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Derivation of diffusion-tensor coefficients for the transport of charged particles in magnetic fields

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The transport of charged particles in various astrophysical environments permeated by magnetic fields is described in terms of a diffusion process, which relies on diffusion-tensor parameters generally inferred from Monte-Carlo simulations. Based on a red-noise approximation to model the two-point correlation function of the magnetic field experienced by charged particles between two successive times, the diffusion-tensor coefficients were previously derived in the case of pure turbulence. In this contribution, the derivation is extended to the case of a mean field on top of the turbulence. In addition, the red-noise approximation is relaxed by modeling the relevant two-point correlation function obtained by Monte-Carlo simulations for a wide range of test-particle rigidities.

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Introduction. In many astrophysical environments, the propagation and acceleration of high-energy charged particles (cosmic rays) are governed by the scattering off magnetic fields, which are described as a turbulence $\delta\mathbf{B}$ on top of a mean field $\mathbf{B}_0$. The turbulence level is generally defined as $\eta = \delta B^2/(\delta B^2 + B_0^2)$. The transport of the particles is then modelled as an anisotropic diffusion process. Under very broad conditions, the coefficients of the diffusion tensor can be related to the velocity correlation function of cosmic rays, $\langle v_{0i} v_j(t)\rangle$, through a time integration [1],

$$D_{ij}(t) = \int_0^t \mathrm{d}t' \, \langle v_{0i} v_j(t')\rangle, \tag{1}$$

in the limit that $t \to \infty$. Here, $v_{0i} \equiv v_i(t=0)$[1] and $\langle\cdot\rangle$ stands for the average quantities, taken over several space and time correlation scales of the turbulent field. Many estimates of these coefficients have been made from numerical simulations exploring wide ranges of particle rigidities and turbulence levels [e.g. 2–10]. Formal estimates, on the other hand, have been presented in [11] in the high rigidity regime, and in [12] in the gyro-resonant regime limited to pure turbulence. In this contribution, we attempt to extend these latter estimates for any turbulence level η. Without loss of generalities, the study is limited to the example of an isotropic 3D turbulence following a Kolmogorov power spectrum without helicity. The setup for the mean field is such that $\mathbf{B}_0 = B_0\mathbf{u}_z$.

We are interested in determining the moments of $v_i(t)$ to derive a workable expression for Eqn. 1. The velocity of each test-particle is governed by the Lorentz-Newton equation of motion,

$$\dot{v}_i(t) = \delta\Omega(t)\ \epsilon_{ijk} v_j(t)\delta b_k(t) + \Omega_0\ \epsilon_{ijk} v_j(t) b_{0k}(t). \tag{2}$$

Here, $\delta\Omega(t) = c^2 Z|e|\delta B(t)/E$ is the gyrofrequency related to the turbulence, Ω_0 that related to the mean field, $Z|e|$ and E the electric charge and the energy of the particle, and $\delta b_k(t) \equiv \delta b_k(\mathbf{x}(t))$ the k-th component of the turbulence (expressed in units of δB) at the spatial coordinate $\mathbf{x}(t)$, which corresponds to the position of the test-particle at time t. A formal solution for $\langle v_i(t)\rangle$ can be obtained by expressing the solution of Eqn. 2 as an infinite number of Dyson series, each combining terms in powers of $\delta\mathbf{b}$ coupled to terms in powers of $\mathbf{B}_0$. Dealing with such an infinite number of Dyson series is however hardly manageable. To circumvent this difficulty, we use the auxiliary variable introduced in [11], $w_i(t) = R_{ij}(t)v_j(t)$, with $\mathbf{R}(t)$ the rotation matrix of angle $\Omega_0 t$ around $\mathbf{u}_z$. The equation of motion for $\mathbf{w}$ is then

$$\dot{w}_i(t) = \delta\Omega(t) R^{-1}_{ij}\epsilon_{jkl}\delta b_l(t) R_{km} w_m(t), \tag{3}$$

the formal solution of which can be expressed as a *single* Dyson series:

$$\begin{aligned}\langle w_{i_0}(t)\rangle = w_{0i_0} + \sum_{p=1}^{\infty} \delta\Omega^p\ \epsilon_{k_1 m_1 n_1}\epsilon_{k_2 m_2 n_2}\dots\epsilon_{k_p m_p n_p} w_{0i_p} \int_0^t \mathrm{d}t_1 \int_0^{t_1}\mathrm{d}t_2 \cdots \int_0^{t_{p-1}}\mathrm{d}t_p \\ \times R^{-1}_{i_0k_1}(t_1)R^{-1}_{i_1k_2}(t_2)\cdots R^{-1}_{i_{p-1}k_p}(t_p)R_{m_1i_1}(t_1)\cdots R_{m_pi_p}(t_p)\langle\delta b_{n_1}(t_1)\dots\delta b_{n_p}(t_p)\rangle.\end{aligned} \tag{4}$$

In the following, we derive the velocity correlation function parallel to the mean field, $\langle v_{0z} v_z(t)\rangle$, based on this equation. The perpendicular (and anti-symmetric) functions can be obtained in a similar way and will be detailed elsewhere.

[1] Since cosmic rays are high-energy relativistic particles, the norm of the velocity is identified to c for convenience.

Red-noise approximation. In the Gaussian approximation, the Wick theorem allows for expressing the expectation value $\langle\delta b_{n_1}(t_1)\ldots\delta b_{n_p}(t_p)\rangle$ in terms of all possible permutations of products of contractions of pairs of $\langle\delta b_{n_i}(t_i)\delta b_{n_j}(t_j)\rangle$, which can be, in the case of 3D isotropic turbulence, written as

$$\langle\delta b_{n_1}(t_i)\delta b_{n_2}(t_j)\rangle = \frac{\delta_{n_1n_2}}{3}\varphi(t_i-t_j). \tag{5}$$

The correlation function $\varphi(t)$, which describes the correlation of the turbulence experienced by a test-particle along its path at two different times, is first modeled as a red-noise process with parameter τ,

$$\varphi(t) = \exp(-t/\tau). \tag{6}$$

The expression of τ, similarly to that found in [3], depends on the regime of rigidity considered. With ρ the Larmor radius of the particle expressed in units of the largest eddy scale $L_{\max}$ of the turbulence, $\tau \simeq L_c/c$ for $\rho \gtrsim \pi L_c/L_{\max}$, where L_c is the coherence scale of the turbulence. This is because in this rigidity regime, particles can travel over a distance L_c undergoing small deflections only. On the other hand, for $\rho \lesssim \pi L_c/L_{\max}$, the following heuristic estimate that makes use of the kinetic energy spectrum of the turbulence $\mathcal{E}(k)$ is observed to reproduce simulations:

$$\tau \simeq \frac{2}{c}\frac{\int_{k_\star}^{k_{\max}}\mathrm{d}k\,k^{-1}\mathcal{E}(k)}{\int_{k_\star}^{k_{\max}}\mathrm{d}k\,\mathcal{E}(k)}. \tag{7}$$

In this regime of rigidity, τ inherits a ρ dependency from that of the lower boundary in wave number $k_\star(\rho) = \rho_\star k_{\min}/\rho$ with $\rho_\star = 2L_c/(\pi L_{\max})$. The truncation in the wavenumber integration range selects modes for which particles do not experience spiral motions around the corresponding large-scale magnetic field lines over several Larmor times, modes that hence prevent decorrelations from occurring on relevant time scales.

To carry out a summation of the Dyson series (Eqn. 4), we resort to a two-step iteration procedure. First, the same partial summation scheme as in [12] is used, retaining the "unconnected" and "nested" classes of diagrams [13] (corresponding formally to the Kraichnan propagator [14]). In fact, it has been shown that the summation of the first two terms (beyond the free propagator) is sufficient to give us a physical solution in the case of pure turbulence [12]. We follow here the same strategy so that, making use of the properties of the Levi-Civita symbols contracted over one index, Eqn. 4 is substituted by

$$\begin{aligned}
\langle w_z^{\mathrm{K}}(t)\rangle = w_{0z} &+ \frac{\delta\Omega^2}{3}\int_0^t\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\,R^{-1}_{zk}(t_1)R_{mi_1}(t_1)R^{-1}_{i_1k}(t_2)R_{mi_2}(t_2)\varphi(t_1-t_2)\langle w_z^{\mathrm{K}}(t-t_1)\rangle \\
&- \frac{\delta\Omega^2}{3}\int_0^t\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\,R^{-1}_{zk_1}(t_1)R_{k_2i_1}(t_1)R^{-1}_{i_1k_2}(t_2)R_{k_1i_2}(t_2)\varphi(t_1-t_2)\langle w_z^{\mathrm{K}}(t-t_1)\rangle \\
&+ \left(\frac{\delta\Omega^2}{3}\right)^2\int_0^t\mathrm{d}t_1\cdots\int_0^{t_3}\mathrm{d}t_4\,\varphi(t_1-t_4)\varphi(t_2-t_3)\langle w_z^{\mathrm{K}}(t-t_1)\rangle\times \\
&(R^{-1}_{zk_1}(t_1)R_{m_1i_1}(t_1)R^{-1}_{i_1k_2}(t_2)R_{m_2i_2}(t_2)R^{-1}_{i_2k_2}(t_3)R_{m_2i_3}(t_3)R^{-1}_{i_3k_1}(t_4)R_{m_1z}(t_4) \\
&- R^{-1}_{zk_1}(t_1)R_{m_1i_1}(t_1)R^{-1}_{i_1k_2}(t_2)R_{m_2i_2}(t_2)R^{-1}_{i_2m_2}(t_3)R_{k_2i_3}(t_3)R^{-1}_{i_3k_1}(t_4)R_{m_1z}(t_4) \\
&- R^{-1}_{zk_1}(t_1)R_{m_1i_1}(t_1)R^{-1}_{i_1k_2}(t_2)R_{m_2i_2}(t_2)R^{-1}_{i_2k_2}(t_3)R_{m_2i_3}(t_3)R^{-1}_{i_3m_1}(t_4)R_{k_1z}(t_4) \\
&+ R^{-1}_{zk_1}(t_1)R_{m_1i_1}(t_1)R^{-1}_{i_1k_2}(t_2)R_{m_2i_2}(t_2)R^{-1}_{i_2m_2}(t_3)R_{k_2i_3}(t_3)R^{-1}_{i_3m_1}(t_4)R_{k_1z}(t_4)),
\end{aligned} \tag{8}$$

where the superscript K stands for “Kraichnan”. Next, some properties of the rotation matrices, in particular $R^{-1}_{zk} = R_{zk} = \delta_{zk}$, $R_{ij}(t_1)R^{-1}_{jk}(t_2) = R_{ik}(t_1 - t_2)$, and $R_{ij}(t_1 - t_2) = R_{ji}(t_2 - t_1)$), allow us to get an explicit form of the equation:

$$\begin{aligned}
\langle w^{\mathrm{K}}_z(t)\rangle &= w_{0z} - \frac{\delta\Omega^2}{3}\int_0^t \mathrm{d}t_1 \int_0^{t_1}\mathrm{d}t_2\, \varphi(t_1-t_2)\cos\left(\Omega_0(t_1-t_2)\right)\langle w^{\mathrm{K}}_z(t-t_1)\rangle \\
&+4\left(\frac{\delta\Omega^2}{3}\right)^2 \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_3}\mathrm{d}t_4\, \varphi(t_1-t_4)\varphi(t_2-t_3)\cos\left(\Omega_0(t_2-t_3)\right)\cos\left(\Omega_0(t_1-t_2+t_3-t_4)\right)\langle w^{\mathrm{K}}_z(t-t_1)\rangle \\
&- 2\left(\frac{\delta\Omega^2}{3}\right)^2 \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_3}\mathrm{d}t_4\, \varphi(t_1-t_4)\varphi(t_2-t_3)\cos\left(\Omega_0(t_1-2t_2+2t_3-t_4)\right)\langle w^{\mathrm{K}}_z(t-t_1)\rangle \\
&+ 2\left(\frac{\delta\Omega^2}{3}\right)^2 \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_3}\mathrm{d}t_4\, \varphi(t_1-t_4)\varphi(t_2-t_3)\cos\left(\Omega_0(t_1-t_2+t_3-t_4)\right)\langle w^{\mathrm{K}}_z(t-t_1)\rangle. \qquad (9)
\end{aligned}$$

To solve this non-linear equation, we proceed with a Laplace transform. The change of variables $t = x + x_1 + \cdots + x_p$, $t_1 = x_1 + \cdots + x_p$, $t_2 = x_2 + \cdots + x_p, \cdots$, $t_p = x_p$ allows for sending all integration boundaries between 0 and $+\infty$ for the x_i variables. After some algebra, the equation for the Laplace transform $\mathcal{L}\left(w(t)\right) \equiv \hat{W}(p)$ reads as

$$\begin{aligned}
\hat{W}^{\mathrm{K}}(p) &= \frac{1}{p} - 2\frac{\delta\Omega^2}{3}\frac{\hat{W}^{\mathrm{K}}(p)}{p}\frac{p+\tau^{-1}}{(p+\tau^{-1})^2+\Omega_0^2} \\
&+ 2\left(\frac{\delta\Omega^2}{3}\right)^2 \frac{\hat{W}^{\mathrm{K}}(p)}{p}\left[\left(\frac{p+2\tau^{-1}}{(p+2\tau^{-1})^2+\Omega_0^2}\right)+\left(\frac{1}{p+2\tau^{-1}}\right)\right]\left(\frac{(p+\tau^{-1})^2-\Omega_0^2}{\left((p+\tau^{-1})^2+\Omega_0^2\right)^2}\right) \\
&- 4\left(\frac{\delta\Omega^2}{3}\right)^2 \frac{\hat{W}^{\mathrm{K}}(p)}{p}\left(\frac{\Omega_0^2(p+\tau^{-1})}{\left((p+2\tau^{-1})^2+\Omega_0^2\right)\left((p+\tau^{-1})^2+\Omega_0^2\right)^2}\right). \qquad (10)
\end{aligned}$$

The solution for $\langle w^{\mathrm{K}}_z(t)\rangle$ is then obtained by making use of the numerical Stehfest scheme of the inverse Laplace transform.

Once $\langle w^{\mathrm{K}}_z(t)\rangle$ is determined, the second step to get an improved estimation of the propagator consists in including “crossed” diagrams (that formally account for any mix of crossed and nested diagrams). To do so, an iterative procedure is designed to carry out this summation. In the Laplace space, the Nth iterated $\langle w^{\mathrm{N}}_z(t)\rangle$ reads as

$$\begin{aligned}
\hat{W}^{(N)}(p) &= \frac{1}{p} - 2\frac{\delta\Omega^2}{3}\frac{\hat{W}^{(N)}(p)}{p}\mathcal{L}\left(\exp\left(-t/\tau\right)w^{(N-1)}(t)\cos\Omega_0 t\right) \\
&+ 2\left(\frac{\delta\Omega^2}{3}\right)^2 \frac{\hat{W}^{(N)}(p)}{p}\mathcal{L}\left(\exp\left(-2t/\tau\right)w^{(N-1)}(t)\right)\mathcal{L}^2\left(\exp\left(-t/\tau\right)w^{(N-1)}(t)\cos\Omega_0 t\right), \qquad (11)
\end{aligned}$$

with the initial iteration $w^{(0)}(t) = w^{\mathrm{K}}(t)$. In practice, convergence is achieved after a few iterations ($N \simeq 5$).

The resulting picture is illustrated in Fig. 1, where the time dependence of the auto-correlation of the particle velocities parallel to the mean field is shown for different values of turbulence level.

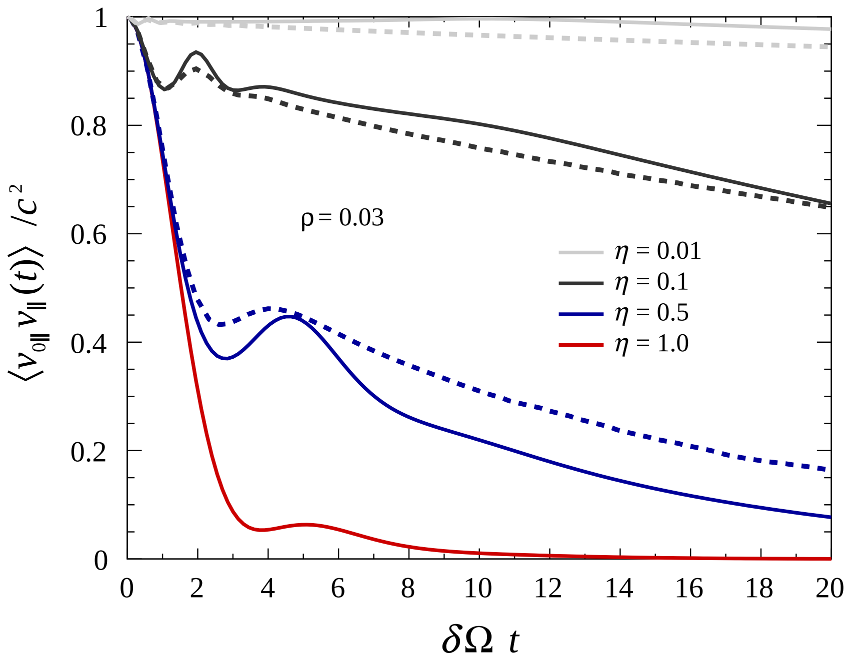

Figure 1: Time dependence of the auto-correlation of the particle velocities parallel to the mean field, for different values of turbulence level. A reduced Larmor radius $\rho = 0.03$ is chosen to illustrate the gyro-resonant regime. The dotted curves are the Monte-Carlo reference.

Solid curves are compared to Monte-Carlo results, shown as dotted curves. The reduced Larmor radius is chosen to be $\rho = 0.03$ so as to explore the gyro-resonant regime. Qualitatively, various features uncovered in Monte-Carlo simulations, such as the turbulence-dependent timescale of the correlation and the structures beyond the exponential falloff, are well reproduced by the calculation presented in this study. Yet, quantitative discrepancies are observed. We are therefore led to observe that the success of the calculation in the purely turbulent case is not as striking when a constant field is added. One simplified ingredient that may source these discrepancies is the red-noise approximation to describe the correlation function $\varphi(t)$ in the presence of a mean field. In the following, we explore a better modeling for $\varphi(t)$.

Beyond the red-noise approximation. The two-point correlation function of the (fluctuating) magnetic field experienced by a particle-test can be conveniently related to the characteristics of the turbulence in Fourier space,

$$\langle \delta b_{n_1}(t_i)\delta b_{n_2}(t_j)\rangle \simeq \iint d\mathbf{k}d\mathbf{k}'\langle \delta b_{n_1}(\mathbf{k})\delta b_{n_2}(\mathbf{k}')\rangle\langle e^{i\mathbf{k}\cdot\mathbf{x}(t)}\rangle, \tag{12}$$

where the Corrsin approximation has been used [15]. The factor $\langle e^{i\mathbf{k}\cdot\mathbf{x}(t)}\rangle$ needs to be estimated beyond the “quasi-linear theory” approximation. To do so, we start from the formal expansion

$$\langle e^{i\mathbf{k}\cdot\mathbf{x}(t)}\rangle = \sum_{n\geq 0}\frac{i^n}{n!}\int_0^t dt_1 \cdots \int_0^t dt_n \langle(\mathbf{k}\cdot\mathbf{v}(t_1))\dots(\mathbf{k}\cdot\mathbf{v}(t_n))\rangle, \tag{13}$$

where the substitution $\mathbf{x}(t) = \int_0^t dt'\ \mathbf{v}(t')$ has been used. Next, the n-point correlation function entering into the integrand expression can be substituted for the sum of all possible contraction of

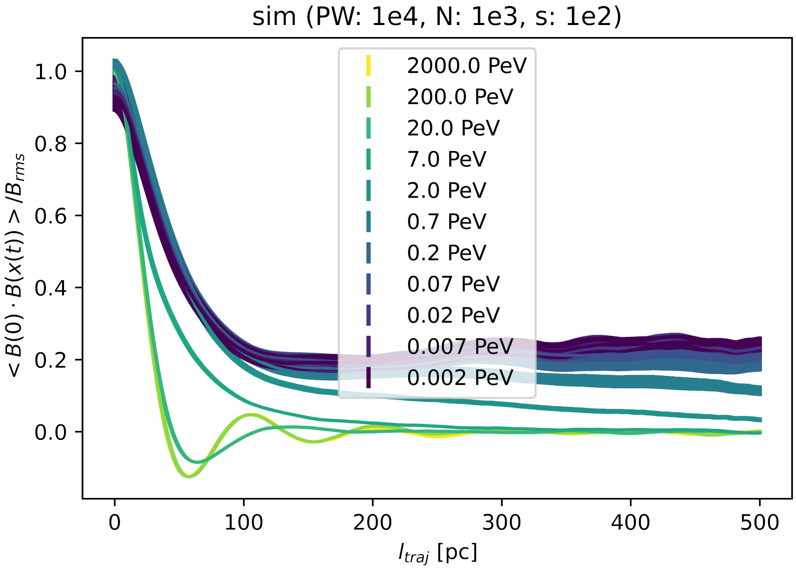

Figure 2: Two-point correlation function of the magnetic field experienced by a test-particle as a function of the distance travelled. Different rigidities are color-coded.

pairs (Wick theorem), which are approximated by

$$\langle (\mathbf{k}\cdot\mathbf{v}(t_1))(\mathbf{k}\cdot\mathbf{v}(t_2))\rangle \simeq \langle \mathbf{k}^2\rangle\langle \mathbf{v}(t_1)\cdot\mathbf{v}(t_2)\rangle \simeq k^2c^2e^{-(t_1-t_2)/\xi(k)}, \tag{14}$$

where the heuristic expression $\xi = \rho/(kc)$ is used. Some algebra then leads to

$$\langle e^{i\mathbf{k}\cdot\mathbf{x}(t)}\rangle \simeq \sum_{p\geq 0}(-k^2c^2)^p \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{2p-1}} dt_{2p} \sum_{\text{pairings}} \prod_{\text{pairs i<j}} e^{-(t_i-t_j)/\xi(k)}, \tag{15}$$

which can be summed using the diagram technique.

A Monte-Carlo calculation of $\langle \delta b_{n_1}(t_i)\delta b_{n_2}(t_j)\rangle$ is shown in Fig. 2 for different rigidities. The corresponding estimation from Eqn. 15 is shown in Fig. 3. Only unconnected diagrams (Bourret propagator) are retained in the summation. The main features uncovered by the Monte-Carlo calculation are quantitatively well reproduced by the estimation at high rigidities (in particular the oscillations around 0 that are absent in the quasi-linear theory or the red-noise approximation) and in the gyro-resonant regime. A numerical summation of Eqn. 4 using this "beyond red-noise" approach for $\langle \delta b_{n_1}(t_i)\delta b_{n_2}(t_j)\rangle$ is under progress and will be reported elsewhere.

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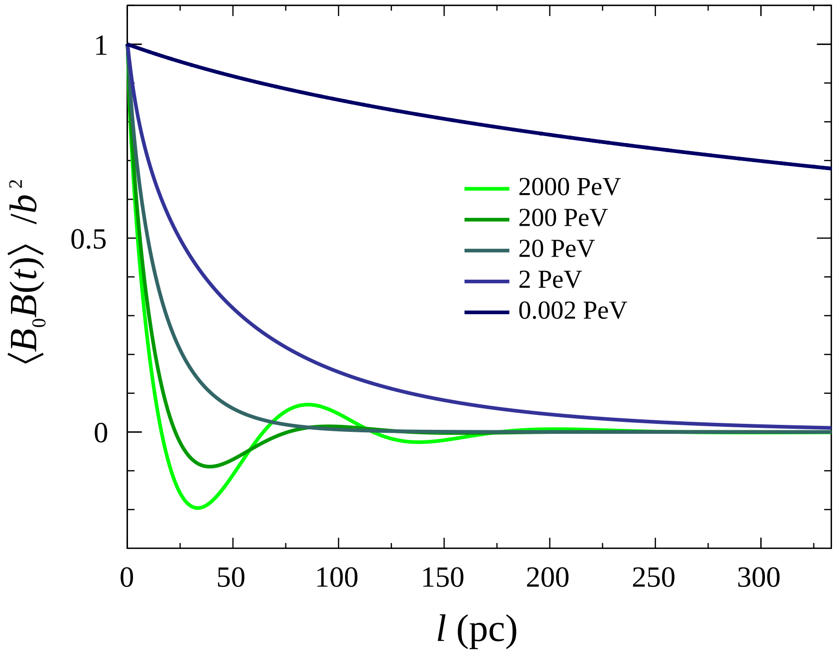

Figure 3: Same as Fig. 2, as predicted by the model.

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