PROCEEDINGS OF SCIENCE

Poisson gauge theory: a review

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In this paper we overview the Poisson gauge theory focusing on the most recent developments. We discuss the general construction and its symplectic-geometric interpretation. We consider explicit realisations of the formalism for all non-commutativities of the Lie algebraic type. We discuss Seiberg-Witten maps between Poisson gauge field-theoretical models.

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1. From non-commutative geometry to Poisson gauge theory.

Various approaches to quantum gravity naturally yield non-commutative geometry, what motivates studies on non-commutative quantum field theory [1]. An important class of field theories on non-commutative space-times is given by non-commutative *gauge* theories, see [2] for the latest review.

Let $\mathcal{M}$ be a flat manifold which represents the space-time. The local coordinates on $\mathcal{M}$ are denoted through x^a, $a = 0, \ldots, n-1$. The non-commutative structure of the space-time is encoded in the Kontsevich star product[1] of smooth functions on $\mathcal{M}$,

$$f(x) \star g(x) = f(x) \cdot g(x) + \frac{\mathrm{i}}{2}\{f(x), g(x)\} + \ldots , \qquad \forall f, g \in C^\infty(\mathcal{M}). \tag{1}$$

In this formula the Poisson bracket is related to the Poisson bivector Θ^{ab},

$$\{f(x), g(x)\} = \Theta^{ab}(x)\ \partial_a f(x)\ \partial_b g(x) , \tag{2}$$

whilst the remaining terms, denoted through "...", contain second and higher derivatives of f an g.

According to the novel approach to non-commutative gauge theories, proposed in [4], non-commutative deformations of the $U(1)$ gauge theory must exhibit the following fundamental properties.

- The infinitesimal gauge transformations should close the non-commutative algebra,

$$[\delta_f, \delta_g] A_a(x) = \delta_{-\mathrm{i}[f(x), g(x)]_\star} A_a(x) . \tag{3}$$

- The commutative limit has to reproduce the usual abelian gauge transformations,

$$\lim_{\Theta \to 0} \delta_f A_a(x) = \partial_a f(x) . \tag{4}$$

Upon the semi-classical limit the star commutator,

$$[f(x), g(x)]_\star = f(x) \star g(x) - g(x) \star f(x) = \mathrm{i}\{f(x), g(x)\} + \ldots ,$$

tends to $\mathrm{i}\{f(x), g(x)\}$, and the full non-commutative algebra (3) reduces to the Poisson gauge algebra,

$$[\delta_f, \delta_g] A_a(x) = \delta_{\{f(x), g(x)\}} A_a(x) . \tag{5}$$

The corresponding gauge theory is called the *Poisson gauge theory* [5]. Physically this theory provides a semi-classical approximation of the non-commutative $U(1)$ gauge theory.

The rest of this paper is organised as follows. In Sec. 2 we describe the general construction of Poisson gauge models. Sec. 3 is devoted to its geometric interpretation. In Sec. 4 we present explicit formulae for non-commutativities of the Lie algebraic type. In Sec. 5 we consider the arbitrariness of the construction and its relation to Seiberg-Witten maps between Poisson gauge models. We round up with the summary and concluding remarks in Sec. 6.

[1] Of course, the star product is not the only way to treat the non-commutative geometry, for example, there is the spectral approach, for a review see [3].

2. General construction.

a. Deformed gauge transformations.

For the canonical non-commutativity, i.e. when $\partial_a \Theta^{bc} = 0$, the required deformed gauge transformations can be easily constructed,

$$\delta_f A_a(x) = \partial_a f(x) + \{A_a(x), f(x)\} \,. \tag{6}$$

Though for non-constant Poisson bivectors Θ^{ab} this simple formula is not compatible with the Poisson gauge algebra (5), it can be generalised in the following way [4, 6],

$$\delta_f A_a(x) = \gamma_a^b(x, A(x))\, \partial_b f(x) + \{A_a(x), f(x)\} \,. \tag{7}$$

The matrix $\gamma(x, p)$, entering this relation, is a solution of the *first master equation,*

$$\gamma_a^b \partial_p^a \gamma_c^d - \gamma_a^d \partial_p^a \gamma_c^b + \Theta^{ba} \partial_a \gamma_c^d - \Theta^{da} \partial_a \gamma_c^b - \gamma_c^a \partial_a \Theta^{bd} = 0, \quad \partial_p^c \equiv \frac{\partial}{\partial p_c}\,, \tag{8}$$

which tends to the identity matrix at the commutative limit,

$$\lim_{\Theta \to 0} \gamma_b^a = \delta_b^a. \tag{9}$$

One can check by direct calculation that these deformed gauge transformations, indeed, close the required non-commutative Poisson gauge algebra (5) and exhibit the correct commutative limit (4).

b. Deformed field strength.

According to the general strategy of [4], the deformed field strength has to transform in a covariant way,

$$\delta_f \mathcal{F}_{ab}(x) = \{\mathcal{F}_{ab}(x), f(x)\} \,, \tag{10}$$

and it has to reproduce the commutative limit correctly,

$$\lim_{\Theta \to 0} \mathcal{F}_{ab}(x) = \partial_a A_b(x) - \partial_b A_a(x) \,. \tag{11}$$

Such a field strength has been obtained in [5] in the following form,

$$\mathcal{F}_{ab}(x) = \rho_a^c(x, A(x))\, \rho_b^d(x, A(x)) \left(\gamma_c^l(x, A(x))\, \partial_l A_d - \gamma_d^l(x, A(x))\, \partial_l A_c + \{A_c\,, A_d\} \right) . \tag{12}$$

The matrix $\rho(x, p)$, which enters this formula, has to obey the *second master equation*,

$$\gamma_d^c \partial_p^d \rho_a^b + \rho_a^d \partial_p^b \gamma_d^c + \Theta^{cd} \partial_d \rho_a^b = 0 \,, \tag{13}$$

and it has to recover the identity matrix at the commutative limit,

$$\lim_{\Theta \to 0} \rho_b^a = \delta_b^a \,. \tag{14}$$

c. Gauge-covariant derivative.

For any field $\psi(x)$, which transforms in a covariant way,

$$\delta_f \psi(x) := \{f(x), \psi(x)\}\,, \tag{15}$$

the gauge-covariant derivative has to transform properly,

$$\delta_f\,(\mathcal{D}_a\psi(x)) = \{\mathcal{D}_a\psi(x), f(x)\}\,, \tag{16}$$

and it has to exhibit a correct commutative limit,

$$\lim_{\Theta\to 0} \mathcal{D}_a\psi(x) = \partial_a\psi(x)\,. \tag{17}$$

A suitable gauge-covariant derivative has been constructed in [5] as follows,

$$\mathcal{D}_a\psi(x) = \rho_a^i(x, A(x))\left(\gamma_i^l(x, A(x))\,\partial_l\psi(x) + \{A_a(x), \psi(x)\}\right). \tag{18}$$

d. Gauge-covariant equations of motion.

The deformed covariant derivative (18) and the deformed field strength (12) allow to write down the "natural" equations of motion (e.o.m.),

$$\mathcal{D}_a\mathcal{F}^{ab} = 0\,, \tag{19}$$

which are manifestly gauge-covariant,

$$\delta_f\left(\mathcal{D}_a\mathcal{F}^{ab}\right) = \{\mathcal{D}_a\mathcal{F}^{ab}, f\}, \tag{20}$$

and which recover the first pair of Maxwell's equations in vacuum at the commutative limit. The gauge-covariance condition (20) insures that the infinitesimal gauge transformations (7) map solutions of the e.o.m. (19) onto other solutions.

For the three-dimensional space-time, equipped with the $\mathfrak{su}(2)$ non-commutativity[2], these "natural" equations of motion are equivalent to the Euler-Lagrange equations, which come out from the gauge-invariant classical action,

$$S[A] = \int_{\mathcal{M}} \mathrm{d}x \left(-\frac{1}{4}\mathcal{F}_{ab}\mathcal{F}^{ab}\right), \tag{21}$$

see [16] for details. A more profound discussion on the gauge-covariant equations of motion can be found in [17].

3. Symplectic geometric interpretation.

The symplectic geometric interpretetion of the Poisson gauge formalism, designed in [16, 18], is based on the two pillars: the symplectic embedding, and the set of constraints in the outcoming symplectic space. Extending the Poisson structure from $\mathcal{M}$ to $T^*\mathcal{M}$,

$$\{x^a, x^b\} = \Theta^{ab}(x)\,, \qquad \{x^a, p_b\} = \gamma_b^a(x, p)\,, \qquad \{p_a, p_b\} = 0\,, \tag{22}$$

[2]Various aspects of the $\mathfrak{su}(2)$ non-commutativity have been studied in [13–15].

we construct the symplectic embedding. One can easily see that the first master equation (8) for γ is equivalent to the Jacobi identity on $T^*\mathcal{M}$. By definition, the set of constraints in this symplectic space reads,

$$\Phi_a(x,p) := p_a - A_a(x), \qquad a = 0,\dots,n-1\,. \tag{23}$$

The constituents of the Poisson gauge formalism exhibit natural expressions in terms of this symplectic geometric construction. In particular, the deformed gauge transformation (6) can be represented through a simple Poisson bracket on $T^*\mathcal{M}$,

$$\delta_f A_a(x) = \{f(x), \Phi_a(x,p)\}_{\Phi(x,p)=0}\,. \tag{24}$$

The deformed field strength (12) and the gauge-covariant derivative (18) can also be represented in a similar manner,

$$\mathcal{F}_{ab}(x) = \{\Phi'_a(x,p), \Phi'_b(x,p)\}_{\Phi'(x,p)=0}\,, \qquad \mathcal{D}_a\psi(x) = \{\psi(x), \Phi'_a(x,p)\}_{\Phi'(x,p)=0}\,, \tag{25}$$

where a new set of the constraints is defined as follows,

$$\Phi'_a(x,p) := \rho_a^b(x,p)\,\Phi_b(x,p)\,. \tag{26}$$

We remind that ρ_b^a obeys the second master equation (13) and exhibits the commutative limit (14). Since ρ is a non-degenerate matrix, the new and the old constraint surfaces coincide,

$$\Phi'_a(x,p) = 0\,, \quad \Leftrightarrow \quad \Phi_a(x,p) = 0\,. \tag{27}$$

4. Lie algebraic non-commutativities and universal solutions.

Consider a class of Poisson bivectors, which are linear in coordinates,

$$\Theta^{ab} = f_c^{ab}\,x^c\,.$$

The constants f_c^{ab} satisfy the Jacobi identity,

$$f_a^{ec}\,f_c^{bd} + f_a^{bc}\,f_c^{de} + f_a^{dc}\,f_c^{eb} = 0\,,$$

therefore these constants can be seen as the structure constants of a Lie algebra. For this purpose we say that the corresponding non-commutativities are of the Lie algebraic type.

In [18] a special solution of the first master equation (8) has been presented in terms of a single matrix-valued function, which is valid for *all* Poisson bivectors of the Lie algebraic type. Introducing the matrix[3],

$$[\hat{p}]_c^b = -\mathrm{i} f_c^{ab}\,p_a,$$

we can represent this universal solution as follows,

$$\gamma(x,p) = G(\hat{p}), \qquad G(z) := \frac{\mathrm{i}\,z}{2} + \frac{z}{2}\cot\frac{z}{2}\,. \tag{28}$$

[3] According to our notations, for any matrix B the upper index enumerates strings, whilst the lower one enumerates columns. In particular for a product of two matrices B and C we write $[B\,C]^i_j = B^i_k\,C^k_j$.

Also the second master equation (13) exhibits the universal solution for ρ, expressed as a matrix valued function of the same matrix variable $\hat{p}$. The explicit formula reads [16],

$$\rho(x,p) = F(\hat{p}), \quad F(z) = \frac{e^{\mathrm{i}z} - 1}{\mathrm{i}\,z}. \tag{29}$$

These universal solutions for γ and ρ exhibit a simple connection [16],

$$\rho^{-1} = \gamma - \mathrm{i}\,\hat{p}. \tag{30}$$

Substituting the universal solutions (28) and (29) in the formulae of Sec. 2, we can build the Poisson gauge model completely: we know the deformed gauge transformations (7), which close the Poisson gauge algebra (5), and we also know the gauge-covariant equations of motion (19).

Example.

In order to illustrate the formalism, we consider the κ-Minkowski non-commutativity, which arises in various contexts, see e.g. [7–12] and for the recent progress. The Poisson bivector, which corresponds to the (generalised) κ-Minkowski non-commutativity, is defined in the following way,

$$\Theta^{ab} = 2(\omega^a x^b - \omega^b x^a)\,, \tag{31}$$

where ω^a, $a = 0, \ldots, n-1$, stand for the deformation parameters. The corresponding structure constants read,

$$f^{ab}_c = 2(\omega^a \delta^b_c - \omega^b \delta^a_c)\,. \tag{32}$$

The universal solutions have been calculated in [16],

$$\begin{aligned} \gamma^a_b(A) &= (\omega\cdot A)\,[1+\coth(\omega\cdot A)]\,\delta^a_b + \frac{1-(\omega\cdot A)-(\omega\cdot A)\coth(\omega\cdot A)}{\omega\cdot A}\,\omega^a A_b\,, \\ \rho^a_b(A) &= \frac{e^{2(\omega\cdot A)}-1}{2(\omega\cdot A)}\,\delta^a_b + \frac{1+2(\omega\cdot A)-e^{2(\omega\cdot A)}}{2(\omega\cdot A)^2}\,\omega^a A_b\,. \end{aligned} \tag{33}$$

These expressions are different from the ones, obtained in the previous studies [19], using different techniques,

$$\begin{aligned} \tilde{\gamma}^a_b(A) &= \left[\sqrt{1+(\omega\cdot A)^2}+(\omega\cdot A)\right]\delta^a_b - \omega^a A_b\,, \\ \tilde{\rho}^a_b(A) &= \left[\sqrt{1+(\omega\cdot A)^2}+(\omega\cdot A)\right]\delta^a_b - \frac{\sqrt{1+(\omega\cdot A)^2}+(\omega\cdot A)}{\sqrt{1+(\omega\cdot A)^2}}\,\omega^a A_b\,. \end{aligned} \tag{34}$$

Therefore we have more than one Poisson gauge model, which corresponds to the Poisson bivector (31). In the next section we take a closer look at the arbitrariness of the construction.

5. Arbitrariness and Seiberg-Witten maps.

Consider a given Poisson bivector $\Theta^{ab}(x)$. Let $\gamma(x,p)$ and $\rho(x,p)$ be solutions of the master equations (8) and (13), which have the commutative limits (9) and (14) respectively. For any invertible field redefinition,

$$A \rightarrow \tilde{A}(A), \tag{35}$$

obeying the condition,

$$\lim_{\Theta\to 0} \tilde{A}(A) = A, \tag{36}$$

the quantities

$$\tilde{\gamma}^a_b(x,\tilde{A}) = \left(\gamma^a_c(x,A)\cdot\frac{\partial \tilde{A}_b}{\partial A_c}\right)\Bigg|_{A=A(\tilde{A})}, \quad \tilde{\rho}^i_a(x,\tilde{A}) = \left(\frac{\partial A_s}{\partial \tilde{A}_i}\cdot\rho^s_a(x,A)\right)\Bigg|_{A=A(\tilde{A})}, \tag{37}$$

are again the solutions of the master equations, which have correct commutative limits, see [5, 16] for details. Therefore, the matrices $\tilde{\gamma}(x,p)$ and $\tilde{\rho}(x,p)$ define one more Poisson gauge model for the same Poisson bivector $\Theta^{ab}(x)$.

The infinitesimal gauge transformation of the new model read,

$$\tilde{\delta}_f \tilde{A}_a(x) = \tilde{\gamma}^i_a(x,\tilde{A}(x))\,\partial_i f(x) + \{\tilde{A}_a(x), f(x)\}\,. \tag{38}$$

Upon the field redefinition the gauge orbits are mapped onto the gauge orbits [5, 16],

$$\tilde{A}\left(A+\delta_f A\right) = \tilde{A}(A) + \tilde{\delta}_f \tilde{A}(A)\,, \tag{39}$$

thus the invertible field redefinition (35) defines a Seiberg-Witten map between the two models.

Example.

In the κ-Minkowski case, the universal solutions (33) and the solutions (34), which have been previously obtained in [19], are connected through the relation (37) and the following Seiberg-Witten map [16],

$$\tilde{A}_a = \frac{\sinh(\omega\cdot A)}{\omega\cdot A}\,A_a \quad \Leftrightarrow \quad A_a = \frac{\operatorname{arcsinh}(\omega\cdot\tilde{A})}{\omega\cdot\tilde{A}}\,\tilde{A}_a\,.$$

6. Summary and concluding remarks.

The main points are the following.

- For a given Poisson bivector Θ^{ab}, defining the non-commutativity, the main constituents of the Poisson gauge formalism, viz the deformed gauge transformations, the deformed field strength and the deformed gauge-covariant derivative, are completely determined by the matrices γ and ρ, which solve the two master equations.
- The Poisson gauge theory exhibits an elegant symplectic-geometric description in terms of symplectic embeddings and constrains in the extended space.
- There exist universal solutions of the master equations, which are valid for *all* non-commutativities of the Lie algebraic type.
- Invertible field redefinitions give rise to new solutions of the master equations. All the outcoming Poisson gauge models are connected with each other through Seiberg-Witten maps.

Of course, our mini-review does not cover all aspects of Poisson gauge theory. In particular, an important connection with the L_∞-algebras has not been discussed. The interested reader is referred to [18] and [20]. See also [21, 22] for applications of the L_∞-structures to generalised gauge symmetries and non-commutative gravity.

In conclusion, we would like to outline a few open questions. First, one has to generalise the present results, introducing the charged matter. Second, it would be very interesting to go beyond the semi-classical approximation towards the full non-commutative gauge algebra (3). Third, one may wonder what are the space-time symmetries of Poisson gauge models. In particular, one may study the fate of the discrete symmetries. In addition to the obvious phenomenological interest[4] in the discrete symmetries breaking, the preserved symmetries such as PT allow the non standard quantum field theories that may play the important role in the dark energy physics [24, 25]. Some aspects of the PT-symmetry in the non-commutative geometric context have already been studied in [26].

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