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Group-Theoretical Classification of $\frac{1}{16}$-BPS states of D=4 Conformal Supersymmetry

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We present explicitly the reduction of supersymmetries of the positive energy unitary irreducible representations of the N-extended D=4 conformal superalgebras su(2,2/N). Further we give the classification of $\frac{1}{16}$-BPS states in general and more explicitly for $su(2,2/4)$ and $su(2,2/8)$.

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1. Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, especially in view of their applications in string theory. From these very important is the AdS/CFT correspondence, namely, the remarkable proposal of Maldacena [1], according to which the large N limit of a conformally invariant theory in d dimensions is governed by supergravity (and string theory) on $d+1$-dimensional AdS space (often called AdS_{d+1}) times a compact manifold. Actually the possible relation of field theory on AdS_{d+1} to field theory on $\mathscr{M}_d$ has been a subject of long interest, cf., e.g., [2–4], and also [5] for discussions motivated by recent developments. The proposal of [1] was elaborated in [6] and [7] where was proposed a precise correspondence between conformal field theory observables and those of supergravity.

In all cases, it was known for a long time that the classification of the UIRs of the conformal superalgebras is of great importance. For some time such classification was known only for the $D=4$ superconformal algebras $su(2,2/1)$ [8] and $su(2,2/N)$ for arbitrary N [9], (see also [10, 11]). Then, more progress was made with the classification for $D=3$ (for even N), $D=5$, and $D=6$ (for $N=1,2$) in [12] (some results being conjectural), then for the $D=6$ case (for arbitrary N) was finalized in [13]. Finally, the cases $D=9,10,11$ were treated by finding the UIRs of $osp(1/2n)$, [14].

After the list of UIRs is found the next problem to address is to find their characters since these give the spectrum which is important for the applications. This problem is solved in principle, though not all formulae are explicit, for the UIRs of $D=4$ conformal superalgebras $su(2,2/N)$ in [15]. From the mathematical point of view this question is clear only for representations with conformal dimension above the unitarity threshold viewed as irreps of the corresponding complex superalgebra $sl(4/N)$ [16–22]. But for $su(2,2/N)$ even the UIRs above the unitarity threshold are truncated for small values of spin and isospin. Furthermore, in the applications the most important role is played by the representations with “quantized" conformal dimensions at the unitarity threshold and at discrete points below.

Especially important in this context are the so-called BPS states, cf., [23–32]. which we consider in the present paper.

These investigations require deeper knowledge of the structure of the UIRs, in particular, more explicit results on the decompositions of long superfields as they descend to the unitarity threshold . Fortunately, most of the needed information is contained in [9–11, 15, 33], see also [34–39].

2. Preliminaries

2.1 Representations of D=4 conformal supersymmetry

The conformal superalgebras in $D=4$ are $\mathscr{G} = su(2,2/N)$. The even subalgebra of $\mathscr{G}$ is the algebra $\mathscr{G}_0 = su(2,2) \oplus u(1) \oplus su(N)$. We label their physically relevant representations of $\mathscr{G}$ by the signature:

$$\chi = [\, d\,;\, j_1\,,\, j_2\,;\, z\,;\, r_1\,, \ldots, r_{N-1}\,] \tag{2.1}$$

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where d is the conformal weight, j_1, j_2 are non-negative (half-)integers which are Dynkin labels of the finite-dimensional irreps of the $D = 4$ Lorentz subalgebra $so(3,1)$ of dimension $(2j_1 + 1)(2j_2 + 1)$, z represents the $u(1)$ subalgebra which is central for $\mathscr{G}_0$ (and is central for $\mathscr{G}$ itself when $N = 4$), and $r_1, \dots, r_{N-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional irreps of the internal (or R) symmetry algebra $su(N)$.

We recall the root system of the complexification $\mathscr{G}^{\boldsymbol{C}}$ of $\mathscr{G}$ (as used in [11]). The positive root system Δ^+ is comprised of α_{ij}, $1 \leq i < j \leq 4+N$. The even positive root system $\Delta^+_{\bar{0}}$ is comprised of α_{ij}, with $i, j \leq 4$ and $i, j \geq 5$; the odd positive root system $\Delta^+_{\bar{1}}$ is comprised of α_{ij}, with $i \leq 4, j \geq 5$. The generators corresponding to the latter (odd) roots will be denoted as $X^+_{i,4+k}$, where $i = 1,2,3,4$, $k = 1, \dots, N$. The simple roots are chosen as in (2.4) of [11]:

$$\gamma_1 = \alpha_{12} \ , \ \gamma_2 = \alpha_{34} \ , \ \gamma_3 = \alpha_{25} \ , \ \gamma_4 = \alpha_{4,4+N} \ , \ \ \gamma_k = \alpha_{k,k+1} \ , \ \ 5 \leq k \leq 3+N. \tag{2.2}$$

Thus, the Dynkin diagram is:

$$\underset{1}{\bigcirc} - \underset{3}{\bigotimes} - \underset{5}{\bigcirc} - \cdots - \underset{3+N}{\bigcirc} - \underset{4}{\bigotimes} - \underset{2}{\bigcirc} \tag{2.3}$$

This is a non-distinguished simple root system with two odd simple roots [41].

Remark: We recall that the group-theoretical approach to $D = 4$ conformal supersymmetry developed in [9–11] involves two related constructions - on function spaces and as Verma modules. The first realization employs the explicit construction of induced representations of $\mathscr{G}$ (and of the corresponding supergroup $G = SU(2,2/N)$) in spaces of functions (superfields) over superspace which are called elementary representations (ER). The UIRs of $\mathscr{G}$ are realized as irreducible components of ERs, and then they coincide with the usually used superfields in indexless notation. The Verma module realization is also very useful as it provides simpler and more intuitive picture for the relation between reducible ERs, for the construction of the irreps, in particular, of the UIRs. For the latter the main tool is an adaptation of the Shapovalov form [40] to the Verma modules [9, 33]. Here we shall need only the second - Verma module - construction. $\diamondsuit$

We use lowest weight Verma modules V^Λ over $\mathscr{G}^{\boldsymbol{C}}$, where the lowest weight Λ is characterized by its values on the Cartan subalgebra $\mathscr{H}$ and is in 1-to-1 correspondence with the signature χ. If a Verma module V^Λ is irreducible then it gives the lowest weight irrep L_Λ with the same weight. If a Verma module V^Λ is reducible then it contains a maximal invariant submodule I^Λ and the lowest weight irrep L_Λ with the same weight is given by factorization: $L_\Lambda = V^\Lambda / I^\Lambda$ [42]. The reducibility conditions were given by Kac [42].

There are submodules which are generated by the singular vectors related to the even simple roots $\gamma_1, \gamma_2, \gamma_5, \dots, \gamma_{N+3}$ [11]. These generate an even invariant submodule I^Λ_c present in all Verma modules that we consider and which must be factored out. Thus, instead of V^Λ we shall consider the factor-modules:

$$\tilde{V}^\Lambda = V^\Lambda / I^\Lambda_c \tag{2.4}$$

The Verma module reducibility conditions for the $4N$ odd positive roots of $\mathscr{G}^{\boldsymbol{C}}$ were derived in [10, 11] adapting the results of Kac [42]:

$$d \;=\; d^1_{Nk} - z\delta_{N4} \qquad (2.5a)$$
$$d^1_{Nk} \;\equiv\; 4 - 2k + 2j_2 + z + 2m_k - 2m/N$$

$$d \;=\; d^2_{Nk} - z\delta_{N4} \qquad (2.5b)$$
$$d^2_{Nk} \;\equiv\; 2 - 2k - 2j_2 + z + 2m_k - 2m/N$$

$$d \;=\; d^3_{Nk} + z\delta_{N4} \qquad (2.5c)$$
$$d^3_{Nk} \;\equiv\; 2 + 2k - 2N + 2j_1 - z - 2m_k + 2m/N$$

$$d \;=\; d^4_{Nk} + z\delta_{N4} \qquad (2.5d)$$
$$d^4_{Nk} \;\equiv\; 2k - 2N - 2j_1 - z - 2m_k + 2m/N$$

where in all four cases of (2.5) $k = 1, \dots, N$, $m_N \equiv 0$, and

$$m_k \equiv \sum_{i=k}^{N-1} r_i \,, \quad m \equiv \sum_{k=1}^{N-1} m_k = \sum_{k=1}^{N-1} k r_k \qquad (2.6)$$

Note that we shall use also the quantity m^* which is conjugate to m :

$$m^* \;\equiv\; \sum_{k=1}^{N-1} k r_{N-k} \;=\; \sum_{k=1}^{N-1} (N-k) r_k \,, \qquad (2.7)$$
$$m + m^* \;=\; N m_1 \,. \qquad (2.8)$$

We need the result of [9] (cf. part (i) of the Theorem there) that the following is the complete list of lowest weight (positive energy) UIRs of $su(2,2/N)$:

$$d \;\geq\; d_{\max} \;=\; \max(d^1_{N1}, d^3_{NN}) \,, \qquad (2.9a)$$
$$d \;=\; d^4_{NN} \geq d^1_{N1} \,, \quad j_1 = 0 \,, \qquad (2.9b)$$
$$d \;=\; d^2_{N1} \geq d^3_{NN} \,, \quad j_2 = 0 \,, \qquad (2.9c)$$
$$d \;=\; d^2_{N1} = d^4_{NN} \,, \quad j_1 = j_2 = 0 \,, \qquad (2.9d)$$

where $d_{\max}$ is the threshold of the continuous unitary spectrum. Note that in case (d) we have $d = m_1$, $z = 2m/N - m_1$, and that it is trivial for $N = 1$.

Next we note that if $d > d_{\max}$ the factorized Verma modules are irreducible and coincide with the UIRs L_Λ. These UIRs are called **long** in the modern literature, cf., e.g., [25, 26, 34–38]. Analogously, we shall use for the cases when $d = d_{\max}$, i.e., (2.9a), the terminology of **semi-short** UIRs, introduced in [25, 34], while the cases (2.9b,c,d) are also called **short** UIRs, cf., e.g., [25, 26, 35–39].

Next consider in more detail the UIRs at the four distinguished reducibility points determining the UIRs list above: d^1_{N1}, d^2_{N1}, d^3_{NN}, d^4_{NN}. The above reducibilities occur for the following odd roots, resp.:

$$\alpha_{3,4+N} \;=\; \gamma_2 + \gamma_4 \,, \quad \alpha_{4,4+N} \;=\; \gamma_4 \,, \quad \alpha_{15} \;=\; \gamma_1 + \gamma_3 \,, \quad \alpha_{25} \;=\; \gamma_3 \,. \qquad (2.10)$$

We note a partial ordering of these four points:

$$d^1_{N1} \;>\; d^2_{N1} \;, \qquad d^3_{NN} \;>\; d^4_{NN} \;. \tag{2.11}$$

Due to this ordering *at most two* of these four points may coincide.

First we consider the situations in which *no two* of the distinguished four points coincide. There are four such situations:

$$\begin{aligned}
&\mathbf{a}: \quad d = d_{\max} = d^1_{N1} = d^a \equiv 2+2j_2+z+2m_1-2m/N > d^3_{NN} && (2.12a)\\
&\mathbf{b}: \quad d \;=\; d^2_{N1} = d^b \equiv z-2j_2+2m_1-2m/N > d^3_{NN} \;,\;\; j_2=0 && (2.12b)\\
&\mathbf{c}: \quad d \;=\; d_{\max} \;=\; d^3_{NN} \;=\; d^c \;\equiv\; 2+2j_1-z+2m/N > d^1_{N1} && (2.12c)\\
&\mathbf{d}: \quad d \;=\; d^4_{NN} = d^d \equiv 2m/N-2j_1-z > d^1_{N1} \;,\;\; j_1=0 && (2.12d)
\end{aligned}$$

where for future use we have introduced notations d^a,d^b,d^c,d^d, the definitions including also the corresponding inequality.

We shall call these cases **single-reducibility-condition (SRC)** Verma modules or UIRs, depending on the context. In addition, as already stated, we use for the cases when $d=d_{\max}$, i.e., (2.12a,c), the terminology of semi-short UIRs, while the cases (2.12b,d), are also called short UIRs.

The factorized Verma modules $\tilde V^\Lambda$ with the unitary signatures from (2.12) have only one invariant odd submodule which has to be factorized in order to obtain the UIRs. These odd embeddings and factorizations are given as follows:

$$\tilde V^\Lambda \;\to\; \tilde V^{\Lambda+\beta} \;, \qquad L_\Lambda \;=\; \tilde V^\Lambda / I^\beta \;, \tag{2.13}$$

where we use the convention [10] that arrows point to the oddly embedded module, and we give only the cases for β that we shall use later:

$$\begin{aligned}
\beta \;&=\; \alpha_{3,4+N} \;, \quad \text{for } (2.12a), \quad j_2>0, && (2.14a)\\
&=\; \alpha_{3,4+N}+\alpha_{4,4+N} \;, \quad \text{for } (2.12a), \quad j_2=0, && (2.14b)\\
&=\; \alpha_{15} \;, \quad \text{for } (2.12c), \quad j_1>0, && (2.14c)\\
&=\; \alpha_{15}+\alpha_{25} \;, \quad \text{for } (2.12c), \quad j_1=0 && (2.14d)
\end{aligned}$$

We consider now the four situations in which *two* distinguished points coincide:

$$\begin{aligned}
&\mathbf{ac}: \quad d \;=\; d_{\max} \;=\; d^{ac} \;\equiv\; 2+j_1+j_2+m_1 = d^1_{N1} = d^3_{NN} && (2.15a)\\
&\qquad z = j_1-j_2-m_1+2m/N\\
&\mathbf{ad}: \quad d \;=\; d^{ad} \;\equiv\; =\; 1+j_2+m_1 \;=\; d^1_{N1} = d^4_{NN} \;,\;\; j_1=0 && (2.15b)\\
&\qquad z = 2m/N-1-j_2-m_1\\
&\mathbf{bc}: \quad d \;=\; d^{bc} \;\equiv\; =\; 1+j_1+m_1 \;=\; d^2_{N1} = d^3_{NN} \;,\;\; j_2=0 && (2.15c)\\
&\qquad z = 2m/N+1+j_1-m_1\\
&\mathbf{bd}: \quad d \;=\; d^{bd} \;\equiv\; =\; m_1 \;=\; d^2_{N1} = d^4_{NN} \;,\;\; j_1=j_2=0 && (2.15d)\\
&\qquad 2m/N-m_1
\end{aligned}$$

We shall call these **double-reducibility-condition (DRC)** Verma modules or UIRs. The cases in (2.15a) are semi-short UIR, while the other cases are short.

- **SRC cases:**

•**a** $d = d^a , \qquad r_1 = 0 .$

•**b** $d = d^b , \qquad r_1 \leq 2 .$

•**c** $d = d^c , \qquad r_{N-1} = 0 .$

•**d** $d = d^d , \qquad r_{N-1} \leq 2 .$

- **DRC cases:**

all non-trivial cases for $N = 1$, while for $N > 1$ the list is:

•**ac** $d = d^{ac} , \qquad r_1 r_{N-1} = 0 .$

•**ad** $d = d^{ad} , \qquad r_{N-1} \leq 2 , \quad r_1 = 0$ for $N > 2$.

•**bc** $d = d^{bc} , \qquad r_1 \leq 2 , \quad r_{N-1} = 0$ for $N > 2$.

•**bd** $d = d^{bd} , \qquad r_1, r_{N-1} \leq 2$ for $N > 2, \quad 1 \leq r_1 \leq 4$ for $N = 2$.

3. Reduction of supersymmetry for $\frac{1}{16}$-BPS states

We need to present explicitly the reduction of the supersymmetries in the irreducible UIRs. This means to give explicitly the number κ of odd generators which are eliminated from the corresponding lowest weight module, (or equivalently, the number of super-derivatives that annihilate the corresponding superfield). Then corresponding state is called $\frac{\kappa}{4N}$-BPS state. First were studied the $\frac{1}{2}$-BPS, the $\frac{1}{4}$-BPS states, $\frac{1}{8}$-BPS states, see [43] and references therein.

Here consider the $\frac{1}{16}$-BPS states, or $\kappa = N/4$.

3.1 R-symmetry scalars

We start with the simpler cases of R-symmetry scalars when $r_i = 0$ for all i, which means also that $m_1 = m = m^* = 0$. These cases are valid also for $N = 1$. More explicitly:

$$\begin{aligned}
\bullet\ \mathbf{a}\quad & d = d^a_{|m=0} = 2 + 2j_2 + z > 2 + 2j_1 - z , \quad j_1 \text{ arbitrary,} \\
& \kappa = N + (1-N)\delta_{j_2,0} , \quad \text{or casewise :} \\
& \kappa = N, \quad \text{if } j_2 > 0, \\
& \kappa = 1, \quad \text{if } j_2 = 0
\end{aligned} \tag{3.1}$$

Here, κ is the number of anti-chiral generators $X^+_{3,4+k}, k = 1, \ldots, \kappa,$ that are eliminated. We need $\kappa = N/4$, which may happen only in the last case $\kappa = 1$ for $N = 4$, i.e., for $su(2,2/4)$.

$$\begin{aligned}
\bullet\ \mathbf{b}\quad & d = d^b_{|m=0} = z > 2 + 2j_1 - z , \quad j_1 \text{ arbitrary,} \quad j_2 = 0, \\
& \kappa = 2N
\end{aligned} \tag{3.2}$$

These short UIRs may be called chiral since they lack all anti-chiral generators $X^+_{3,4+k}$, $X^+_{4,4+k}$, $k = 1,\ldots,N$. However, there do not occur $\frac{1}{16}$-BPS states.

$$\begin{aligned}
\bullet\ \mathbf{c}\quad & d \ = \ d^c_{|m=0} \ = \ 2+2j_1-z \ > \ 2+2j_2+z\ , \quad j_2 \text{ arbitrary}, \\
& \kappa \ = \ N+(1-N)\delta_{j_1,0}\ , \quad \text{or casewise}: \\
& \kappa = N, \quad j_1>0, \\
& \kappa = 1, \quad j_1 = 0
\end{aligned} \tag{3.3}$$

Here, κ is the number of chiral generators $X^+_{1,4+k}, k = 1,\ldots,\kappa,$ that are eliminated. This is similar to case a), $\frac{1}{16}$-BPS states happen only in the last case for $\kappa = 1$ and $N = 4$, i.e., for $su(2,2/4)$.

$$\begin{aligned}
\bullet\ \mathbf{d}\quad & d \ = \ d^d_{|m=0} \ = \ -z \ > \ 2+2j_2+z\ , \quad j_2 \text{ arbitrary}, \quad j_1 = 0, \\
& \kappa = 2N
\end{aligned} \tag{3.4}$$

These short UIRs may be called anti-chiral since they lack all chiral generators $X^+_{1,4+k}$, $X^+_{2,4+k}$, $k = 1,\ldots,N$. As in case b) there do not occur as $\frac{1}{16}$-BPS states.

$$\begin{aligned}
\bullet\ \mathbf{ac}\quad & d \ = \ d^{ac}_{|m=0} \ = \ 2+j_1+j_2\ , \ z \ = \ j_1-j_2\ , \\
& \kappa \ = \ 2N+(1-N)(\delta_{j_1,0}+\delta_{j_2,0}), \quad \text{or casewise}: \\
& \kappa = 2N, \text{ if } \ j_1, j_2 > 0, \\
& \kappa = N+1, \text{ if } \ j_1 > 0,\ j_2 = 0, \\
& \kappa = N+1, \text{ if } \ j_1 = 0,\ j_2 > 0, \\
& \kappa = 2, \text{ if } \ j_1 = j_2 = 0.
\end{aligned} \tag{3.5}$$

Here, κ is the number of mixed elimination: chiral generators $X^+_{1,4+k}$, $(k = 1,\ldots,N+(1-N)\delta_{j_1,0})$, and anti-chiral generators $X^+_{3,4+k}$, $(k = 1,\ldots,N+(1-N)\delta_{j_2,0})$. We need $\kappa = N/4$, which may happen only in the last case $\kappa = 2$ for $N = 8$, i.e., for $su(2,2/8)$.

$$\begin{aligned}
\bullet\ \mathbf{ad}\quad & d \ = \ d^{ad}_{|m=0} \ = \ 1+j_2 \ = \ -z\ , \quad j_1 = 0, \\
& \kappa \ = \ 3N+(1-N)\delta_{j_2,0}, \quad \text{or casewise}: \\
& \kappa = 3N, \quad j_2 > 0, \\
& \kappa = 2N+1, \quad j_2 = 0.
\end{aligned} \tag{3.6}$$

Here, κ is the number of mixed elimination: both types chiral generators $X^+_{1,4+k}$, $X^+_{2,4+k}$, $(k = 1,\ldots,N)$, and anti-chiral generators $X^+_{3,4+k}$, $(k = 1,\ldots,N+(1-N)\delta_{j_2,0})$. However, there do not occur $\frac{1}{16}$-BPS states.

$$\begin{aligned}
\bullet\ \mathbf{bc}\quad & d \ = \ d^{bc}_{|m=0} \ = \ 1+j_1 \ = \ z\,, \quad j_2=0, \\
& \kappa \ = \ 3N+(1-N)\delta_{j_1,0}, \quad \text{or casewise :} \\
& \kappa = 3N, \quad j_1>0, \\
& \kappa = 2N+1, \quad j_1=0\ .
\end{aligned} \tag{3.7}$$

Here, κ is the number of mixed elimination: chiral generators $X^+_{1,4+k}$, $(k=1,\dots,N+(1-N)\delta_{j_1,0})$ and both types anti-chiral generators $X^+_{3,4+k}$, $X^+_{2,4+k}$, $(k=1,\dots,N)$. However, there do not occur $\frac{1}{16}$-BPS states.

The case $\bullet$**bd** for *R*-symmetry scalars is trivial, since also all other quantum numbers are zero ($d=j_1=j_2=z=0$).

3.2 R-symmetry non-scalars

Here we need some additional notation. Let $N>1$ and let i_0 be an integer such that $0\leq i_0\leq N-1$, $r_i=0$ for $i\leq i_0$, and if $i_0<N-1$ then $r_{i_0+1}>0$. Let now i'_0 be an integer such that $0\leq i'_0\leq N-1$, $r_{N-i}=0$ for $i\leq i'_0$, and if $i'_0<N-1$ then $r_{N-1-i'_0}>0$.[1]

With this notation the cases of *R*-symmetry scalars occur when $i_0+i'_0=N-1$, thus, from now on we have the restriction:

$$0\leq i_0+i'_0\leq N-2 \tag{3.8}$$

Now we can make a list for the values of κ, with the same interpretation as in the previous subsection, only the last case is added here.

$$\begin{aligned}
\bullet\ \mathbf{a}\quad & d \ = \ d^a = 2+2j_2+z+2m_1-2m/N > 2+2j_1-z+2m/N\,, \\
& j_1, j_2 \text{ arbitrary}, \\
& \kappa = 1+i_0(1-\delta_{j_2,0}) \leq N-1\ .
\end{aligned} \tag{3.9}$$

Here are eliminated the anti-chiral generators $X^+_{3,4+k}$, $k\leq\kappa$. We need $\kappa=N/4$, thus, $N=4+i_0(1-\delta_{j_2,0})$. We have the following $\frac{1}{16}$-BPS cases:

$$j_2=0 \ \Rightarrow\ N=4 \tag{3.10a}$$

$$j_2\neq 0 \ \Rightarrow\ N=4+4i_0 \ \Rightarrow\ N=4s, s=1,2,..., \ i_0=s-1 \tag{3.10b}$$

$$\begin{aligned}
\bullet\ \mathbf{b}\quad & d \ = \ d^b = z+2m_1-2m/N > 2+2j_1-z+2m/N\,, \\
& j_2=0\,, \quad j_1 \text{ arbitrary}, \\
& \kappa = 2+2i_0 \leq 2N-2\ .
\end{aligned} \tag{3.11}$$

[1] Both definitions are formally valid for $N=1$ with $i_0=0$ since $r_0\equiv 0$ by convention and with $i'_0=0$ since $r_N\equiv 0$ by convention.

Here are eliminated the anti-chiral generators $X^+_{3,4+k}$, $X^+_{3,4+k}$, $k \leq 1+i_0$. We need $\kappa = N/4$, thus, we have:

$$N = 8 + 8i_0, \ \Rightarrow \ N = 8s, \ s = 1,2,..., \ i_0 = s-1 \tag{3.12}$$

- **c** $d = d^c = 2 + 2j_1 - z + 2m/N > 2 + 2j_2 + z + 2m_1 - 2m/N$,
  j_1, j_2 arbitrary,
  $$\kappa = 1 + i'_0(1 - \delta_{j_1,0}) \leq N - 1 \ . \tag{3.13}$$

Here are eliminated the chiral generators $X^+_{1,4+k}$, $k \leq \kappa$. We need $\kappa = N/4$, thus, $N = 4 + i'_0(1 - \delta_{j_1,0})$. We have the following cases:

$$j_1 = 0 \ \Rightarrow \ N = 4 \tag{3.14a}$$
$$j_1 \neq 0 \ \Rightarrow \ N = 4 + 4i'_0 \ \Rightarrow \ N = 4s, s = 1,2,..., \ i'_0 = s-1 \tag{3.14b}$$

- **d** $d = d^d = 2m/N - z > 2 + 2j_2 + z + 2m_1 - 2m/N$,
  $j_1 = 0$, j_2 arbitrary,
  $$\kappa = 2 + 2i'_0 \leq 2N - 2 \ . \tag{3.15}$$

Here are eliminated the chiral generators $X^+_{1,4+k}$, $X^+_{2,4+k}$, $k \leq 1+i'_0$. We need $\kappa = N/4$, thus, we have:

$$N = 8 + 8i'_0, \ \Rightarrow \ N = 8s, s = 1,2,..., \ i'_0 = s-1 \tag{3.16}$$

- **ac** $d = d^{ac}$, $z = j_1 - j_2 + 2m/N - m_1$, j_1, j_2 arbitrary,
  $$\kappa = 2 + i_0(1-\delta_{j_2,0}) + i'_0(1-\delta_{j_1,0}) \leq N \ . \tag{3.17}$$

Here are eliminated chiral generators $X^+_{1,4+k}$, $k \leq 1 + i'_0(1-\delta_{j_1,0})$, and anti-chiral generators $X^+_{3,4+k}$, $k \leq 1 + i_0(1-\delta_{j_2,0})$.
We have the following $\frac{1}{16}$-BPS cases:

$$j_1 = j_2 = 0, \ \Rightarrow \ N = 8 \tag{3.18a}$$
$$j_1 = 0, j_2 \neq 0, \ \Rightarrow \ N = 8 + 4i_0 \tag{3.18b}$$
$$j_1 \neq 0, j_2 = 0, \ \Rightarrow \ N = 8 + 4i'_0 \tag{3.18c}$$
$$j_1 \neq 0, j_2 \neq 0, \ \Rightarrow \ N = 8 + 4i_0 + 4i'_0 \tag{3.18d}$$

- **ad** $d = d^{ad}$, $j_1 = 0$, $z = 2m/N - m_1 - 1 - j_2$, j_2 arbitrary,
  $$\kappa = 3 + i_0(1-\delta_{j_2,0}) + 2i'_0 \leq 1 + N + i'_0 \leq 2N - 1 \ . \tag{3.19}$$

Here are eliminated chiral generators $X^+_{1,4+k}, X^+_{2,4+k}$, $k \leq 1+i'_0$, and anti-chiral generators $X^+_{3,4+k}$, $k \leq 1+i_0(1-\delta_{j_2,0})$.
We have the following $\frac{1}{16}$-BPS cases:

$$j_2 = 0, \ \Rightarrow \ N = 12+8i'_0 \tag{3.20a}$$

$$j_2 \neq 0, \ \Rightarrow \ N = 12+4i_0+8i'_0 \tag{3.20b}$$

- **bc** $d = d^{bc}$, $j_2 = 0$, $z = 2m/N - m_1 + 1 + j_1$, j_1 arbitrary,

$$\kappa = 3+2i_0+i'_0(1-\delta_{j_1,0}) \leq 1+N+i_0 \leq 2N-1 \ . \tag{3.21}$$

Here are eliminated chiral generators $X^+_{1,4+k}$, $k \leq 1+i'_0(1-\delta_{j_1,0})$, and anti-chiral generators $X^+_{3,4+k}$, $X^+_{4,4+k}$, $k \leq 1+i_0$.
We have the following $\frac{1}{16}$-BPS cases:

$$j_1 = 0, \ \Rightarrow \ N = 12+8i_0 \tag{3.22a}$$

$$j_1 \neq 0, \ \Rightarrow \ N = 12+8i_0+4i'_0 \tag{3.22b}$$

- **bd** $d = d^{bd} = m_1$, $j_1 = j_2 = 0$, $z = 2m/N - m_1$,

$$\kappa = 4+2i_0+2i'_0 \leq 2N \ . \tag{3.23}$$

Here are eliminated chiral generators $X^+_{1,4+k}$, $X^+_{2,4+k}$, $k \leq 1+i'_0$, and anti-chiral generators $X^+_{3,4+k}$, $X^+_{3,4+k}$, $k \leq 1+i_0$.
We have the following $\frac{1}{16}$-BPS cases:

$$N = 16+8i_0+8i'_0 \tag{3.24}$$

In the next Section we shall use the above classification in order to resent explicitly the more interesting $\frac{1}{16}$-BPS cases.

4. Explicit presentation of $\frac{1}{16}$-BPS states

As we saw in the previous section we need to have $N \leq 4$ in order to have $\frac{1}{16}$-BPS states. Thus, below se start with $N = 4$.

4.1 SU(2,2/4)

The most interesting case is when $N = 4$, then $\kappa = N/4 = 1$. Group-theoretically the case $N = 4$ is special also since the $u(1)$ subalgebra carrying the quantum number z becomes central and one can invariantly set $z = 0$, i.e., consider the case $PSU(2,2/4)$.

We give now the explicit list of these states.

•**a** $d \ = \ d^1_{41} \ = \ 2+2j_2+2m_1+z-\frac{1}{2}m \ > \ d^3_{44}$. The last inequality leads to the restriction:

$$2j_2+r_1+z > 2j_1+r_3 \ . \tag{4.1}$$

Thus, for the signature of the $\frac{1}{16}$-BPS states we have:

$$\chi \ = \ [d = 2+2m_1+z-\tfrac{1}{2}m\,;\, j_1\,,0\,;z\,;r_1\,,r_2\,,r_3], \quad j_2 = 0,\ \ z > 2j_1+r_3-r_1 \tag{4.2}$$

Here is annihilated the anti-chiral generators $X^+_{3,5}$. This case is possible also for $PSU(2,2/4)$, i.e., to set $z=0$ except for R-symmetry scalars (since that would require $0 > j_1$).

•**c** $d \ = \ d^3_{44} \ = \ 2+2j_1-z+\frac{1}{2}m \ > \ d^1_{41} \quad \Longrightarrow$

$$2j_1+r_3-z > 2j_2+r_1 \ , \tag{4.3}$$

Thus, for the signature of the $\frac{1}{16}$-BPS states we have:

$$\chi \ = \ [d = 2+2j_1-z+\tfrac{1}{2}m\,;\, j_1\,,0\,;z\,;r_1\,,r_2\,,r_3], \quad j_1 = 0, -z > 2j_2+r_1-r_3 \ , \tag{4.4}$$

Here is annihilated the chiral generator $X^+_{1,5}$. This case is possible also for $PSU(2,2/4)$, i.e., to set $z=0$ except for R-symmetry scalars (since that would require $0 > j_2$).

4.2 SU(2,2/8)

The $\frac{1}{16}$-BPS cases for $SU(2,2/8)$ are given as follows:

$$\begin{aligned} \bullet\ \mathbf{a}\ \ d \ &= \ d^a = 2+2j_2+z+2m_1-m/4 > 2+2j_1-z+m/4\ , \\ &j_1, j_2 \text{ arbitrary}, \\ &\kappa = 1+i_0(1-\delta_{j_2,0})\ . \end{aligned} \tag{4.5}$$

Here are eliminated the anti-chiral generators $X^+_{3,4+k}\,,\, k\leq\kappa$. We have $\kappa = N/4 = 2$, $i_0 = 1$, $j_2 \neq 0$. We have the following $\frac{1}{16}$-BPS cases:

$$\chi \ = \ [d = d^a\,;\, j_1\,,\, j_2 \neq 0\,; z\,; 0\,, r_2 \neq 0\,, ..., r_7] \tag{4.6}$$

$$\begin{aligned} \bullet\ \mathbf{b}\ \ d \ &= \ d^b = z+2m_1-m/4 > 2+2j_1-z+m/4\ , \\ &j_2 = 0\ ,\ i_0 = 0,\ \ j_1 \text{ arbitrary}, \\ &\kappa = N/4 = 2\ . \end{aligned} \tag{4.7}$$

Here are eliminated the anti-chiral generators $X^+_{3,5}\,,\ X^+_{4,5}$. The signature is:

$$\chi \ = \ [d = d^b\,;\, j_1\,,0\,;z\,;r_1 \neq 0\,,r_2\,,...,r_7] \tag{4.8}$$

$$\bullet\ \mathbf{c}\quad d \ = \ d^c = 2+2j_1 - z + m/4 > 2+2j_2+z+2m_1 - m/4 \ ,$$
$$j_1, j_2 \text{ arbitrary},$$
$$\kappa = 1 + i'_0(1-\delta_{j_1,0}) \ . \tag{4.9}$$

Here are eliminated the chiral generators $X^+_{1,4+k}$, $k \leq \kappa$. We have $\kappa = N/4 = 2$, $i'_0 = 1$, $j_1 \neq 0$. We have the following $\frac{1}{16}$-BPS cases:

$$\chi \ = \ [d = d^c\, ;\, j_1 \neq 0\, ,\, j_2\, ;\, z;\, r_1\, ,...,r_6 \neq 0, r_7 = 0] \tag{4.10}$$

$$\bullet\ \mathbf{d}\quad d \ = \ d^d = m/4 - z > 2+2j_2+z+2m_1 - m/4 \ ,$$
$$j_1 = 0,\ i'_0 = 0,\ j_2 \text{ arbitrary},$$
$$\kappa = N/4 = 2 \ . \tag{4.11}$$

Here are eliminated the chiral generators $X^+_{1,5}$, $X^+_{2,5}$. The signature is:

$$\chi \ = \ [d = d^d\, ;\, 0\, ,\, j_2\, ;\, z;\, r_1\, ,...,r_7 \neq 0] \tag{4.12}$$

$$\bullet\ \mathbf{ac}\quad d \ = \ d^{ac} = 2 + j_1 + j_2 + m_1\ , \quad z \ = \ j_1 - j_2 + m/4 - m_1\ ,$$
$$\kappa = 2 + i_0(1-\delta_{j_2,0}) + i'_0(1-\delta_{j_1,0}) \leq N \ . \tag{4.13}$$

Here are eliminated chiral generators $X^+_{1,4+k}$, $k \leq 1 + i'_0(1-\delta_{j_1,0})$, and anti-chiral generators $X^+_{3,4+k}$, $k \leq 1 + i_0(1-\delta_{j_2,0})$.
We have the following $\frac{1}{16}$-BPS cases:

$$j_1 = j_2 = 0, \tag{4.14a}$$
$$j_1 = 0,\ i_0 = 0 \tag{4.14b}$$
$$j_2 = 0,\ i'_0 = 0 \tag{4.14c}$$
$$i_0 = 0,\ i'_0 = 0 \tag{4.14d}$$

In all cases from (4.13) follows that $\kappa = 2$. The signatures are correspondingly as follows

$$\chi^{ac}_a \ = \ [d = d^{ac}\, ;\, 0\, ,\, 0\, ;\, z;\, r_1\, ,...,r_7] \tag{4.15a}$$
$$\chi^{ac}_b \ = \ [d = d^{ac}\, ;\, 0\, ,\, j_2\, ;\, z;\, r_1 \neq 0\, ,...,r_7] \tag{4.15b}$$
$$\chi^{ac}_c \ = \ [d = d^{ac}\, ;\, j_1\, ,\, 0\, ;\, z;\, r_1\, ,...,r_7 \neq 0] \tag{4.15c}$$
$$\chi^{ac}_d \ = \ [d = d^{ac}\, ;\, j_1\, ,\, j_2\, ;\, z;\, r_1 \neq 0\, ,...,r_7 \neq 0] \tag{4.15d}$$

Case χ^{ac}_a is possible for R-symmetry scalars.

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