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Computing Jet Transport Coefficients On The Lattice

Amit Kumar,[a,b] **Abhijit Majumder,**[b] **Ismail Soudi**[b] **and Johannes H. Weber**[c,*]

[a] *Department of Physics, McGill University, Montreal, QC H3A-2T8, Canada*

[b] *Department of Physics and Astronomy, Wayne State University, Detroit, MI 48201, USA*

[c] *Institut für Physik, & IRIS Adlershof & RTG2575, Humboldt-Universität zu Berlin, D-12489 Berlin, Germany*

E-mail: johannes.weber@physik.hu-berlin.de

The leading jet transport coefficients $\hat{q}$ or $\hat{e}_2$ encode transverse or longitudinal momentum broadening of a hard parton traversing a hot medium. Understanding their temperature dependence is key to appreciating the observed suppression of high-transverse momentum probes at RHIC or LHC collision energies. We present a first continuum extrapolated result of $\hat{q}$ computed on pure SU(3) lattices with non-trivial temperature dependence different from the weak-coupling expectation.

We discuss the formalism published in Refs. [1, 2] and its challenges and status in view of obtaining $\hat{e}_2$ or of unquenching the calculation. We consider a hard quark subject to a single scattering on the plasma. The transport coefficients are factorized in terms of matrix elements given as integrals of non-local gauge-covariant gluon field-strength field-strength correlators. After the analytic continuation to the deep-Euclidean region, the hard scale permits to recast these as a series of local, gauge-invariant operators. The renormalized leading-twist term in this expansion is closely related to static quantities, and is computed on pure SU(3) lattices ($N_\tau = 4,\ 6,\ 8$, and 10) for a wide range of temperatures, ranging from 200MeV < T < 1GeV. Our estimate for the unquenched result in 2 + 1-flavor QCD has very similar features.

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*Speaker

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1. Introduction

Jets, heavy quarks and quarkonia are the key observables at sufficiently hard scales that permit probing the properties of the hot nuclear medium produced in heavy-ion collisions at short distance and time scales. These hard probes accumulate modifications corresponding to the different stages of the hot medium, as the primordial fireball evolves from pre-equilibrated early-time dynamics, through a locally equilibrated plasma stage, to an extended late-time stage of an inviscid nuclear liquid. One usually attempts to capture the mercurial and complex in-medium dynamics of such probes in terms of a small number of transport coefficients. Once these are determined either from experimental observation or theoretical calculation, the kinematics of the hard probes are otherwise treated in a simplified, or even classical manner. Given the influence of the strongly-coupled late-time stage, it is clear that weak-coupling calculation cannot successfully accommodate the underlying physics entering these transport coefficients.

For a jet, the leading transport coefficient related to in-medium energy loss is the one due to transverse momentum broadening per unit path length, namely, $\hat{q} \equiv \frac{\langle k_\perp^2 \rangle_{T,L}}{L}$, which coincides with the second (transverse) moment of the collision kernel $\frac{d^4 W(k)}{d^4 k}$. Another jet transport coefficient that is generally considered as subleading, is the one due to longitudinal momentum broadening per unit length, coined $\hat{e}_2 \equiv \frac{\langle k_z^2 \rangle_{T,L}}{L}$ (or $\hat{q}_L$), wherein we have assumed a jet traveling almost along the light cone in the negative z direction. On the level of these transport coefficients, very different models can be compared to perturbative or non-perturbative calculations. These coefficients may serve as free parameters or input in phenomenological descriptions of heavy-ion collision events such as in the JETSCAPE framework [3].

2. Hard parton at leading order

Both coefficients are obtained from t-channel processes, and involve at tree-level scattering mediated by one-parton-exchange; at leading order in the weak-coupling approach, $\hat{q}$ for a hard parton in representation R is a function of the UV and IR cutoffs [4][1]

$$\begin{aligned}
\hat{q}(\mu_{\mathrm{UV}}) &= C_R \sum_{s=\pm} \Xi_s \mathcal{I}_s(\mu_{\mathrm{UV}}) \frac{g^4 T^3}{\pi^2} \,, \\
\mathcal{I}_s(\mu_{\mathrm{UV}}) &\simeq \frac{\zeta_s(3)}{2\pi} \ln\left(\frac{\mu_{\mathrm{UV}}}{\mu_{\mathrm{IR}}}\right) + \Delta\mathcal{I}_s \,, \\
\Delta\mathcal{I}_s &= \frac{\zeta_s(2) - \zeta_s(3)}{2\pi} \left[\ln\left(\frac{T}{\mu_{\mathrm{IR}}}\right) + \frac{1}{2} - \gamma_E + \ln(2)\right] - \frac{\sigma_s}{2\pi} \,, \\
\Xi_+ &= 2N_c,\ \Xi_- = 2N_f,\ \zeta_\pm(n) = \sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1}}{k^n} \,.
\end{aligned} \tag{1}$$

where the IR cutoff is taken to be the Debye mass $\mu_{\mathrm{IR}} = m_D$, and strict weak-coupling hierarchies are assumed: $\Lambda_{\mathrm{QCD}} \ll m_D \sim gT \ll T$. With each logarithm one associated power of g^2 is at the harder and another at the softer scale. Thus, as the UV cutoff μ_{UV} is sent to infinity, i.e. the parton is

[1]The constants $\sigma_\pm$ are given in Ref. [4], but are of no importance for the current discussion.

considered to be infinitely hard, the result remains finite and is effectively promoted to order $O(g^2)$, since the logarithmic divergence cancels against a running coupling $g^2(\mu_{\rm UV})$ at leading order [4, 5]:

$$\lim_{\mu_{\rm UV}\to\infty} \ln\left(\frac{\mu_{\rm UV}}{\mu_{\rm IR}}\right) g^2(\mu_{\rm UV}) g^2(\mu_{\rm IR}) = \frac{g^2(\mu_{\rm IR})}{-2\bar{b}_0} \quad \text{with} \quad \bar{b}_0 = -\frac{11N_c - 2N_f}{N_C(4\pi)^2} \,. \tag{2}$$

As such the result in the limit of a hard parton is given up to terms of order $O(g^4)$ by

$$\begin{aligned} \frac{\hat{q}(\infty)}{T^3} \equiv \lim_{\mu_{\rm UV}\to\infty} \frac{\hat{q}(\mu_{\rm UV})}{T^3} &= C_R \sum_{s=\pm} \frac{\Xi_s \zeta_s(3)}{-4\pi^3 \bar{b}_0} g^2(\mu_{\rm IR}) \\ &= C_R \sum_{s=\pm} \frac{\Xi_s \zeta_s(3)}{\Xi_s \zeta_s(2)} \frac{2N_c \frac{\pi^2}{6}}{-4\pi^3 \bar{b}_0} \left(\frac{m_D(\mu_{\rm IR})}{T}\right)^2 \,. \end{aligned} \tag{3}$$

For a SU(3) pure gauge plasma, i.e. $N_f = 0$, the result becomes particularly simple[2], and is related to the $N_f = 0$ entropy density in the Stefan-Boltzmann limit, $s_{\rm SB} = \frac{32}{45}\pi^2 T^3$, as

$$\frac{\hat{q}(\infty)}{T^3} = \frac{21}{176} C_R \left(\frac{m_D(\mu_{\rm IR})}{T}\right)^2 \frac{s_{\rm SB}}{T^3} = \frac{21}{44}\pi\, C_R\, \alpha_s(\mu_{\rm IR}) \frac{s_{\rm SB}}{T^3} \,. \tag{4}$$

This form is very suggestive: there is a coefficient of order one, a representation-dependent Casimir, C_R, a factor that accounts for medium-modified interactions in any single scattering event, $\left(\frac{m_D(\mu_{\rm IR})}{T}\right)^2$ or $\alpha_s(\mu_{\rm IR})$, and a term from the equation of state (EoS) that accounts for the density of available scattering centers, $\frac{s_{\rm SB}}{T^3}$. The counterparts of the latter two are accessible in non-perturbative lattice calculations. In the physical world, however, i.e. $N_f > 0$, N_f dependent terms break the simple relation with the EoS. Do similar results arise, if weak coupling—clearly inappropriate for media of phenomenological interest at $T \simeq gT \simeq \Lambda_{\rm QCD}$—does not apply?

3. Formalism

We use the setup of a hard parton propagating in the negative z-direction with light-cone momentum q^- and follow the formalism outlined in Refs. [1, 2]. At leading virtuality the transport coefficient for j-momentum broadening is obtained (in $A^- = 0$ gauge, assuming ergodicity, dropping subleading virtualities, promoting $\partial_j \to D_j$, and enforcing an on-shell condition) as

$$\hat{q}_j(q^-) = \sum_n \frac{e^{-\beta E_n}}{ZT_I} \int d^4k\, k_j^2 \frac{d^4W(k)}{d^4k} \tag{5}$$

$$\simeq c_{0R}\, g^2 \int \frac{dy^- d^2y_\perp d^2k_\perp}{(2\pi)^3} e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{y}_\perp - i\frac{\boldsymbol{k}_\perp^2}{q^-}y^-} \left\langle \mathrm{Tr}\left[F^{+j}(0) F_j^{+}(y^-, y_\perp)\right]\right\rangle_T \,, \tag{6}$$

where $c_{0R} = \sqrt{2}C_R$, and g^2 is the squared gauge coupling at an appropriate thermal IR scale $\mu_{\rm IR}$. Obviously, the full product must be renormalized consistently. The thermal correlator $\left\langle \mathrm{Tr}\left[F^{+j}(0) F_j^{+}(y^-, y_\perp)\right]\right\rangle_T$ is gauge covariant, but at an almost light-cone separation. The necessary infinitely extended light-cone and transverse Wilson lines [6] imply that a Euclidean definition of this quantity—and in particularly one on a hypercubic lattice—is far from straightforward.

[2]We have used the Ramanujan series truncated at the leading term for Apéry's constant, $\zeta(3) = \frac{7\pi^3}{180} \simeq 1.202\ldots$.

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For this reason we define a generalized coefficient without enforcing the on-shell condition,

$$\hat{Q}_j(q^+,q^-) \simeq c_{0R}g^2 \int \frac{d^4y d^4k}{(2\pi)^4} \frac{e^{ik\cdot y}\, 2q^-}{(q+k)^2+i\varepsilon} \left\langle \mathrm{Tr}\left[F^{+j}(0)F^{+}_{j}(y)\right]\right\rangle_T , \tag{7}$$

whose thermal discontinuity at $q^+ \simeq \pm T$ corresponds to $\hat{q}_j(q^-)$. For space-like momenta $q^+ \simeq -q^-$ there is no nearby d iscontinuity, and the ratio can be expanded as a geometric series in $\frac{k_3}{q^-}$ (neglecting terms of order k^2). After promoting $\partial_3 \to D_3$, both integrals can be performed and we obtain

$$\hat{Q}_j(q^+=-q^-,q^-) \simeq \frac{c_{0R}g^2}{q^-} \sum_{n=0}^{\infty} \left(\tfrac{\nu}{q^-}\right)^n \left\langle \mathrm{Tr}\left[F^{+j}(0)\Delta^n F^{+}_{j}(0)\right]\right\rangle_T , \quad \Delta \equiv \frac{i\sqrt{2}D_3}{\nu} , \tag{8}$$

where we have introduced a intermediate scale between the thermal IR scale and the hard scale, $\mu_{\mathrm{IR}} \lesssim \nu \ll q^-$. For a medium that is invariant under parity, odd orders vanish; and for a medium at rest there are no mixed terms à la $S^3 = F^{0j}F^3_j$.

While complex contour integration of $\frac{\hat{Q}_j(q^+,q^-)}{q^-+q^+}$ along a small circle around $q^+ \simeq -q^-$ yields $\hat{Q}_j(q^+=-q^-,q^-)$, the same integral yields for a contour deformed over the whole complex plane the contributions from the two branch cuts of $\hat{Q}_j(q^+,q^-)$, the thermal one of width[3] $\delta_T T \simeq 2\sqrt{2}T$ at $q^+ \simeq \pm T$, and another one at $q^+ \geq 0$ for a time-like parton undergoing vacuum-like forward splitting. The latter vanishes at $n=0$ (at $q^- \to \infty$) for a single radiated on-shell gluon; beyond that approximation (or beyond $n=0$), it can be tamed to some extent through vacuum subtraction, where there is no thermal discontinuity. Putting everything together we obtain

$$\frac{\hat{q}_j(q^-)}{T^3} \simeq c_{0R}g^2 \frac{T}{T_\delta} \sum_{n=0}^{\infty} \left(\frac{\nu}{q^-}\right)^{2n} \frac{1}{T^4} \left\langle \mathrm{Tr}\left[F^{+j}\Delta^{2n}F^{+}_{j}\right]\right\rangle_{(T-V)} . \tag{9}$$

The right hand side is a series of vacuum-subtracted, gauge-invariant local operators, and thus, in principle, amenable to a lattice calculation. ν could be any scale of the order of the inverse lattice spacing, i.e. the temperature $T=\frac{1}{aN_\tau}$. Thus, for $q^- \to \infty$, the continuum limit at fixed temperature can be defined (after appropriate renormalization).

For $q^- < \infty$ there are two major problems: first, sending the typical scale of the hardest modes, i.e. the cutoff a^{-1}, to infinity, implies that the geometric series in $\frac{k_3}{q^-}$ cannot be used. Second, the mixing of the temperature-dependent higher-twist operators ($n>0$) with temperature-dependent lower dimensional operators multiplied by powers of the lattice cutoff cannot be canceled by the vacuum subtraction. To date, this unsolved problem is a limitation of our formalism.

4. Numerical Results

After a Wick rotation, $x^0 \to -ix^4$, $A^{0,a} \to iA^{4,a} \Longrightarrow F^{0j,a} \to iF^a_{4j}$, which takes[4] Eq. (9) into

$$\frac{\hat{q}_j(q^-)}{T^3} \simeq \frac{c_{0R}}{4\delta_T} \sum_{n=0}^{\infty} \left(\frac{T}{q^-}\right)^{2n} [g^2\widehat{O}_{j;n}]^{(R)} \;,\; \widehat{O}_{j;n} \equiv \frac{1}{T^4} \left\langle \left[F^a_{3j}\Delta^{2n}F^a_{3j} - F^a_{4j}\Delta^{2n}F^a_{4j}\right]\right\rangle_{(T-V)} , \tag{10}$$

we evaluate the leading operator ($\widehat{O}_{j;0}$) on the lattice.

[3]The factor $\sqrt{2}$ arises due to light-cone coordinates.

[4]The factor $\frac{1}{4}$ is due to the color trace of the generators and the reversal from light-cone to cartesian coordinates.

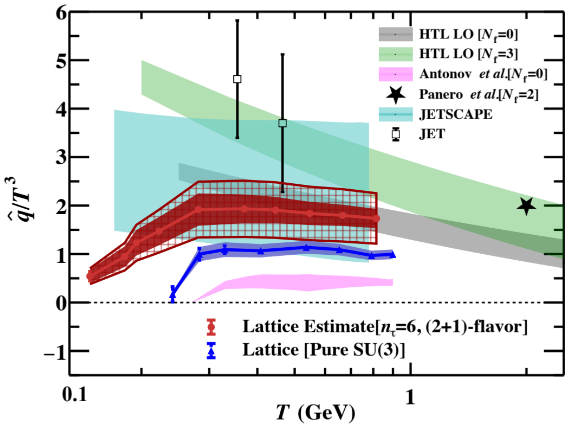

Figure 1: $\frac{\hat{q}(\infty)}{T^3}$ calculated on a lattice rises similarly to $\frac{s}{T^3}$ during the QCD crossover/transition, and exhibits a rather flat temperature dependence above $T \gtrsim 300$ MeV. Phenomenological determinations [3, 13] are quantitatively close, and a stochastic vacuum model finds a similar trend [11]. LO HTL at $q^- = 100$ GeV assuming constant $g^2(\mu_{\rm IR})$, i.e. the $O(g^4)$ contribution in Eq. (1), rises logarithmically towards lower temperatures and is compatible only at $T \gtrsim 2$ GeV, where the NNLO EQCD result [12] is similar, too.

In pure SU(3) theory, the transverse sum ($j = 1, 2$) of this operator coincides with the bare, vacuum-subtracted energy-momentum tensor (EMT) $T^{(3)}_{G,34} = \frac{1}{T^4}\sum_\mu \left\langle \left[F^a_{3\mu}F^a_{3\mu} - F^a_{4\mu}F^a_{4\mu}\right]\right\rangle_{(T-V)}$ (in temperature units) in triplet representation[5]. We use plaquette action with $N_\tau = 4,\ 6,\ 8$, and 10, aspect ratio $\frac{N_\sigma}{N_\tau} = 4$, and the field strength's clover discretization, which is a combination whose multiplicative renormalization constant $Z^{(3)}_T$ is known from finite momentum Ward identities [7, 8]. In the rest frame, $T^{(3,R)}_{G,34} = Ts$ with the entropy density s. Thus, we obtain after a few approximations

$$\frac{\hat{q}(\infty)}{T^3} \simeq \frac{c_{0R}}{4\delta_T}[g^2]^{(R)}\frac{s}{T^3} = \frac{1}{8}\,C_R\,[g^2]^{(R)}\frac{s}{T^3}\quad , \tag{11}$$

which is extremely similar to the LO result of Eq. (4). Since the entropy density is a (scheme-independent) physical observable, the scheme choice for the squared coupling introduces a scheme dependence for $\hat{q}(\infty)$. We use $N_f = 0$ $\overline{\rm MS}$ one-loop coupling at the scale $\mu_{\rm IR} = (2\ldots4)\pi T$.

In full QCD, the identification with the EMT fails, since $[\sum_j \widehat{O}_{j;0}]^{(R)}$ corresponds only to the renormalized EMT's gauge part. In light of the LO results, lack of a simple relation to the EoS is unsurprising. Bare gauge and quark parts mix upon renormalization, where not all mixing matrix elements could be fixed from Ward identities. They depend on the details of gauge and quark actions, and are unknown[6] for the combination of actions we use, namely, Lüscher-Weisz gauge action and (2+1)-flavors of highly improved staggered quarks (HISQ) (physical strange quark, and $m_\pi = 161$ MeV in the continuum limit). We use $N_\tau = 4,\ 6$, and 8, aspect ratio $\frac{N_\sigma}{N_\tau} = 4$, and again the clover discretization. For approximate renormalization we apply tadpole improvement [10] of the QCD results, and estimate the missing contributions are with two complementary arguments. Overall, this leads to a 30% systematic uncertainty; see Ref. [1] for details. We use $N_f = 3$ $\overline{\rm MS}$ one-loop coupling at the scale $\mu_{\rm IR} = (2\ldots4)\pi T$, and show our estimate (at $N_\tau = 6$) together with the $N_f = 0$ continuum result in Fig. 1.

[5]On a lattice, the continuum EMT's nonet representation breaks apart into a sextet and a triplet.
[6]It is known at one-loop order for some other actions, see [9]. We use this to estimate the size of missing contributions.

Applying the same formalism, we may obtain $\hat{e}_2$ by setting $j = 3$. The first (magnetic) term of $\widehat{O}_{3;0}$ vanishes exactly, and we have to recombine the second (electric) one to produce two scale-independent contributions in pure SU(3) theory. As a consequence, $\hat{e}_2$ also receives a contribution from the singlet representation of the vacuum-subtracted EMT that does not share the scheme dependence of $\hat{q}(\infty)$, namely,

$$\frac{\hat{e}_2(\infty)}{T^3} \simeq \frac{1}{4}\frac{\hat{q}(\infty)}{T^3} + \frac{2\pi^2}{3N_C b_0}\frac{T_G^{(1)}}{T^4} \ , \tag{12}$$

which is also accessible through lattice calculations, and subject of ongoing work.

5. Outlook and discussion

We have demonstrated that both $\hat{q}$ and $\hat{e}_2$ are in principle amenable to a lattice calculation for $q^- \to \infty$, and can be related to the EoS in pure SU(3) theory. The formal similarity to LO results, Eq. (4) and the closeness to phenomenology are astonishing. The logarithmic rise of HTL appears to be suppressed in the lattice result due to a diminishing number of scatterers at $T \lesssim 300$ MeV.

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