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Proposal of a gauge-invariant treatment of $l = 0, 1$-mode perturbations on the Schwarzschild background spacetime

Kouji Nakamura[a,*]

[a] *Gravitational-Wave Science Project, National Astronomical Observatory of Japan, 2-21-1, Osawa, Mitaka, Tokyo 181-8588, Japan*

E-mail: dr.kouji.nakamura@gmail.com

A gauge-invariant perturbation theory on a generic background spacetime is developing from 2003 and "zero-mode problem" for linear metric perturbations was proposed as the essential problem of this theory. In the perturbation theory on the Schwarzschild background spacetime, $l = 0, 1$ modes correspond to the above "zero-mode" and the gauge-invariant treatments of these modes is a famous non-trivial problem in perturbation theories on the Schwarzschild background spacetime. Due to this situation, a gauge-invariant treatment for these $l = 0, 1$-mode perturbations is proposed. Through this gauge-invariant treatment, the solutions to the linearized Einstein equation for these modes with a generic matter field are derived. In the vacuum case, the linearized version of uniqueness theorem of Kerr spacetime is confirmed in a gauge-invariant manner. In this sense, our proposal is reasonable.

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*Speaker

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PoS(ICRC2023)1543

1. Introduction ——— From the direct observation of gravitational waves [1], in 2015, the gravitational-wave astronomy and multi-messenger astronomy including gravitational waves began. One of future directions of gravitational-wave astronomy is the development as a precise science by the detailed studies of source science and the tests of general-relativity. To support such precise sciences, higher-order perturbation theories in general relativity are useful.

Among future targets of gravitational-wave sources, the Extreme-Mass-Ratio-Inspiral (EMRI) is one of the targets of the Laser Interferometer Space Antenna [2]. The EMRI is a source of gravitational waves, which is the motion of a stellar mass object around a supermassive black hole, and black hole perturbation theories are used to describe this EMRI. Therefore, theoretical sophistications of black hole perturbation theories and their higher-order extensions are necessary.

Although realistic black holes have their angular momentum and we must consider the perturbation theory of a Kerr black hole for direct applications to the EMRI, further sophistication is possible even in perturbation theories on the Schwarzschild spacetime. Based on the pioneering works by Regge and Wheeler, and Zerilli [3], there have been many studies on the perturbations of the Schwarzschild spacetime. Because the Schwarzschild spacetime has the spherical symmetry, we decompose perturbations through the spherical harmonics Y_{lm} and classify them into odd- and even-modes based on their parity. However, $l = 0$ and $l = 1$ modes should be separately treated, and "*gauge-invariant*" treatments for $l = 0$ and $l = 1$ even-modes remain unknown.

In this situation, we proposed a gauge-invariant treatment of $l = 0, 1$-modes and derived the solutions to the linearized Einstein equations for these modes [4]. The obtained solutions [4] are physically reasonable. For this reason, we may say that our proposal is also reasonable. In addition, owing to our proposal, the formulation of higher-order gauge-invariant perturbation theory developed in [5–7] becomes applicable to any-order perturbations on the Schwarzschild background spacetime [8]. In this manuscript, we briefly explain these issues.

2. Brief review of general-relativistic gauge-invariant perturbation theory ——— General relativity is a theory based on general covariance, and that covariance is the reason that the notion of "gauge" has been introduced into the theory. In particular, in general relativistic perturbations, *the second-kind gauge* appears in perturbations [9]. In general-relativistic perturbation theory, we usually treat the one-parameter family of spacetimes $\{(\mathscr{M}_\lambda, Q_\lambda)|\lambda \in [0,1]\}$ to discuss differences between the background spacetime $(\mathscr{M}, Q_0) = (\mathscr{M}_{\lambda=0}, Q_{\lambda=0})$ and the physical spacetime $(\mathscr{M}_{\rm ph}, \bar{Q})$ $= (\mathscr{M}_{\lambda=1}, Q_{\lambda=1})$. Here, λ is the infinitesimal parameter for perturbations, $\mathscr{M}_\lambda$ is a spacetime manifold for each λ, and Q_λ is the collection of the tensor fields on $\mathscr{M}_\lambda$. Since each $\mathscr{M}_\lambda$ is a different manifold, we have to introduce the point identification map $\mathscr{X}_\lambda : \mathscr{M} \to \mathscr{M}_\lambda$ to compare tensor fields on different manifolds. This point-identification is *the gauge choice of the second kind*. Since we have no guiding principle by which to choose identification map $\mathscr{X}_\lambda$ due to the general covariance, we may choose a different point-identification $\mathscr{Y}_\lambda$ from $\mathscr{X}_\lambda$. This degree of freedom in the gauge choice is *the gauge degree of freedom of the second kind. The gauge-transformation of the second kind* is a change of this identification map. We note that this second-kind gauge is a different notion of the degree of freedom of coordinate choices on a single manifold, which is called *the gauge of the first kind* [9].

Once we introduce the second-kind gauge choice $\mathscr{X}_k : \mathscr{M} \to \mathscr{M}_\lambda$, we can compare the tensor fields on different manifolds $\{\mathscr{M}_\lambda\}$, and *perturbations* of a tensor field Q_λ are represented by the

PoS(ICRC2023)1543

difference $\mathscr{X}_\lambda^* Q_\lambda - Q_0$, where $\mathscr{X}_\lambda^*$ is the pull-back induced by the gauge choice $\mathscr{X}_\lambda$ and Q_0 is the background value of the variable Q_λ. This representation of perturbations completely depends on $\mathscr{X}_\lambda$. If we change the gauge choice from $\mathscr{X}_\lambda$ to $\mathscr{Y}_\lambda$, the pulled-back variable of Q_λ is represented by $\mathscr{Y}_\lambda^* Q_\lambda$. These different representations are related through the gauge-transformation rule

$$\mathscr{Y}_\lambda^* Q_\lambda = \Phi_\lambda^* \mathscr{X}_\lambda^* Q_\lambda, \quad \Phi_\lambda := \mathscr{X}_\lambda^{-1} \circ \mathscr{Y}_\lambda. \tag{1}$$

Φ_λ is a diffeomorphism on the background spacetime $\mathscr{M}$.

In the perturbative approach, we treat the perturbations of the pulled-back variable $\mathscr{X}_\lambda^* Q_\lambda$ through the Taylor series with respect to the infinitesimal parameter λ as

$$\mathscr{X}_\lambda^* Q_\lambda =: \sum_{n=0}^{k} \frac{\lambda^n}{n!} {}^{(n)}_{\mathscr{X}}Q + O(\lambda^{k+1}), \tag{2}$$

where ${}^{(n)}_{\mathscr{X}}Q$ is the representation of the kth-order perturbation of Q_λ under $\mathscr{X}_\lambda$ with ${}^{(0)}_{\mathscr{X}}Q = Q_0$.

Similarly, we can have the representation of the perturbation of Q_λ under the different gauge choice $\mathscr{Y}_\lambda$ from $\mathscr{X}_\lambda$. Since these different representations are related to the gauge-transformation rule (1), the order-by-order gauge-transformation rule between ${}^{(n)}_{\mathscr{X}}Q$ and ${}^{(n)}_{\mathscr{Y}}Q$ is given from the Taylor expansion of Eq. (1). In general, Φ_λ is given by a *knight diffeomorphism* [5]: *Let Φ_λ be a one-parameter family of diffeomorphisms, and T a tensor field such that $\Phi_\lambda^* T$ is of class C^k. Then, $\Phi_\lambda^* T$ can be expanded around $\lambda = 0$ as*

$$\Phi_\lambda^* T = \sum_{n=0}^{k} \lambda^n \sum_{\{j_i\} \in J_n} C_{n,\{j_i\}} \pounds_{\xi_{(1)}}^{j_1} \cdots \pounds_{\xi_{(n)}}^{j_n} T + O(\lambda^{k+1}). \tag{3}$$

Here, $J_n := \left\{ \{j_i\} \middle| {}^\forall i \in \mathbb{N}, j_i \in \mathbb{N}, s.t. \sum_{i=1}^{\infty} i j_i = n \right\}$ *and* $C_{n,\{j_i\}} := \prod_{i=1}^{n} \frac{1}{(i!)^{j_i} j_i!}$. *The vector fields $\xi_{(1)}$, ..., $\xi_{(k)}$ in Eq. (3) are called the generators of Φ_λ.*

Substituting Eqs. (2) and (3) into Eq. (1), we obtain the order-by-order gauge-transformation rules between ${}^{(n)}_{\mathscr{X}}Q$ and ${}^{(n)}_{\mathscr{Y}}Q$ as

$${}^{(n)}_{\mathscr{Y}}Q - {}^{(n)}_{\mathscr{X}}Q = \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j_i\} \in J_l} C_{l,\{J_i\}} \pounds_{\xi_{(1)}}^{j_1} \cdots \pounds_{\xi_{(l)}}^{j_l} {}^{(n-l)}_{\mathscr{X}}Q. \tag{4}$$

Inspecting the gauge-transformation rule (4), we first defined gauge-invariant variables for metric perturbations [5]. We consider the metric $\bar{g}_{ab}$ on $(\mathscr{M}_{\rm ph}, \bar{Q}) = (\mathscr{M}_{\lambda=1}, Q_{\lambda=1})$, and we expand the pulled-back metric $\mathscr{X}_\lambda^* \bar{g}_{ab}$ to $\mathscr{M}$ through a gauge choice $\mathscr{X}_k$ as

$$\mathscr{X}_\lambda \bar{g}_{ab} = \sum_{n=0}^{k} \frac{\lambda^n}{n!} {}^{(n)}_{\mathscr{X}}g_{ab} + O(\lambda^{k+1}), \tag{5}$$

where $g_{ab} := {}^{(0)}_{\mathscr{X}}g_{ab}$ is the metric on $\mathscr{M}$. The expansion (5) of the metric depends entirely on $\mathscr{X}_\lambda$. Nevertheless, henceforth, we do not explicitly express the index of the gauge choice $\mathscr{X}_\lambda$ if there is no possibility of confusion. In [5], we proposed a procedure to construct gauge-invariant variables for higher-order perturbations. Our starting point of this construction was the following conjecture for the linear metric perturbation $h_{ab} := {}^{(1)}g_{ab}$:

PoS(ICRC2023)1543

Conjecture 1. *If the gauge-transformation rule for a pulled-back tensor field h_{ab} from $\mathscr{M}_{\rm ph}$ to $\mathscr{M}$ is given by ${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = \pounds_{\xi_{(1)}} g_{ab}$ with the metric g_{ab} on $\mathscr{M}$, there then exist a tensor field $\mathscr{F}_{ab}$ and a vector field Y^a such that h_{ab} is given by $h_{ab} =: \mathscr{F}_{ab} + \pounds_Y g_{ab}$, where $\mathscr{F}_{ab}$ and Y^a are transformed as ${}_{\mathscr{Y}}\mathscr{F}_{ab} - {}_{\mathscr{X}}\mathscr{F}_{ab} = 0$ and ${}_{\mathscr{Y}}Y^a - {}_{\mathscr{X}}Y^a = \xi^a_{(1)}$ under the gauge transformation, respectively.*

We call $\mathscr{F}_{ab}$ and Y^a as the *gauge-invariant* and *gauge-variant* parts of h_{ab}, respectively.

Based on Conjecture 1, in [7], we found that the nth-order metric perturbation ${}^{(n)}_{\mathscr{X}}g_{ab}$ is decomposed into its gauge-invariant and gauge-variant parts as [1]

$$
{}^{(n)}g_{ab} = {}^{(n)}\mathscr{F}_{ab} - \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j_i\}\in J_l} C_{l,\{j_i\}} \pounds^{j_1}_{-{}^{(1)}Y} \cdots \pounds^{j_l}_{-{}^{(l)}Y} {}^{(n-l)}g_{ab}. \tag{6}
$$

Furthermore, through the gauge-variant variables ${}^{(i)}Y^a$ $(i = 1, ..., n)$, we also found the definition of the gauge-invariant variable ${}^{(n)}\mathscr{Q}$ for the nth-order perturbation ${}^{(n)}Q$ of an arbitrary tensor field Q. This definition of the gauge-invariant variable ${}^{(n)}\mathscr{Q}$ implies that the nth-order perturbation ${}^{(n)}Q$ of any tensor field Q is always decomposed into its gauge-invariant part and gauge-variant part as

$$
{}^{(n)}Q = {}^{(n)}\mathscr{Q} - \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j_i\}\in J_l} C_{l,\{j_i\}} \pounds^{j_1}_{-{}^{(1)}Y} \cdots \pounds^{j_l}_{-{}^{(l)}Y} {}^{(n-l)}Q. \tag{7}
$$

For example, the perturbative expansion of the Einstein tensor and the energy-momentum tensor, which are pulled back through $\mathscr{X}_\lambda$, are given by

$$
\mathscr{X}^*_\lambda \bar{G}_a{}^b = \sum_{n=0}^{k} \frac{\lambda^n}{n!} {}^{(n)}_{\mathscr{X}}G_a{}^b + O(\lambda^{k+1}), \quad \mathscr{X}^*_\lambda \bar{T}_a{}^b = \sum_{n=0}^{k} \frac{\lambda^n}{n!} {}^{(n)}_{\mathscr{X}}T_a{}^b + O(\lambda^{k+1}). \tag{8}
$$

Then, the nth-order perturbation ${}^{(n)}_{\mathscr{X}}G_a{}^b$ of the Einstein tensor and the nth-order perturbation ${}^{(n)}_{\mathscr{X}}T_a{}^b$ of the energy-momentum tensor are also decomposed as

$$
{}^{(n)}G_a{}^b = {}^{(n)}\mathscr{G}_a{}^b - \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j_i\}\in J_l} C_{l,\{j_i\}} \pounds^{j_1}_{-{}^{(1)}Y} \cdots \pounds^{j_l}_{-{}^{(l)}Y} {}^{(n-l)}G_a{}^b, \tag{9}
$$

$$
{}^{(n)}T_a{}^b = {}^{(n)}\mathscr{T}_a{}^b - \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j_i\}\in J_l} C_{l,\{j_i\}} \pounds^{j_1}_{-{}^{(1)}Y} \cdots \pounds^{j_l}_{-{}^{(l)}Y} {}^{(n-l)}T_a{}^b. \tag{10}
$$

Through the lower-order Einstein equation ${}^{(k)}_{\mathscr{X}}G_a{}^b = 8\pi {}^{(k)}_{\mathscr{X}}T_a{}^b$ [2] with $k \leq n-1$, the nth-order Einstein equation ${}^{(n)}_{\mathscr{X}}G_a{}^b = 8\pi {}^{(n)}_{\mathscr{X}}T_a{}^b$ is automatically given in the gauge-invariant form

$$
{}^{(n)}\mathscr{G}_a{}^b = {}^{(1)}\mathscr{G}_a{}^b\left[{}^{(n)}\mathscr{F}\right] + {}^{(\rm NL)}\mathscr{G}_a{}^b\left[\left\{{}^{(i)}\mathscr{F}\middle| i<n\right\}\right] = 8\pi {}^{(n)}\mathscr{T}_a{}^b, \tag{11}
$$

where ${}^{(1)}\mathscr{G}_a{}^b$ is the gauge-invariant part of the linear-order perturbation of the Einstein tensor. Explicitly, ${}^{(1)}\mathscr{G}_a{}^b[A]$ for an arbitrary tensor field A_{ab} of the second rank is given by [5]

$$
{}^{(1)}\mathscr{G}_a{}^b[A] := {}^{(1)}\Sigma_a{}^b[A] - \frac{1}{2}\delta_a{}^{b\,(1)}\Sigma_c{}^c[A], \tag{12}
$$

$$
{}^{(1)}\Sigma_a{}^b[A] := -2\nabla_{[a}H_{d]}{}^{bd}[A] - A^{cb}R_{ac}, \quad H_{ba}{}^c[A] := \nabla_{(a}A_{b)}{}^c - \frac{1}{2}\nabla^c A_{ab}. \tag{13}
$$

[1] Precisely speaking, to reach to the decomposition formula (6), we have to confirm Conjecture 4.1 in Ref. [7] in addition to Conjecture 1.

[2] We use the unit $G = c = 1$, where G is Newton's constant of gravitation, and c is the velocity of light.

PoS(ICRC2023)1543

As derived in [5], when the background Einstein tensor vanishes, we obtain the identity $\nabla_a {}^{(1)}\mathscr{G}_b{}^a[A] = 0$ for an arbitrary tensor field A_{ab} of the second rank.

We emphasize that Conjecture 1 was the important premise of the above framework of the higher-order perturbation theory.

3. Linear perturbations on spherically symmetric background ——— We use the 2+2 formulation of the perturbations on spherically symmetric spacetimes. The topological space of spherically symmetric spacetimes is $\mathscr{M} = \mathscr{M}_1 \times S^2$, and the metric on this spacetime is

$$g_{ab} = y_{ab} + r^2\gamma_{ab}, \quad y_{ab} = y_{AB}(dx^A)_a(dx^B)_b, \quad \gamma_{ab} = \gamma_{pq}(dx^p)_a(dx^q)_b, \tag{14}$$

where $x^A = (t, r)$, $x^p = (\theta, \phi)$, and γ_{pq} is a metric of the unit sphere. In the Schwarzschild spacetime, $y_{ab} = -f(dt)_a(dt)_b + f^{-1}(dr)_a(dr)_b$ with $f = 1 - 2M/r$.

On this $(\mathscr{M}, g_{ab})$, we consider the components of the metric perturbation as

$$h_{ab} = h_{AB}(dx^A)_a(dx^B)_b + 2h_{Ap}(dx^A)_{(a}(dx^p)_{b)} + h_{pq}(dx^p)_a(dx^q)_b. \tag{15}$$

In Ref. [4], we proposed the decomposition of these components as

$$h_{AB} = \sum_{l,m} \tilde{h}_{AB} S_\delta, \quad h_{Ap} = r\sum_{l,m}\left[\tilde{h}_{(e1)A}\hat{D}_p S_\delta + \tilde{h}_{(o1)A}\varepsilon_{pq}\hat{D}^q S_\delta\right], \tag{16}$$

$$h_{pq} = r^2\sum_{l,m}\left[\frac{1}{2}\gamma_{pq}\tilde{h}_{(e0)}S_\delta + \tilde{h}_{(e2)}\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right)S_\delta + 2\tilde{h}_{(o2)}\varepsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta\right], \tag{17}$$

where $\hat{D}_p$ is the covariant derivative associated with the metric γ_{pq} on S^2, $\hat{D}^p := \gamma^{pq}\hat{D}_q$, and $\varepsilon_{pq} = \varepsilon_{[pq]}$ is the totally antisymmetric tensor on S^2.

The decomposition (16)–(17) implicitly state that the Green functions of the derivative operators $\hat{\Delta} := \hat{D}^r\hat{D}_r$ and $\hat{\Delta} + 2 := \hat{D}^r\hat{D}_r + 2$ should exist if we require the one-to-one correspondence between $\{h_{Ap}, h_{pq}\}$ and $\{\tilde{h}_{(e1)A}, \tilde{h}_{(o1)A}, \tilde{h}_{(e0)}, \tilde{h}_{(e2)}, \tilde{h}_{(o2)}\}$. Because the eigenvalue of the operator $\hat{\Delta}$ on S^2 is $-l(l+1)$, the kernels of the operators $\hat{\Delta}$ and $\hat{\Delta}+2$ are $l = 0$ and $l = 1$ modes, respectively. Thus, the one-to-one correspondence between $\{h_{Ap}, h_{pq}\}$ and $\{\tilde{h}_{(e1)A}, \tilde{h}_{(o1)A}, \tilde{h}_{(e0)}, \tilde{h}_{(e2)}, \tilde{h}_{(o2)}\}$ is not guaranteed for $l = 0, 1$ modes in Eqs. (16)–(17) with $S_\delta = Y_{lm}$. To recover this one-to-one correspondence, we consider the scalar harmonics [4]

$$S_\delta = \left\{Y_{lm} \text{ for } l \geq 2; \quad k_{(\hat{\Delta}+2)m} \text{ for } l = 1; \quad k_{(\hat{\Delta})} \text{ for } l = 0\right\}. \tag{18}$$

As the explicit functions of $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)m}$, we employ

$$k_{(\hat{\Delta})} = 1 + \delta\ln\left(\frac{1-z}{1+z}\right)^{1/2}, \quad k_{(\hat{\Delta}+2)m=0} = z + \delta\left(\frac{z}{2}\ln\frac{1+z}{1-z} - 1\right), \tag{19}$$

$$k_{(\hat{\Delta}+2)m=\pm1} = (1-z^2)^{1/2}\left\{1 + \delta\left(\frac{1}{2}\ln\frac{1+z}{1-z} + \frac{z}{1-z^2}\right)\right\}e^{\pm i\phi}, \tag{20}$$

where $\delta \in \mathbb{R}$ and $z = \cos\theta$. This choice guarantees the linear-independence of the set

$$\left\{S_\delta, \hat{D}_p S_\delta, \varepsilon_{pq}\hat{D}^q S_\delta, \frac{1}{2}\gamma_{pq}S_\delta, \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right)S_\delta, 2\varepsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta\right\} \tag{21}$$

of the harmonic functions including $l = 0, 1$ modes if $\delta \neq 0$, but is singular if $\delta \neq 0$. On the other hand, when $\delta = 0$, we have $k_{(\hat{\Delta})} \propto Y_{00}$ and $\hat{k}_{(\hat{\Delta}+2)m} \propto Y_{1m}$.

Through the above harmonics functions S_δ, in Ref. [4], we proposed the following strategy:

PoS(ICRC2023)1543

Proposal 1. *We decompose the metric perturbations h_{ab} on the background spacetime with the metric (14), through Eqs. (16)–(17) with the harmonic functions S_δ given by Eq. (18). After deriving the mode-by-mode field equations such as linearized Einstein equations using S_δ, we choose $\delta = 0$ when we solve these field equations as the regularity of solutions.*

Once we accept Proposal 1, we can justify Conjecture 1 for the linear-order perturbation h_{ab} on spherically symmetric background spacetimes [4]. Then, we showed that above our formulation of a gauge-invariant perturbation theory is applicable to perturbations on the Schwarzschild spacetime including $l = 0,1$ modes, and derived the $l = 0,1$ solutions to the linearized Einstein equation [4].

From Eq. (11), the linearized Einstein equation ${}^{(1)}G_a{}^b = 8\pi{}^{(1)}T_a{}^b$ for $h_{ab} = \mathscr{F}_{ab} + \pounds_Y g_{ab}$ with the vacuum background Einstein equation $G_a{}^b = 8\pi T_a{}^b = 0$ is given by

$$ {}^{(1)}\mathscr{G}_a{}^b[\mathscr{F}] = 8\pi{}^{(1)}\mathscr{T}_a{}^b, \tag{22}$$

and the linear-order continuity equations of the energy-momentum tensor is given by

$$ \nabla^{a\,(1)}\mathscr{T}_a{}^b = 0. \tag{23}$$

We decompose the components of the linear perturbation of ${}^{(1)}\mathscr{T}_{ac}$ as

$$\begin{aligned} {}^{(1)}\mathscr{T}_{ac} =& \sum_{l,m}\tilde{T}_{AC}S_\delta(dx^A)_a(dx^C)_c + 2r\sum_{l,m}\left\{\tilde{T}_{(e1)A}\hat{D}_pS_\delta + \tilde{T}_{(o1)A}\varepsilon_{pq}\hat{D}^qS_\delta\right\}(dx^A)_{(a}(dx^p)_{c)} \\ &+r^2\sum_{l,m}\left\{\tilde{T}_{(e0)}\frac{1}{2}\gamma_{pq}S_\delta + \tilde{T}_{(e2)}\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right)S_\delta + \tilde{T}_{(o2)}\varepsilon_{s(p}\hat{D}_{q)}\hat{D}^sS_\delta\right\}(dx^p)_a(dx^q)_c. \end{aligned} \tag{24}$$

Since we impose $\delta = 0$ after deriving mode-by-mode perturbative Einstein equations, we may choose $\tilde{T}_{(e2)} = \tilde{T}_{(o2)} = 0$ for $l = 0,1$ modes, and $\tilde{T}_{(e1)A} = 0 = \tilde{T}_{(o1)A}$ for $l = 0$ modes. This choice and Eq. (23) leads $\tilde{T}_{(e0)} = 0$ for $l = 0$ mode. Then, we derived the $l = 0,1$-mode solutions to Eq. (22) [4]:

For $l = 1$ $m = 0$ odd-mode perturbations, we derived

$$ 2{}^{(1)}\mathscr{F}_{Ap}(dx^A)_{(a}(dx^p)_{b)} = \left(6Mr^2\int dr\frac{1}{r^4}a_1(t,r)\right)\sin^2\theta(dt)_{(a}(d\phi)_{b)} + \pounds_{V_{(1,o1)}}g_{ab}, \tag{25}$$

$$ V_{(1,o1)a} = \left(\beta_1(t) + W_{(1,o)}(t,r)\right)r^2\sin^2\theta(d\phi)_a. \tag{26}$$

Here, $\beta_1(t)$ is an arbitrary function of t. The function $a_1(t,r)$ is the solution to Eq. (22) given by

$$ a_1(t,r) = -\frac{16\pi}{3M}r^3f\int dt\tilde{T}_{(o1)r} + a_{10} = -\frac{16\pi}{3M}\int dr r^3\frac{1}{f}\tilde{T}_{(o1)t} + a_{10}, \tag{27}$$

where a_{10} is the constant of integration which corresponds to the Kerr parameter perturbation. On the other hand, $rf\partial_rW_{(1,o)}$ of the variable $W_{(1,o)}$ in Eq. (26) is determined by the evolution equation

$$ \partial_t^2(rf\partial_rW_{(1,o)}) - f\partial_r(f\partial_r(rf\partial_rW_{(1,o)}) + \frac{1}{r^2}f[3f-1](rf\partial_rW_{(1,o)}) = 16\pi f^2\tilde{T}_{(o1)r}. \tag{28}$$

For the $l = 0$ even-mode perturbation, we have

$$\begin{aligned} {}^{(1)}\mathscr{F}_{ab} =& \frac{2}{r}\left(M_1 + 4\pi\int dr\left[\frac{r^2}{f}\tilde{T}_{tt}\right]\right)\left((dt)_a(dt)_b + \frac{1}{f^2}(dr)_a(dr)_b\right) \\ &+2\left[4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right)\right](dt)_{(a}(dr)_{b)} + \pounds_{V_{(1,e0)}}g_{ab}, \end{aligned} \tag{29}$$

$$ V_{(1,e0)a} := \left(\frac{1}{4}f\Upsilon_1 + \frac{1}{4}rf\partial_r\Upsilon_1 + \gamma_1(r)\right)(dt)_a + \frac{1}{4f}r\partial_t\Upsilon_1(dr)_a, \tag{30}$$

PoS(ICRC2023)1543

where M_1 is the linear-order Schwarzschild mass parameter perturbation, $\gamma_1(r)$ is an arbitrary function of r. The variable ${}^{(1)}\tilde{F} := \partial_t \Upsilon_1$ in the generator (30) satisfies the following equation:

$$-\frac{1}{f}\partial_t^2 \tilde{F} + \partial_r(f\partial_r \tilde{F}) + \frac{1}{r^2}3(1-f)\tilde{F} = -\frac{8}{r^3}m_1(t,r) + 16\pi\left[-\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right], \tag{31}$$

where

$$m_1(t,r) = 4\pi\int dr\left[\frac{r^2}{f}\tilde{T}_{tt}\right] + M_1 = 4\pi\int dt\left[r^2 f\tilde{T}_{rt}\right] + M_1, \quad M_1 \in \mathbb{R}. \tag{32}$$

For the $l = 1$ $m = 0$ even-mode perturbation, we have

$$\begin{aligned}
{}^{(1)}\mathscr{F}_{ab} = \frac{16\pi r^2}{1-f}\Bigg\{ & -\frac{f^2}{3}\left[\frac{1+f}{2}\tilde{T}_{rr} + rf\partial_r\tilde{T}_{rr} - \tilde{T}_{(e0)} - 4\tilde{T}_{(e1)r}\right](dt)_a(dt)_b \\
& + \left[(1-f)\tilde{T}_{tr} - \frac{2r}{3f}\partial_t\tilde{T}_{tt}\right](dt)_{(a}(dr)_{b)} + \frac{1-3f}{2f^2}\left[\tilde{T}_{tt} - \frac{2rf}{3(1-3f)}\partial_r\tilde{T}_{tt}\right](dr)_a(dr)_b \\
& - \frac{r^2\tilde{T}_{tt}}{3}\gamma_{ab}\Bigg\}\cos\theta + \pounds_{V_{(1,e1)}} g_{ab},
\end{aligned} \tag{33}$$

$$V_{(1,e1)a} := -r\partial_t\Phi_{(e)}\cos\theta(dt)_a + \left(\Phi_{(e)} - r\partial_r\Phi_{(e)}\right)\cos\theta(dr)_a - r\Phi_{(e)}\sin\theta(d\theta)_a, \tag{34}$$

where $\Phi_{(e)}$ satisfies the following equation

$$\begin{aligned}
& -\frac{1}{f}\partial_t^2\Phi_{(e)} + \partial_r\left[f\partial_r\Phi_{(e)}\right] - \frac{1-f}{r^2}\Phi_{(e)} = 16\pi\frac{r}{3(1-f)}S_{(\Phi_{(e)})}, \\
& S_{(\Phi_{(e)})} := \frac{3(1-3f)}{4f}\tilde{T}_{tt} - \frac{1}{2}r\partial_r\tilde{T}_{tt} + \frac{1+f}{4}f\tilde{T}_{rr} + \frac{1}{2}f^2 r\partial_r\tilde{T}_{rr} - \frac{f}{2}\tilde{T}_{(e0)} - 2f\tilde{T}_{(e1)r}.
\end{aligned} \tag{35}$$

4. Extension to the higher-order perturbations ——— As shown in Sec. 2, the n-th order Einstein equation is given in Eq. (11), which we rewrite as

$${}^{(1)}\mathscr{G}_a{}^b\left[{}^{(n)}\mathscr{F}\right] = -{}^{(\mathrm{NL})}\mathscr{G}_a{}^b\left[\left\{{}^{(i)}\mathscr{F}_{cd}\middle| i<n\right\}\right] + 8\pi{}^{(n)}\mathscr{T}_a{}^b =: 8\pi{}^{(n)}\mathbb{T}_a{}^b. \tag{36}$$

Here, the left-hand side in Eq. (36) is the linear term of ${}^{(n)}\mathscr{F}_{ab}$ and the first term in the right-hand side is the non-linear term consists of the lower-order metric perturbation ${}^{(i)}\mathscr{F}_{ab}$ with $i < n$. The right-hand side $8\pi{}^{(n)}\mathbb{T}_a{}^b$ of Eq. (36) is regarded an effective energy-momentum tensor for the nth-order metric perturbation ${}^{(n)}\mathscr{F}_{ab}$.

The vacuum background condition $G_a{}^b = 0$ implies the identity $\nabla_a{}^{(1)}\mathscr{G}_b{}^a[A] = 0$ and Eq. (36) implies $\nabla^a{}^{(n)}\mathbb{T}_a{}^b = 0$. This equation gives consistency relations which should be confirmed. Note that ${}^{(n)}\mathbb{T}_a{}^b$ does not include ${}^{(n)}\mathscr{F}_{ab}$, since the terms $-{}^{(\mathrm{NL})}\mathscr{G}_a{}^b\left[\left\{{}^{(i)}\mathscr{F}_{cd}\middle| i<n\right\}\right]$ and ${}^{(n)}\mathscr{T}_a{}^b$ in Eq. (36) don't include ${}^{(n)}\mathscr{F}_{ab}$ due to the vacuum background condition. This situation is same as that when we solved the linear equations (22)–(23). Furthermore, we decompose ${}^{(n)}\mathbb{T}_{ab}$ as

$$\begin{aligned}
{}^{(1)}\mathbb{T}_{ab} =: & \sum_{l,m}\tilde{\mathbb{T}}_{AB}S_\delta(dx^A)_a(dx^B)_b + 2r\sum_{l,m}\left\{\tilde{\mathbb{T}}_{(e1)A}\hat{D}_p S_\delta + \tilde{\mathbb{T}}_{(o1)A}\varepsilon_{pq}\hat{D}^q S_\delta\right\}(dx^A)_{(a}(dx^p)_{b)} \\
& + r^2\sum_{l,m}\left\{\tilde{\mathbb{T}}_{(e0)}\frac{1}{2}\gamma_{pq}S_\delta + \tilde{\mathbb{T}}_{(e2)}\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right)S_\delta + \tilde{\mathbb{T}}_{(o2)}\varepsilon_{s(p}\hat{D}_{q)}\hat{D}^s S_\delta\right\}(dx^p)_a(dx^q)_b.
\end{aligned} \tag{37}$$

Then, the replacements $\tilde{T}_{AB} \to \tilde{\mathbb{T}}_{AB}$, $\tilde{T}_{(e1)A} \to \tilde{\mathbb{T}}_{(e1)A}$, $\tilde{T}_{(o1)A} \to \tilde{\mathbb{T}}_{(o1)A}$, $\tilde{T}_{(e0)} \to \tilde{\mathbb{T}}_{(e0)}$, $\tilde{T}_{(e2)} \to \tilde{\mathbb{T}}_{(e2)}$, $\tilde{T}_{(o2)} \to \tilde{\mathbb{T}}_{(o2)}$ in the solutions (25)–(35) yield the solutions to Eq. (36).

PoS(ICRC2023)1543

5. Summary ——— We proposed a gauge-invariant treatment of the $l = 0, 1$-mode perturbations on the Schwarzschild background spacetime as the Proposal 1. Following this proposal, we derived the $l = 0, 1$-mode solutions to the Einstein equations with the general linear perturbations of the energy-momentum tensor in the gauge-invariant manner.

The derived solution in the $l = 1$ odd mode actually realizes the linearized Kerr solution in the vacuum case. Furthermore, we also derived the $l = 0, 1$ even-mode solutions to the Einstein equations. In the vacuum case, in which all components of ${}^{(1)}\mathscr{T}_{ab}$ vanish, the $l = 0$ even-mode solution realizes the only the additional mass parameter perturbation of the Schwarzschild spacetime. These results are the realization of the linearized gauge-invariant version of uniqueness theorem of Kerr black hole and these solutions are physically reasonable. Owing to this realization, we may say that our proposal is also physically reasonable. Details of our discussions are given in Ref. [9].

The fact that we confirmed Conjecture 1 for the linear-metric perturbations in the Schwarzschild background case including the $l = 0, 1$ modes implies that the extension to any-order perturbations through our gauge-invariant formulation [7] was possible, at least, in the case of the Schwarzschild background case. Thus, we can develop a higher-order gauge-invariant perturbation theory on the Schwarzschild background spacetime [8].

We leave the development for specific astrophysical situations such as gravitational-wave astronomy through our formulation as future works.

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PoS(ICRC2023)1543