - J_-)^{-1}$. The complexified Kähler form is [26, §5.2]

$$B' + \sqrt{-1}\,\omega' = -\sqrt{-1}\left(\frac{w_+ + w_-}{w_+ - w_-}\omega_I + \sqrt{-1}\frac{w_+w_- + 1}{w_+ - w_-}\omega_J + \frac{w_+w_- - 1}{w_+ - w_-}\omega_K\right).$$

Moreover, the twisted theory is anomaly-free for all pairs $(w_+, w_-) \in \mathbb{C}P^1 \times \mathbb{C}P^1$.

Suppose the worldsheet Σ has a boundary $\partial\Sigma$ and the boundary condition is given by a brane with a line bundle L over its support $N \subset M$. Consider the topological A- and B-models in the absence of a B-field. Consistency with the respective supersymmetry in the bulk imposes constraints on the geometry of the branes (cf. [1]). For example, in the A-model, N is Lanrangian and L is flat (A-branes), whereas in the B-model, N is complex and L is holomorphic (B-branes). In special situations, there can be coisotropic A-branes [25]. With a B-field and a bi-Hermitian or generalised complex target space M, the condition for supersymmetry is that N is a generalised complex submanifold of M [5, 55]. This means that the generalised complex structure $\mathcal{J}$ preserves the direct sum of TN and its annihilator $(TN)^\circ \subset T^*M$ [16, §7]. Note that in general N is not generalised complex on its own, for neither is $TN \oplus (TN)^\circ$ the generalised tangent space of N. When M is symplectic or complex, the condition reduces to the one for A- or B-branes.

3. Dimensional reduction of the Kapustin-Witten theory

3.1 Reduction along orientable surfaces

We consider gauge theory on a spacetime manifold $X = \Sigma \times C$, where C is a closed orientable surface of small size while the surface Σ is also orientable and is either open or closed but of large size. At low energies, fields on X have to achieve minimal energy along C but can be slowly varying along Σ. So a gauge theory on X reduces to a sigma-model on the worldsheet Σ [3, 18]. For the twisted $N = 4$ gauge theory of Kapustin and Witten [26], the equations for minimal energy configurations along C are, for all $t \in \mathbb{C}P^1$ parametrising the theory, Hitchin's equations

$$F_A = \tfrac{1}{2}[\phi, \phi], \quad d_A * \phi = 0, \quad d_A\,\phi = 0,$$

where A is a connection on a principal G-bundle P over C and $\phi \in \Omega^1(C, \mathrm{ad}\,P)$. The Hitchin moduli space $\mathcal{M}_{\mathrm{H}}(C, G)$ is the space of pairs (A, ϕ) satisfying the above equations modudo the group of gauge transformations [20]. Among the low energy degrees of freedom is a map $u\colon \Sigma \to \mathcal{M}_{\mathrm{H}}(C, G)$, which reconstructs a G_{ad}-bundle $(u \times \mathrm{id}_C)^*\mathcal{U}$ (and thus almost the G-bundle) over X in gauge theory, where $\mathcal{U} \to \mathcal{M}_{\mathrm{H}}(C, G) \times C$ is the universal bundle of structure group G_{ad} [26, §7.2]. The G_{ad}-bundle lifts to a G-bundle along C, because of $\mathcal{M}_{\mathrm{H}}(C, G)$, and in fact along $\gamma \times C$ for any curve $\gamma \subset \Sigma$, but not necessarily along Σ.

We match the topological sectors in the gauge theory on $X = \Sigma \times C$ and the sigma model on Σ with target $\mathcal{M}_{\mathrm{H}}(C, G)$ in a way that is generally covariant on the worldsheet Σ; for the matching in the canonical formalism when Σ has a spacetime splitting, see [26, §7.2]. Suppose Σ and C are both closed and orientable. In gauge theory, the topology of the G-bundles over X are classified by elements in $H^2(X, \pi_1(G)) \cong H^2(C, \pi_1(G)) \oplus H^1(\Sigma, H^1(C, \pi_1(G))) \oplus H^0(C, \pi_1(G))$ as well as in $H^4(X, \pi_3(G)) \cong H^2(C, \pi_3(G))$. Topological sectors of the sigma model are classified by $[\Sigma, \mathcal{M}_{\mathrm{H}}(C,G)]$, the set of homotopy classes of the map u. We recall the homotopy groups $\pi_k(\mathcal{M}_{\mathrm{H}}(C,G))$ for small k; see [49, §A.4] and references therein for details. First, $\pi_0(\mathcal{M}_{\mathrm{H}}(C,G)) = H^2(C,\pi_1(G))$, i.e., the connected components $\mathcal{M}_{\mathrm{H}}^{m_0}(C,G)$ of $\mathcal{M}_{\mathrm{H}}(C,G)$ are classified by the topology $m_0 = \xi(P)$ of the G-bundle P over C. Furthermore, for each $m_0 \in H^2(C,\pi_1(G))$, the fundamental group $\pi_1(\mathcal{M}_{\mathrm{H}}^{m_0}(C,G)) \cong H^1(C,\pi_1(G))$ is Abelian and it acts trivially on $\pi_2(\mathcal{M}_{\mathrm{H}}^{m_0}(C,G))$. The latter fits in the exact sequence

$$0 \to H^2(C,\pi_3(G)) \to \pi_2(\mathcal{M}_{\mathrm{H}}^{m_0}(C,G)) \to H^0(C,\pi_1(G_{\mathrm{ad}})) \to 0.$$

By the Puppe sequence (applied to $M = \mathcal{M}_{\mathrm{H}}^{m_0}(C,G)$), we have an exact sequence [49, §3.2]

$$0 \to \pi_2(\mathcal{M}_{\mathrm{H}}^{m_0}(C,G)) \to [\Sigma, \mathcal{M}_{\mathrm{H}}^{m_0}(C,G)] \to H^1(\Sigma, H^1(C,\pi_1(G))) \to 0.$$

Thus the topological sectors of the two theories match except for the parts $H^0(C,\pi_1(G)) \cong H^2(\Sigma,\pi_1(G))$ in four dimensions and $H^0(C,\pi_1(G_{\mathrm{ad}})) \cong H^2(\Sigma,\pi_1(G_{\mathrm{ad}}))$ in two dimensions. This discrepancy is consistent with the observation that u recovers only a G_{ad}-bundle on Σ. We should restrict to a subset of maps u that gives rise to honest G-bundles. Field-theoretically, this can be achieved by allowing discrete B-fields on the target and summing over them, much like the lift of G_{ad} to G in gauge theory (§2.1). More precisely, for each $e_0 \in Z(G)^\vee$, we have a flat B-field $e_0(\bar\xi(\mathcal{U})^{2,0}) \in H^2(\mathcal{M}_{\mathrm{H}}(C,G),\mathrm{U}(1))$ on $\mathcal{M}_{\mathrm{H}}(C,G)$, where $\bar\xi(\mathcal{U})^{2,0}$ is the $(2,0)$-component of $\bar\xi(\mathcal{U}) \in H^2(\mathcal{M}_{\mathrm{H}}(C,G)\times C, Z(G))$ obtained from $\xi(\mathcal{U}) \in H^2(\mathcal{M}_{\mathrm{H}}(C,G)\times C,\pi_1(G_{\mathrm{ad}}))$ by the map $\pi_1(G_{\mathrm{ad}}) \to Z(G)$ of coefficients. The B-field contributes a phase $u^*(e_0(\bar\xi(\mathcal{U})^{2,0})) \in H^2(\Sigma,\mathrm{U}(1)) \cong \mathrm{U}(1)$ in the path integral whereas summing over $e_0 \in Z(G)^\vee$ selects the desired subset of maps u [49, §3.2]. However if we fix $(m_0,e_0) \in \pi_1(G)\times Z(G)^\vee$, the 4-dimensional theory does not have a standard G-bundle: it is an honest G-bundle along C with the topology $m_0 \in H^2(C,\pi_1(G)) \cong \pi_1(G)$ but is twisted along Σ by $Z(G)$-gerbes $\zeta \in H^2(\Sigma, Z(G)) \cong Z(G)$ [28, 48], each weighted by a phase $e_0(\zeta)$.

In four dimensions, the 1-form symmetry group is $H^1(X,Z(G))$. Since $X = \Sigma\times C$, we have $H^1(X,Z(G)) \cong H^1(\Sigma,Z(G)) \oplus H^1(C,Z(G))$. In the two dimensional theory, $H^1(C,Z(G))$ becomes an ordinary 0-form, internal symmetry, acting on the target space $\mathcal{M}_{\mathrm{H}}(C,G)$. The part $H^1(\Sigma,Z(G))$ that remains a 1-form symmetry in two dimensions is related to sectors of the low energy theory that have so far remained hidden. The map $u\colon \Sigma \to \mathcal{M}_{\mathrm{H}}(C,G)$ pulls back the universal G_{ad}-bundle and the latter should be lifted to a G-bundle (especially along Σ) as explained above. However, for a general worldvolume Σ, the lift is not unique and different lifts are parametrised by elements of $H^2(\Sigma,\pi_1(G))$ that are in the image of the Bockstein map $\beta^1_\Sigma\colon H^1(\Sigma,Z(G)) \to H^2(\Sigma,\pi_1(G))$. So $H^1(\Sigma,Z(G))$ acts among the sectors with discrete parameters in $\mathrm{im}(\beta^1_\Sigma)$ and each sector is weighted by a phase determined by the fractional part of the instanton number, as required by gauge theory. If Σ is an orientable worldsheet, the situation simplifies because the lift of a G_{ad}-bundle on Σ to a G-bundle is always unique and the 1-form symmetry $H^1(\Sigma,Z(G))$ acts within the only sector.

For simplicity, we always assume that C is of genus $g(C) > 1$ and focus on the smooth part of $\mathcal{M}_{\mathrm{H}}(C,G)$ which we denote by the same notation. A generic Hitchin pair (A,ϕ) on C is irreducible and represents a smooth point on the moduli space. The space $\mathcal{M}_{\mathrm{H}}(C,G)$, of real dimension $4(g(C)-1)\dim G$, is a hyper-Kähler quotient of an infinite dimensional affine space

[20]. Following [26, §4.1], we let I be the complex structure on $\mathcal{M}_{\mathrm{H}}(C,G)$ induced by that on C, J be the rotation from δA to $\delta\phi$ (both are in $\Omega^1(C,\mathrm{ad}P)$), and $K=IJ$. At low energies, the sigma-model metric on the target space $\mathcal{M}_{\mathrm{H}}(C,G)$ is $\frac{4\pi}{e^2}=\mathrm{Im}\,\tau$ times the standard hyper-Kähler metric whereas the theta term in the four-dimensional gauge theory reduces to a globally defined closed B-field $B_\theta=-\frac{\theta}{2\pi}\omega_I=-(\mathrm{Re}\,\tau)\,\omega_I$ on $\mathcal{M}_{\mathrm{H}}(C,G)$. In fact, B_θ is the integration of the 4-form $\frac{\theta}{8\pi^2}\,\mathrm{tr}\,F^{\mathcal{U}}\wedge F^{\mathcal{U}}$ on $\mathcal{M}_{\mathrm{H}}(C,G)\times C$ along C, where $F^{\mathcal{U}}$ is the curvature of the universal bundle [49, §A.5]. This continuous B-field on $\mathcal{M}_{\mathrm{H}}(C,G)$ is in addition to the above discrete B-fields from various $e_0\in Z(G)^\vee$.

The twisted $N=4$ gauge theory parametrised $t\in\mathbb{C}P^1$ reduces to the topological sigma model with the hyper-Kähler target $\mathcal{M}_{\mathrm{H}}(C,G)$ parametrised by $(w_+,w_-)=(-t,t^{-1})$ [26, §5.1], which is anomaly-free as expected. The corresponding generalised complex structure on the target is

$$\mathcal{J}_t=\frac{1}{1+\bar{t}t}\begin{pmatrix}-\sqrt{-1}(t-\bar{t})J & -(\mathrm{Im}\,\tau)^{-1}((1-\bar{t}t)\omega_I^{-1}-(t+\bar{t})\omega_K^{-1})\\ \mathrm{Im}\,\tau((1-\bar{t}t)\omega_I-(t+\bar{t})\omega_K) & \sqrt{-1}(t-\bar{t})J^t\end{pmatrix}.$$

Following the general pattern (cf. §2.4), if $t=\pm\sqrt{-1}$, then $w_+=w_-=\mp\sqrt{-1}$, and the 2-dimensional theory is a B-model in the complex structures $\pm J$. If $t\in\mathbb{R}\cup\{\infty\}$, then the theory is an A-model; for example, the symplectic structure is $\pm(\mathrm{Im}\,\tau)\,\omega_K$ if $t=\mp1$ and $\pm(\mathrm{Im}\,\tau)\,\omega_I$ if $t=0,\infty$. For other values of t, the theory is an A-model with the symplectic form ω_t upon a B-field transform by B_t, where [26, §5.2]

$$\omega_t:=(\mathrm{Im}\,\tau)\frac{1-\bar{t}^2t^2}{(1+t^2)(1+\bar{t}^2)}\Big(\omega_I-\frac{t+\bar{t}}{1-\bar{t}t}\,\omega_K\Big),\qquad B_t:=-(\mathrm{Im}\,\tau)\frac{\sqrt{-1}(t^2-\bar{t}^2)}{(1+t^2)(1+\bar{t}^2)}\Big(\omega_I+\frac{1-\bar{t}t}{t+\bar{t}}\,\omega_K\Big).$$

Combining the B-field B_θ and the complexified Kähler form $B_t+\sqrt{-1}\,\omega_t$, we obtain the cohomology class $[B_\theta+B_t+\sqrt{-1}\,\omega_t]=-\Psi[\omega_I]$, showing that Ψ is the relevant parameter in the 2-dimensional theory as well. These 2-dimensional theories depend only on J, ω_I, ω_K that are defined without choosing a complex structure on C. This reflects the metric independence of the 4-dimensional topological theories.

With $(w_+,w_-)=(-t,t^{-1})$, the S-duality of gauge theory in four dimensions is compatible with the mirror symmetry of sigma models in two dimensions. For example, if $t=\sqrt{-1}$ and τ is purely imaginary, the sigma model is a B-model in the complex structure J. Its mirror, with ${}^Lt=1$, is an A-model with the symplectic form $(\mathrm{Im}\,\tau)\omega_K$. This is the important special case in which S-duality in four dimensions gives rise to the geometric Langlands programme when reduced to two dimensions [26]. S-duality in gauge theory maps $(m_0,e_0)\mapsto(e_0,-m_0)$ [26, §7.2], a discrete analogue of the Hodge star operation. This is possible because of the isomorphisms $\pi_1({}^LG)\cong Z(G)^\vee$, $Z({}^LG)\cong\pi_1(G)^\vee$. For sigma models, the target space $\mathcal{M}_{\mathrm{H}}^{m_0}(C,G)$ with a B-field given by ${}^Lm_0=e_0$ is mirror to $\mathcal{M}_{\mathrm{H}}^{e_0}(C,{}^LG)$ with a B-field given by ${}^Le_0=-m_0$. Mathematically [8, 19], the mirror symmetry is a T-duality in the sense of Strominger, Yau and Zaslow [37], extended by [7, 21] in the presence of B-fields. When G is not simply connected, the transformation $T\colon\tau\mapsto\tau+1$ brings a non-trivial phase in the sigma model as well because $[\omega_I/2\pi]$ need not be integral [26, §4.1].

3.2 Reduction along non-orientable surfaces

We consider the $N=4$ gauge theory on a particular closed orientable 4-manifold X that is not a product of two surfaces but contains embedded non-orientable surfaces. Let the non-orientable surface C' be a connected sum of $g(C')$ copies of $\mathbb{R}P^2$ and let $\pi\colon C \to C'$ be its orientation double cover. Then C is orientable and is of genus $g(C) = g(C') - 1$ and the non-trivial deck transformation ι on C is an involution reversing its orientation. Let $\tilde{\Sigma}$ be another closed orientable surface with an orientation reversing involution, also denoted by ι, and let $\Sigma = \tilde{\Sigma}/\iota$. If the fixed point set $\tilde{\Sigma}^\iota$ of ι is non-empty, then Σ is an orientable surface whose boundary $\partial\Sigma$ is identified with $\tilde{\Sigma}^\iota$. The 4-manifold is $X = \tilde{\Sigma} \times_\iota C$. Since the diagonal action of ι on the closed orientable 4-manifold $\tilde{\Sigma} \times C$ is fixed-point free and orientation preserving, X is smooth, orientable and closed [49, §4.1]. Globally, X is not a product of two surfaces, but there is a projection map $\pi_X\colon X \to \Sigma$ (by forgetting C). If $\sigma \in \Sigma^\circ$, the interior of Σ, then $\pi_X^{-1}(\sigma)$ is a copy of C. But if $\sigma \in \partial\Sigma$, then $\pi_X^{-1}(\sigma)$ is a copy of C' embedded in X.

It will be useful to recall the cohomology groups of surfaces with coefficients in any Abelian group A [49, §A.8]. By the naturality of the universal coefficient formula, the map $\pi^*\colon H^2(C', A) \to H^2(C, A)$ is zero. On the other hand, we have $\ker(\pi^*\colon H^1(C', A) \to H^1(C, A)) = A_{[2]}$, the 2-torsion subgroup of A. Let $\partial\Sigma$ be the disjoint union of $h = h(\partial\Sigma)$ copies of the circle. Then in the long exact sequence

$$0 \to H^0(\Sigma, A) \to H^0(\partial\Sigma, A) \to H^1((\Sigma, \partial\Sigma), A) \to H^1(\Sigma, A) \to H^1(\partial\Sigma, A) \to H^2((\Sigma, \partial\Sigma), A) \to 0,$$

the map $H^0(\Sigma, A) \cong A \to H^0(\partial\Sigma, A) \cong A^{\oplus h}$ is the diagonal map, $H^1(\partial\Sigma, A) \cong A^{\oplus h} \to H^2((\Sigma, \partial\Sigma), A) \cong A$ sums over the components in $A^{\oplus h}$, and $H^1((\Sigma, \partial\Sigma), A) \cong H^1(\hat{\Sigma}, A) \oplus \operatorname{coker}(A \to A^{\oplus h}) \to H^1(\Sigma, A) \cong H^1(\hat{\Sigma}, A) \oplus \ker(A^{\oplus h} \to A)$ is block diagonal, being the identity isomorphism on $H^1(\hat{\Sigma}, A)$ and zero on its complement. Here $\hat{\Sigma}$ is the closed surface obtained from Σ by attaching a disk to each boundary circle.

For a gauge theory on X with gauge group G, the topological sectors are labelled by the instanton numbers in $H^4(X, \pi_3(G)) \cong \pi_3(G)$ and the discrete fluxes in $H^2(X, \pi_1(G))$. To compute the cohomology groups $H^\bullet(X, A)$ for any Abelian group A, consider a map of pairs $(\Sigma, \partial\Sigma) \times C \to (X, \partial\Sigma \times C')$ and the commutative diagramme

$$\begin{tikzcd}
\cdots \arrow[r] & H^{k-1}(\partial\Sigma \times C', A) \arrow[r, "\delta_X^{k-1}"] \arrow[d, "(\mathrm{id}_{\partial\Sigma} \times \pi)^*"'] & H^k((X, \partial\Sigma \times C'), A) \arrow[r] \arrow[d, "\theta^k"'] & H^k(X, A) \arrow[r] \arrow[d] & H^k(\partial\Sigma \times C', A) \arrow[r, "\delta_X^k"] \arrow[d, "(\mathrm{id}_{\partial\Sigma} \times \pi)^*"'] & H^{k+1}((X, \partial\Sigma \times C'), A) \arrow[r] \arrow[d, "\theta^{k+1}"'] & \cdots \\
\cdots \arrow[r] & H^{k-1}(\partial\Sigma \times C, A) \arrow[r, "\delta_{\Sigma\times C}^{k-1}"] & H^k((\Sigma, \partial\Sigma) \times C, A) \arrow[r] & H^k(\Sigma \times C, A) \arrow[r] & H^k(\partial\Sigma \times C, A) \arrow[r, "\delta_{\Sigma\times C}^k"] & H^{k+1}((\Sigma, \partial\Sigma) \times C, A) \arrow[r] & \cdots
\end{tikzcd}$$

in which the rows are the long exact sequences of the pairs and the vertical maps θ^k are excision isomorphisms [49, §A.9]. Thus $H^k(X, A)$ fits in the short exact sequence

$$0 \to \operatorname{coker}(\delta_X^{k-1}) \to H^k(X, A) \to \ker(\delta_X^k) \to 0.$$

The kernel and cokernel of δ_X^k are isomorphic to those of the composite map $\delta_{\Sigma\times C}^k \circ (\mathrm{id}_{\partial\Sigma} \times \pi)^*$. Using these, we conclude [49, §A.9] that the graded components of $H^1(X, A)$ are $A_{[2]}^{\oplus h}$, $H^1(\Sigma, A)$, $\pi^* H^1(C', A)$ whereas those of $H^2(X, A)$ are $A_{[2]}^{\oplus h}$, $H^1(C, A)^{\oplus(h-1)}$, $H^1(\hat{\Sigma}, H^1(C, A))$, $(A/2A)^{\oplus h}$. So the topological sectors in the gauge theory are labelled by elements in $\pi_3(G)$, $\pi_1(G)_{[2]}^{\oplus h}$, $H^1(C, \pi_1(G))^{\oplus(h-1)}$, $H^1(\hat{\Sigma}, H^1(C, \pi_1(G)))$, $(\pi_1(G)/2\pi_1(G))^{\oplus h}$.

We now consider the reduction to two dimensions. If Σ is large while C and C' are small, the fields A, ϕ in the low energy theory satisfy Hitchin's equations along the fibres C or C' of π_X. The gauge theory on X reduces to a sigma-model on Σ: the interior Σ° is mapped to $\mathcal{M}_{\mathrm{H}}(C,G)$ by u while the boundary $\partial\Sigma$ is mapped to $\mathcal{M}_{\mathrm{H}}(C',G)$ by u'. Here $\mathcal{M}_{\mathrm{H}}(C',G)$ is the Hitchin moduli space for the non-orientable surface C' [23]. The map $p\colon \mathcal{M}_{\mathrm{H}}(C',G) \to \mathcal{M}_{\mathrm{H}}(C,G)$ that pulls back bundles, connections and sections from C' to C is a regular $Z(G)_{[2]}$-covering onto its image $\mathcal{N}(C,G)$, which is in the ι-invariant subspace $\mathcal{M}_{\mathrm{H}}(C,G)^\iota$ of the same dimension. At low energies, the map u from Σ° extends to Σ and satisfies $u|_{\partial\Sigma} = p\circ u'$. In this way, the boundary lives on a brane which wraps around non-trivially on $\mathcal{N}(C,G)$. We have a sigma-model on the worldsheet Σ with boundary, with a pair of maps $(u,u')\colon (\Sigma,\partial\Sigma)\to(\mathcal{M}_{\mathrm{H}}(C,G),\mathcal{M}_{\mathrm{H}}(C',G))$. The maps yield two G_{ad}-bundles: $(u\times\mathrm{id}_C)^*\mathcal{U}$ on $\Sigma^\circ\times C$ and $(u'\times\mathrm{id}_{C'})^*\mathcal{U}'$ on $\partial\Sigma\times C'$, where $\mathcal{U}'\to\mathcal{M}_{\mathrm{H}}(C',G)\times C$ is the universal bundle for the non-orientable C'. They match to form a G_{ad}-bundle on X [49, §4.1].

If C' is the connected sum of $g(C') > 2$ copies of $\mathbb{R}P^2$, then a generic Hitchin pair on C' is irreducible and represents a smooth point on $\mathcal{M}_{\mathrm{H}}(C',G)$. The moduli space $\mathcal{M}_{\mathrm{H}}(C',G)$ is of real dimension $2(g(C')-2)\dim G$ and is Kähler in J. The homotopy groups $\pi_k(\mathcal{M}_{\mathrm{H}}(C',G))$ for small k are yet to be worked out fully, but it is expected (see [49, §A.4] and references therein) that $\pi_0(\mathcal{M}_{\mathrm{H}}(C',G)) \cong H^2(C',\pi_1(G)) \cong \pi_1(G)/2\pi_1(G)$, $\pi_1(\mathcal{M}_{\mathrm{H}}(C',G)) \cong H^1(C',\pi_1(G))$ and $\pi_2(\mathcal{M}_{\mathrm{H}}(C',G)) \cong H^0(C',\pi_1(G_{\mathrm{ad}})) \cong \pi_1(G_{\mathrm{ad}})$. Shortening the notations to $\mathcal{M} = \mathcal{M}_{\mathrm{H}}(C,G)$, $\mathcal{M}' = \mathcal{M}_{\mathrm{H}}(C',G)$, the long exact sequence of relative homotopy groups (with $p\colon \mathcal{M}'\to\mathcal{M}$) is

$$0\to H^2(C,\pi_3(G))\to\pi_2(\mathcal{M},\mathcal{M}')\to H^1(C',\pi_1(G))\to H^1(C,\pi_1(G))\to\pi_1(\mathcal{M},\mathcal{M}')\to H^2(C',\pi_1(G))\to 0.$$

This yields $\mathrm{coker}(\pi_2(\mathcal{M})\to\pi_2(\mathcal{M},\mathcal{M}'))\cong\pi_1(G)_{[2]}$ and a short exact sequence [49, §A.10]

$$0\to\pi_3(G)\to\pi_2(\mathcal{M},\mathcal{M}')\to\pi_1(G)_{[2]}\to 0.$$

We now compute the set of topological sectors of the sigma model when the worldsheet Σ has a boundary. First, suppose $\partial\Sigma$ is a single circle, i.e., $h=1$. Then the relative Puppe sequence yields

$$0\to\pi_2(\mathcal{M},\mathcal{M}')\to[(\Sigma,\partial\Sigma),(\mathcal{M},\mathcal{M}')]_0\to H^1(\hat{\Sigma},H^1(C,\pi_1(G)))\to 0.$$

If we consider maps without base points, there is an additional set of parameters from $\pi_0(\mathcal{M}') = \pi_1(G)/2\pi_1(G)$. Therefore the graded components of $[(\Sigma,\partial\Sigma),(\mathcal{M},\mathcal{M}')]$ are $\pi_3(G)$, $\pi_1(G)_{[2]}$, $H^1(\hat{\Sigma},H^1(C,\pi_1(G)))$, $\pi_1(G)/2\pi_1(G)$. This agrees with the gauge-theoretic calculation when $h=1$ [49, §4.3]. Now suppose $\partial\Sigma$ consists of $h>1$ copies of the circle. Each time h increases by 1, the new circle is mapped to a connected component of $\mathcal{M}'$ labelled by an element in $\pi_0(\mathcal{M}') = \pi_1(G)/2\pi_1(G)$. The image of the circle can wind around in $\mathcal{M}'$ in ways labelled by an element in $\ker(\pi_1(\mathcal{M}')\to\pi_1(\mathcal{M})) = \pi_1(G)_{[2]}$. Finally, the cylindrical neck in Σ near the new circle can wind around in $\mathcal{M}$ in ways labelled by $\pi_1(\mathcal{M}) = H^1(C,\pi_1(G))$. Combining the above, the graded components of $[(\Sigma,\partial\Sigma),(\mathcal{M},\mathcal{M}')]$ are $\pi_3(G)$, $\pi_1(G)_{[2]}^{\oplus h}$, $H^1(C,\pi_1(G))^{\oplus(h-1)}$, $H^1(\hat{\Sigma},H^1(C,\pi_1(G)))$, $(\pi_1(G)/2\pi_1(G))^{\oplus h}$, in perfect agreement with the topological sectors in 4-dimensional gauge theory (§3.1). The matching of discrete parameters in the canonical formalism (where Σ is a cylinder) can also be accurately made [49, §4.2–3].

Note that the image of $p\colon\mathcal{M}_{\mathrm{H}}(C',G)\to\mathcal{M}_{\mathrm{H}}(C,G)$ is always in the component $m_0=0$, as the pull-back map $\pi^*\colon H^2(C',\pi_1(G))\to H^2(C,\pi_1(G))$ is zero. Since $\mathcal{M}_{\mathrm{H}}^{m_0=0}(C,G)$ is the only component that can support branes, m_0 is no longer a discrete parameter of the theory. That neither is non-zero e_0 allowed when $\partial\Sigma\neq\emptyset$ is another interesting phenomenon. For the untwisted theory whose worldsheet boundary lives on a brane, the Freed-Witten anomaly-free condition [10] involves the topology of the B-field on the support of the brane and that of the normal bundle. In our twisted theory, the anomaly-free condition is simply the triviality of the B-field on the support of the brane [49, §4.1]. This implies $e_0 = 0$ and hence the absence of e_0. The disappearance of flat gerbes is consistent with the fact that the $G_{\rm ad}$-bundle on X from the pair (u, u') always lifts to a G-bundle even along Σ because $H^2(\Sigma, \pi_1(G)) = 0$. It is also reflected in the calculation of the relative homotopy group. Although $\pi_1(G_{\rm ad})$ appears in $\pi_2(\mathcal{M}')$, it does not in $\pi_2(\mathcal{M}, \mathcal{M}')$ and hence there is no mismatch between the topological sectors of the gauge theory and of the sigma model.

Among the graded components of the 1-form symmetry $H^1(X, Z(G))$ in four dimensions, we have $H^1(\Sigma, Z(G))$, whose role was explained in §3.1, and $\pi^* H^1(C', Z(G))$, which is a subgroup of $H^1(C, Z(G))$. The latter is now a 0-form symmetry acting on the target $\mathcal{M} = \mathcal{M}_{\rm H}(C, G)$ preserving the support $\mathcal{N} = \mathcal{N}(C, G)$ of the branes. The group $H^1(C', Z(G))$ is the extension of $\pi^* H^1(C', Z(G))$ by $Z(G)_{[2]}$, which acts as deck transformations of $p\colon \mathcal{M}' \to \mathcal{N}$. As mentioned above, the connected components of $\mathcal{M}' = \mathcal{M}_{\rm H}(C', G)$ are labelled by $m_2 \in H^2(C', \pi_1(G)) = \pi_1(G)/2\pi_1(G)$. Naively, the branes are classified by (m_2, e_2) because each $e_2 \in Z(G)^\vee_{[2]}$ defines a flat line bundle $\mathcal{M}' \times_{e_2} \mathbb{C}$ over $\mathcal{N}$. However, a group element $z \in Z(G)_{[2]}$ changes m_2 to $m_2 + \beta_2(z)$, where $\beta_2\colon Z(G)_{[2]} \to \pi_1(G)/2\pi_1(G)$ is the map induced by the exact sequence $0 \to \pi_1(G) \to \pi_1(G_{\rm ad}) \to Z(G) \to 1$. In a sector with a fixed m_2, the unbroken subgroup is $\ker(\beta_2)$. Thus instead of e_2, we should take $\bar e_2 \in \ker(\beta_2)^\vee$. On the other hand, any two choices of m_2 related by a symmetry $z \in Z(G)_{[2]}$ are equivalent in the sense that the coverings are isomorphic. So we should take $\bar m_2 \in \operatorname{coker}(\beta_2)$ as the true discrete parameter. In fact, $\mathcal{N}$ is a disjoint union of connected components $\mathcal{N}^{\bar m_2}$ labelled by $\bar m_2$ [49, §A.10], and for each $\bar e_2$, there is a flat line bundle $L^{\bar e_2}$ over $\mathcal{N}^{\bar m_2}$. We thus find branes $\mathcal{B}^{\bar m_2, \bar e_2}$ labelled by $(\bar m_2, \bar e_2)$ [49, §4.3].

The presence of these branes supported on $\mathcal{N}(C, G)$ is compatible with the supersymmetry δ_t that is made a BRST transformation in the topological field theory. When $t = \pm\sqrt{-1}$ or equivalently when $\Psi = \infty$, the reduction to two dimensions is a B-model in the complex structure $\pm J$. For other values of t or Ψ, the reduction is an A-model (possibly after a B-field transform) with the symplectic form ω_t, which is a linear combination of ω_I and ω_K. Happily, $\mathcal{N}(C, G)$ is both a complex submanifold in J and a Lagrangian submanifold in ω_I and ω_K and hence also in ω_t. This can also be phrased as a statement in generalised geometry. Since the involution ι preserves J but reverses ω_I, ω_K, the tangent bundle of $\mathcal{N}(C, G) \subset \mathcal{M}_{\rm H}(C, G)^\iota$ plus its annihilator, which is $\ker(\iota_* - 1) \oplus \ker(\iota^* + 1) \subset (T \oplus T^*)\mathcal{M}_{\rm H}(C, G)$, is preserved by the generalised complex structures $\mathcal{J}_t$ in §3.1. Therefore $\mathcal{M}_{\rm H}(C, G)^\iota$, and hence $\mathcal{N}(C, G)$, is a generalised complex submanifold with respect to $\mathcal{J}_t$ for all $t \in \mathbb{C}P^1$, and hence supersymmetry remains in the presence of such branes.

Finally, the mirror symmetry between $\mathcal{M}_{\rm H}(C, G)$ and $\mathcal{M}_{\rm H}(C, {}^L G)$ from S-duality in the 4-dimensional gauge theory is enriched by the presence of branes constructed above. First, the relevant sector $(m_0, e_0) = (0, 0)$ when $\partial\Sigma \neq \emptyset$ is indeed mirror to $({}^L m_0, {}^L e_0) = (0, 0)$ in the dual theory. As for the other discrete parameters $m_2 \in \pi_2(G)/2\pi_1(G)$, $e_2 \in Z(G)^\vee_{[2]}$, ${}^L m_2 \in \pi_2({}^L G)/2\pi_1({}^L G)$, $e_2 \in Z({}^L G)^\vee_{[2]}$, we indeed have the natural isomorphisms of the 2-torsion groups $Z({}^L G)^\vee_{[2]} \cong \pi_2(G)/2\pi_1(G)$ and $\pi_2({}^L G)/2\pi_1({}^L G) \cong Z({}^L G)^\vee_{[2]}$. However, the true parameters of the branes are $\bar m_2 \in \operatorname{coker}(\beta_2)$ and $\bar e_2 \in \ker(\beta_2)^\vee$. Fortunately, under the above isomorphisms, the map ${}^L\beta_2$ in the dual theory is the Pontryagin dual of β_2. Therefore ${}^L\bar m_2 \in \operatorname{coker}({}^L\beta_2) \cong \ker(\beta_2)^\vee$ and ${}^L\bar e_2 \in \ker({}^L\beta_2)^\vee \cong \operatorname{coker}(\beta_2)$. The mirror of the brane $\mathcal{B}^{\bar m_2, \bar e_2}$ is ${}^L\mathcal{B}^{\bar e_2, \bar m_2}$ [49, §4.4].

4. Summary and outlook

We considered the reduction of the twisted $N = 4$ gauge theory on an orientable space-time 4-manifold containing embedded non-orientable surfaces to a 2-dimensional sigma model on a worldsheet with boundary. The boundary conditions of the sigma-model are specified by branes constructed from the Hitchin moduli space of the non-orientable surface. The resulting 2-dimensional theory is anomaly-free. We matched the discrete fluxes together with the instanton number in the gauge theory with the homotopy classes of relative maps in the sigma model in a way that is manifestly covariant on the worldsheet. The agreement of topological sectors and their duality is another non-trivial test of S-duality using the known results on the topology of Hitchin moduli spaces associated to orientable and non-orientable surfaces. Conversely, S-duality provides evidence for some of the anticipated results on the homotopy groups of the moduli spaces.

The interplay between S-duality of gauge theory on our 4-manifold and the mirror symmetry of the low energy sigma model can shed new insight on the geometric Langlands programme for a non-orientable surface and its orientation double cover. The 4-manifold containing embedded non-orientable surfaces can be used in the dimensional reduction of other 4-dimensional theories.

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References

[1] P. S. Aspinwall, T. Bridgeland, A. Craw, M. R. Douglas, M. Gross, A. Kapustin, G. W. Moore, G. Segal, B. Szendrői and P. M. H. Wilson, Dirichlet branes and mirror symmetry, Clay Math. Monographs, Vol. 4, Amer. Math. Soc., Providence (2009)

[2] A. A. Belavin, A. M. Polyakov, A. S Schwartz and Yu. S. Tyupkin, Pseudoparticle solutions of the Yang-Mills equations, Phys. Lett. B 59 (1975) 85-87

[3] M. Bershadsky, A. Johansen, V Sadov and C. Vafa, Topological reduction of 4D SYM to 2D σ-models, Nucl. Phys. B 448 (1995) 166–186, `arXiv:hep-th/9501096`

[4] P. Bouwknegt, J. Evslin and V. Mathai, Topology and H-flux of T-dual manifolds, Phys. Rev. Lett. 92 (2004) 181601, 3 pp., `arXiv:hep-th/0312052`

[5] A. Bredthauer, U. Lindström, J. Persson and M. Zabzine, Generalized Kähler geometry from supersymmetric sigma models, Lett. Math. Phys. 77 (2006) 291–308, `arXiv:hep-th/0603130`

[6] T. H. Buscher, Path-integral derivation of quantum duality in nonlinear sigma-models, Phys. Lett. B 201 (1988) 466–472

[7] R. Donagi and T. Pantev, Torus fibrations, gerbes, and duality, Mem. Amer. Math. Soc. 193, no. 901 (2008) 90 pp., `arXiv:math/0306213 [math.AG]`

[8] R. Donagi and T. Pantev, Langlands duality for Hitchin systems, Invent. Math. 189 (2012) 653–735, arXiv:math/0604617 [math.AG]

[9] D. S. Freed, G. W. Moore and G. Segal, The uncertainty of fluxes, Commun. Math. Phys. 271 (2007) 247–274, arXiv:hep-th/0605198

[10] D.S. Freed and E. Witten, Anomalies in string theory with D-branes, Asian J. Math. 3 (1999) 819–851, arXiv:hep-th/9907189

[11] D. Friedan, Nonlinear models in $2+\varepsilon$ dimensions, Phys. Rev. Lett. 45 (1980) 1057–1060

[12] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, Generalized global symmetries, J. High Energy Phys. 02 (2015) 172, 62 pp., arXiv:1412.5148 [hep-th]

[13] S. J. Gates, Jr., C. M. Hull and M. Roček, Twisted multiplets and new supersymmetric non-linear-σ-models, Nucl. Phys. B 248 (1984) 157–186

[14] M. Gell-Mann and M. Lévy, The axial vector current in beta decay, Nuovo Cimento 16 (1960) 705–726

[15] P. Goddard, J. Nuyts and D. I. Olive, Gauge theories and magnetic charge, Nucl. Phys. B 125 (1977) 1–28

[16] M. Gualtieri, Generalized complex geometry, D. Phil. thesis, Oxford University (2003), arXiv:math/0401221 [math.DG]

[17] R. S. Hamilton, Three-manifolds with positive Ricci curvature, J. Diff. Geom. 17 (1982) 255–306

[18] J. A. Harvey, G. W. Moore and A. Strominger, Reducing S duality to T duality, Phys. Rev. D 52 (1995) 7161–7167, arXiv:hep-th/9501022

[19] T. Hausel and M. Thaddeus, Mirror symmetry, Langlands duality, and the Hitchin system, Invent. Math. 153 (2003) 197–229, arXiv:math/0205236 [math.AG]

[20] N. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. 55 (1987) 59–126

[21] N. Hitchin, Lectures on special Lagrangian submanifolds, in: Winter school on mirror symmetry, vector bundles and Lagrangian submanifolds (Cambridge, MA, 1999), AMS/IP Stud. Adv. Math., 23, pp. 151–182, Amer. Math. Soc., Providence, RI (2001), arXiv:math/9907034 [math.DG]

[22] N. Hitchin, Generalized Calabi-Yau manifolds, Quart. J. Math. 54 (2003) 281–308, arXiv:math/0209099 [math.DG]

[23] N.-K. Ho, G. Wilkin and S. Wu, Hitchin's equations on a nonorientable manifold, Commun. Anal. Geom. 26 (2018) 857–886, arXiv:1211.0746 [math.DG]

[24] A. Kapustin and Y. Li, Topological sigma-models with H-flux and twisted generalized complex manifolds, Adv. Theor. Math. Phys. 11 (2007) 261–290, arXiv:hep-th/0407249

[25] A. Kapustin and D. Orlov, Remarks on A-branes, mirror symmetry, and the Fukaya category, J. Geom. Phys. 48 (2003) 84–99 arXiv:hep-th/0109098

[26] A. Kapustin and E. Witten, Electric-magnetic duality and the geometric Langlands program, Commun. Num. Theor. Phys. 1 (2007) 1–236, arXiv:hep-th/0604151

[27] K. H. Look, The Yang-Mills fields and the connections of principal fibre bundles (in Chinese with English abstract), Acta Phys. Sinica 23 (1974) 249–263

[28] M. Mackaay, A note on the holonomy of connections in twisted bundles, Cah. Topol. Géom. Différ. Catég. 44 (2003) 39–62, arXiv:math/0106019 [math.DG]

[29] N. Marcus, The other topological twisting of $N = 4$ Yang-Mills, Nucl. Phys. B 452 (1995) 331–345, arXiv:hep-th/9506002

[30] C. Montonen and D. I. Olive, Magnetic monopoles as gauge particles?, Phys. Lett. B 72 (1977) 117–120

[31] V. Pustun, Topological strings in generalized complex space, Adv. Theor. Math. Phys. 11 (2007) 399–450, arXiv:hep-th/0603145

[32] N. Seiberg, Exact results on the space of vacua of four-dimensional SUSY gauge theories, Phys. Rev. D 49 (1994) 6857–6863, arXiv:hep-th/9402044

[33] N. Seiberg and E. Witten, Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory, Nucl. Phys. B 426 (1994) 19–52; Erratum: ibid. 430 (1994) 485, arXiv:hep-th/9407087

[34] A. Sen, Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space, and $SL(2, \mathbb{Z})$ invariance in string theory, Phys. Lett. B 329 (1994) 217–221, arXiv:hep-th/9402032

[35] E. H Spanier, Algebraic topology, Springer-Verlag, New York (1966)

[36] N. Steenrod, The topology of fibre bundles, Princeton Univ. Press, Princeton (1951)

[37] A. Strominger, S.-T. Yau and E. Zaslow, Mirror symmetry is T-duality, Nucl. Phys. B479 (1996) 243–259, arXiv:hep-th/9606040

[38] G. 't Hooft, A property of electric and magnetic flux in non-abelian gauge theories, Nucl. Phys. B 153 (1979) 141–160

[39] R. Utiyama, Invariant theoretical interpretation of interaction, Phys. Rev. 101 (1956) 1597–1607

[40] C. Vafa and E. Witten, A strong coupling test of S-duality, Nucl. Phys. B 431 (1994) 3–77, arXiv:hep-th/9408074

[41] E. Witten, Topological quantum field theory, Commun. Math. Phys. 117 (1988) 353–386

[42] E. Witten, Topological sigma models, Commun. Math. Phys. 118 (1988) 411–449

[43] E. Witten, Mirror manifolds and topological field theory, in: Essays on mirror manifolds, ed. S.T. Yau, International Press, Hong Kong (1992), pp. 120–158, `arXiv:hep-th/9112056`

[44] E. Witten, Monopoles and four manifolds, Math. Res. Lett. 1 (1994) 769–796, `arXiv:hep-th/9411102`

[45] E. Witten, Dynamics of quantum field theory, in: Quantum fields and strings: a course for mathematicians, Vols. II, eds. P. Deligne et al, pp. 1119–1424, Amer. Math. Soc., Providence (1999)

[46] E. Witten, More on gauge theory and geometric Langlands, Adv. Math. 327 (2018) 624–707, `arXiv:1506.04293 [hep-th]`

[47] S. Wu, S-duality in Vafa-Witten theory for non-simply laced gauge groups, J. High Energy Phys. 05 (2008) 009, 17 pp., `arXiv:0802.2047 [hep-th]`

[48] S. Wu, Twisted character varieties, covering spaces, and gerbes, Theor. Math. Phys. 185 (2015) 1769–1788

[49] S Wu, Non-orientable surfaces and electric-magnetic duality, J. High Energy Phys. 10 (2018) 169, 61 pp, `arXiv:1804.11343 [hep-th]`

[50] S Wu, Testing S-duality with non-orientable surfaces, PoS (ICHEP2018) 505, 6 pp., `arXiv:1811.12571 [hep-th]`

[51] S Wu, Discrete fluxes, higher-form symmetries and Ward identities, to appear (2023)

[52] T. T. Wu and C. N. Yang, Concept of nonintegrable phase factors and global formulation of gauge fields, Phys. Rev. D12 (1975) 3845–3957

[53] J. P. Yamron, Topological action from twisted supersymmetric theories, Phys. Lett. B 213 (1988) 325–330

[54] C. N. Yang and R. L. Mills, Conservation of isotopic spin and isotopic gauge invariance, Phys. Rev. 96 (1954) 191–195

[55] M. Zabzine, Geometry of D-branes for general $N = (2, 2)$ sigma models, Lett. Math. Phys. 70 (2004) 211–221, `arXiv:hep-th/0405240`