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Quasi Linear Theory of Pitch-angle Scattering: an Alternative Formulation

József Kóta[a,*]

[a] *The University of Arizona, Lunar and Planetary Laboratory, 1649 E University Blvd., Tucson, AZ 85721-0092, USA*

E-mail: jkota@arizona.edu

A cardinal question in the transport of cosmic rays and solar energetic particles is the description of scattering in the fluctuating magnetic fields of the heliosphere. Most of the initial descriptions assumed an isotropic scattering time. A major breakthrough in this field, the concept of quasi-linear theory (QLT) was introduced by Jokipii in the nineteen sixties. The classic QLT considered slab turbulence with a uniform background field with small fluctuations superimposed. Later efforts extended the linear perturbation model to include second order effects and non-slab fields. In this work we put forward an alternative approach: the pitch-angle of particles will be taken relative to the actual local field (opposed to relative to the guiding field). The structure of the fluctuations is described by three complex quantities corresponding to curvature, divergence, twist, and shear of the field lines. We focus on the role of shear close to perpendicular pitch angles, that has been a sore point of the QLT.

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*Speaker

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1. Introduction

Particles in stochastic magnetic fields undergo random scattering. Of particular interest is the scattering in the cosine of the pitch angle, μ, that is described by the $D_{\mu\mu}$ scattering coefficient. The connection between the spectrum of irregular field and the appropriate $D_{\mu\mu}$ was derived by the pioneering work of [1] in a quasi linear approach. The Quasi Linear Theory (QLT) assumes that the random field, δB is small relative to the regular background field B_0, and integrates the cumulative effect of the small random fields along unperturbed helical trajectories. The sore point of the theory is at $\mu = 0$, where the unperturbed trajectories stand in place. Several work has been carried out to extend the QLT. JUST NAME A FEW.

The present work outlines an alternative approach that is based on the actual pitch-angle rather than that relative to the mean guiding field, B_0. Curvilinear coordinates will be attached to the actual field lines instead of the rectangular frame attached to the mean field, B_0. This, among others, may allow the guiding field to turn without significantly changing the field strength. We shall introduce coefficients describing the curvature, divergence, twist, and shear of the field lines. Of particular attention will be paid to the effects of the shear, that is responsible for the random separation and mixing of field lines. This is a project of the author started but not completed study decades ago [6]. This is an effort to revitalize the project and provide an alternative look at the pitch-angle scattering from a different angle. Results, of course should be the same as those obtained from other approaches.

2. Triad formulation

It is convenient to use the unit vector $l_i = B_i/B$ pointing in the direction of the actual field, and compose a complex unit vector $m_i = (a_i + iB_i)/\sqrt{2}$ with a_i and b_i being unit vectors perpendicular to both l_i and each other. Then, obviously $l^2 = m\bar{m} = 1, lm = m^2 = 0$, and the Kronecker delta is $\delta_{ij} = l_i l_j + m_i \bar{m}_j + \bar{m}_i m_j$ We shall also introduce the derivations in the direction along the field and perpendicular to it as $D = l_i \partial/\partial x_i$, and $\delta = m_i \partial/\partial x_i$, respectively. Because of the curvature of the field lines, these derivatives will not, in general, be commutative. Double indices refer to summation, and bars stand for complex conjugates throughout the present work.

Next we consider how the pitch angle relative to the actual field will change. The equation of motion in a static field is

$$w\frac{d\mu}{dt} = \frac{dw_i}{dt} l_i + w_i \frac{dl_i}{dt} = w_i w_j \frac{\partial l_i}{\partial x_j} \tag{1}$$

where w_i and w refer to the the velocity of the charged particle,

Since the Lorentz-force is perpendicular to the field, it does not appear directly. The effect of the non-uniform field fluctuations will appear as inertial forces due to the change of the frame. Projecting the derivatives to the triad of l_i, m_i, $\bar{m}_m$) we obtain

$$w\frac{d\mu}{dt} = w_i w_j l_i, j = -2Re\Big(w^2 \frac{(1-\mu^2)}{2}\rho + w\mu(w_k \bar{m}_k))\,\kappa + (w_k \bar{m}_k)^2 \sigma\Big) \tag{2}$$

The change of the field appears in the derivatives of the unit vector l_i, which are given by three complex quantities:

$$\kappa = -\frac{\partial l_i}{\partial x_j} m_i\, l_j \qquad \rho = -\frac{\partial l_i}{\partial x_j} m_i\, \bar{m}_j \qquad \sigma = -\frac{\partial l_i}{\partial x_j} m_i\, m_j \tag{3}$$

Among these, the real and imaginary part of ρ represent the divergence and twist of the field lines, respectively. The divergence is directly connected with the change of field strength along field lines. Thus this term in Eq.(2) is responsible for adiabatic focusing. Twist is connected with helicity. The complex κ is associated with curvature, while σ describes the shear of field lines. Shear is the sole source of the separation and mixing of field lines, so it must have an important role in perpendicular diffusion.

Notice that the conditions above still allow the m_i vector to freely rotate as we move along the the field line. We shall fix the direction of m_i by requiring $Dm_i = -(Dl_j m_j)l_i$ around the field line considered

Since the perpendicular velocity $w_i m_i$ rotates at the gyro-frequency, ω, the curvature term in Eq.(2) which is linear in (Wm) will select the resonant wave-number, $(\omega/w\mu)$ while the quadratic shear-term will select the double wave-number $(2\omega/v\mu)$ (REF). In a slab model, the curvature yields the only first-order contribution in $(\delta B/B)$, the others are of second order.

The complex distance, $\eta_= x + iy$ between two adjacent lines varies along the field line according to

$$D\xi = \bar{\rho}\xi + \sigma\bar{\xi} \tag{4}$$

implying that shear is solely responsible for the random separation of originally close field lines; without shear ($\sigma = 0$) adjacent field lines would remain close to each other (not counting a possible uniform expansion of the field). The resulting rate of separation is in accord with [7].

3. Discussion

In this section we overview the role of the four coefficients in the transport of charged particles. In each example we show a geometry with constant coefficients and also with coefficients that change along the field line, so that their change may resonantly interact with the charged particles spiralling along the given field line. Here we assume that particle has small gyro-radius so that it remains on the same field line. The three coefficients resonate with different wave-numbers in accord with the different frequencies in the three terms.

3.1 Curvature

The classic QLT [1],[2] considered SLAB turbulence where the fluctuating field components induce pitch-angle scattering via resonant scattering with the wave of $k = \omega/w\mu$, where ω is the gyro-frequency in the guiding field. Figure 1 exhibits two cases of real κ values. The left panel shows the bare case when the curvature is constant, the right panel depicts a case of a periodically changing curvature, which can be in resonance with the gyrating particle and leading to pitch-angle scattering, while in a field configuration of constant κ the pitch-angle scattering would average out.

Since this term is slab. the cross-section of an initial circle at the bottom of the plot remains circle preserving it size.

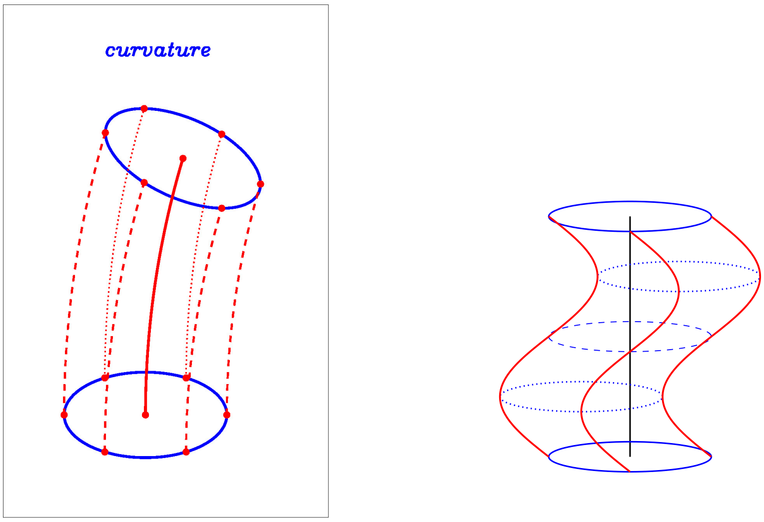

Figure 1: Figure depicting curved field lines in a slab model. The left panel shows a basic situation when κ is real and constant along the field line. The right panel shows field lines with periodically changing their curvature, which can resonate with particles of the proper gyro-frequency. A particle of cosine of pitch angle, μ will resonate with the wave-number $k = \omega/ww\mu$. Considering the circle at the bottom of the plot, it will always remain a same-size circle (being slab model).

3.2 Divergence and Twist:

The real part of ρ accounts for the divergence/convergence of the field lines which leads to adiabatic mirroring or focusing. This mode describes a symmetric mode, an initial circle will remain a circle but its size will decrease/increase as the field gets stronger/weaker.

It is important that this symmetric configuration changes the pitch angle, but preserves the adiabatic invariant $p_\perp^2/B$, without any random scattering. The original pitch angle returns if the field strength goes back to its original value(except for possible mirroring) . Figure 3 illustrates the twist of field lines. This mode does not change the pitch angle, the imaginary part of ρ cancels out in Eq.(2)

3.3 Shear:

Shear is the only mode that involves deformation. It is primarily responsible for the random separation of initially close field lines. Shear will not change the area of the initial circle but deforms it. In a linear approximation circles become elongated ellipses. Shear is in a remarkable pairing with ρ. Any compression is likely to be anisotropic. For instance, a compression in the x direction can be viewed as a combination of isotropic compression plus shear. On average compression and shear tend to have equal powers.

Eq. (2) yields the variation of μ for a particle moving along a field of line. In addition to the regular focusing/defocusing we obtan the random scattering as:

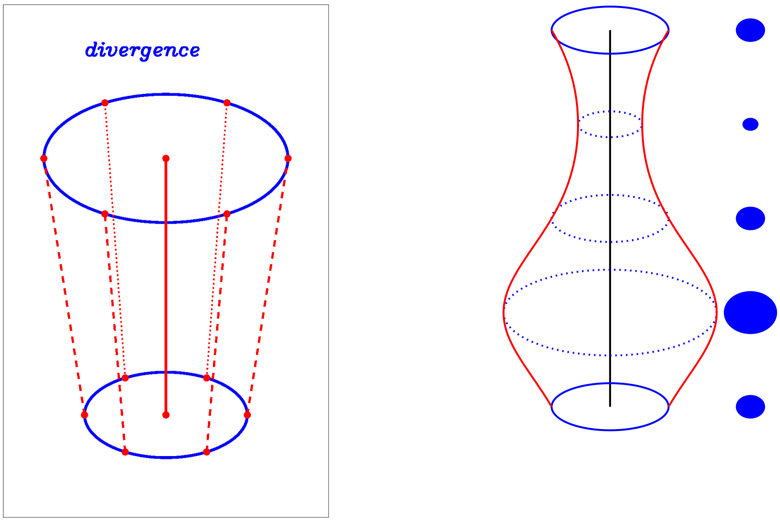

Figure 2: Basic cases of diverging/converging field lines, described by the real part of the parameter, ρ. The left panel depicts the with a constant ρ, the right panel shows a case where ρ varies sinusoidally along the field lines. The blue circles on the right indicate how the bottom circle develops: it always remains a circle, but its size varies in accord with the change of field strength. The divergence will change the pitch angle, but preserves the adiabatic invariant, so the absolute value of the original pitch angle returns if the field strength is the same (mirroring possible).

$$D_{\mu\mu} = 2\,w^2(1-\mu^2)\int_{-\infty}^{\infty}\left(\mu^2\langle\kappa(0)\kappa(\tau)\rangle e^{i\omega\tau} + (1-\mu^2)\,\langle\sigma(0)\sigma(\tau)\rangle e^{2i\omega\tau}\right)d\tau \qquad (5)$$

Eq.(4) shows, after substituting $\tau = \Delta z/w\mu$, that the resonant wave-number, k, for scattering due to curvature and shear are $k_1 = \omega/(w\mu)$ and $k_2 = 2\omega/(w\mu)$, respectively. Note that the kinematical factors are different: $\mu 2$ versus $(1-\mu 2)$, implying that scattering due to shear becomes increasingly important near $\mu = 0$.

We note that almost all types of compression are anisotropic containing both compression and deformation. For instance a transverse compression can be viewed as a combination of isotropic compression (ρ) and deformation ($sigma$). In a statistically homogeneous field these two modes (divergence and shear) have, on average, the same power.

4. Summary

This project is still in an initial stage, after a long inactivity. The main intention of this contribution is to outline the formalism, that may be of interest even if do not seek final results. Here we demonstrated that the shear of the field lines (expressed by σ) may have an important, perhaps dominating role near $\mu = 0$. The full exploration of this potential role is challenging and

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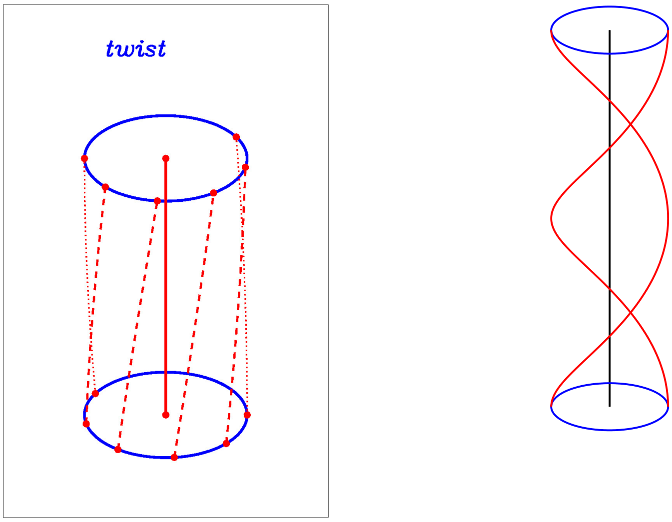

Figure 3: Two basic cases of twisted field lines, that is described by the constant or sinusoidally varying value of the imaginary part of ρ.

shall be addressed in future work. Shear is omnipresent, it is a part of any deformation in the fluid carrying the field.

References

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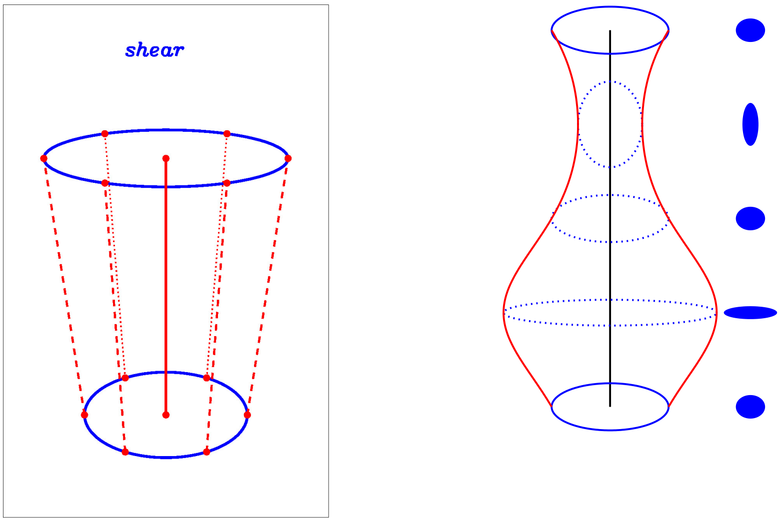

Figure 4: Basic cases of shearing field lines. Shear is the mode that does not change the area of the bottom circle but deforms it to an elliptical shape as shown in the far right of the figure. Shear is the only mode that deforms the original circle and is sole source of the separation of originally adjacent field lines.