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Monte Carlo analyses of the shower-disc structure for gamma-ray shower

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Shower-disc structures of the gamma-ray shower are investigated by analyzing Monte Carlo simulated showers based on the multiple scattering theory of Yang for actual path-length. We investigated the radial distribution of shower electrons with given delays of path-length, and found the density of the relevant electrons keeps almost constant near the shower axis and drops with the increase of radial distance reaching rapidly to null at the edge of shower-disc, We propose structure functions of the shower-disc, composed of the longitudinal distribution function with the radial distribution integrated and the radial distribution function with the delay of path-length fixed.

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1. Introduction

The diffusion equation for the actual-path-length distribution of electrons traversing through matters was proposed by C.N.Yang [1] and was generally solved by us [2]. A Monte Carlo (abbreviated as MC, hereafter) code to obtain the arrival time distribution of cascade shower particles was developed applying the above solution, and some results were reported at Bangalore [3] and at La Jolla (as a lated paper [4]) conferences. This time we investigate the shower-disc structure of particles by carrying out the code in usual PCs.

2. Location distribution of electrons in the gamma-ray shower

We investigate the location distribution of electrons in our MC events of gamma-induced cascade shower (abbreviated as gamma-ray shower, hereafter).

First, we define the "geometrical front" of shower at the penetration depth of t by the surface of sphere with the radius of t from the start point of gamma-ray shower, as indicated in Fig. 1. Next, we define the "front plane" of shower by the plane vertical to the shower axis and tangential to the geometrical front. We define the "delay Δ" of the shower particle by the distance from the front plane of the shower. Then the delay Δ of the shower particle on the geometrical front satisfies

$$\Delta = t - \sqrt{t^2 - r^2} \simeq r^2/(2t), \tag{2.1}$$

where t, r, and Δ are all measured in the radiation length [5, 6].

We carried MC simulations of gamma-ray shower, in the air of uniform density with the radiation length X_0 of 308 m and the critical energy ε of 81 MeV, with the incident energy $W_0 = 10^4\varepsilon$ of primary gamma-ray and the threshold energy of $10^{-1}\varepsilon$ (called as G41 series, hereafter). We plot MC shower electrons in Fig. 3 at the penetration depths t of 5, 10, 15, and 20 with the horizontal and the vertical coordinates of (r^2,Δ), where the crowded rates of the plots toward the horizontal direction and the vertical direction become proportional to the areal density $dn/(\pi dr^2)$ and the delay density $dn/d\Delta$ of electrons. We draw the geometrical front by the solid line, upward from which electrons locate in the figure.

When we fix the square of the radial distance r^2, we see the density of electrons increases very rapidly upward after the geometrical front, it reaches to some peak, and it decreases slowly with the increase of Δ. On the contrary when we fix the delay Δ, the density shows almost constant near the shower axis ($r^2 \ll 1$) and it decreases slowly with the increase of r^2 up to the geometrical front. To confirm this character precise, we convert Fig. 3 to Fig. 4 by plotting the electrons with the coordinates of $(r^2/(2t\Delta),\Delta)$, with the horizontal coordinate normalized by the square of the radius of the geometrical front $2t\Delta$. Then we find in Fig. 4 the density of shower electrons falls almost similarly with the delay Δ of vertical coordinate irrespective of the horizontal coordinate of $r^2/2t\Delta$, though with Δ fixed the density keeps almost constant near the shower axis ($r^2/2t\Delta \ll 1$) and decreases very rapidly just before reaching to the geometrical front ($r^2/2t\Delta \to 1$). We analyze disc structures of gamma-ray shower, mainly according to these characters.

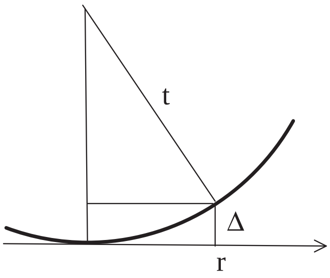

Figure 1: The geometrical front of shower electrons at the penetrating depth of t. The geometrical front shows trivial delay of $\Delta \simeq r^2/(2t)$ at r from the tangential plane at the shower front.

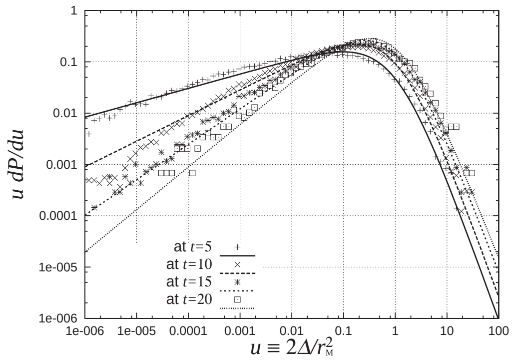

Figure 2: The u-weighted probability densities, $u\,dP/du$, for the delay $u \equiv 2\varepsilon^2\Delta/E_s^2$ at $t = 5$, 10, 15, and 20.

3. The structure functions of shower electrons

Based on the investigations in the last section, we assume the infinitesimal probability $d^2\rho$ of the shower electrons to delay Δ and to spread $\vec{r}$ from the shower axis as

$$d^2\rho = \frac{dP(\Delta;t)}{d\Delta}d\Delta\frac{dQ(r;\Delta)}{d\vec{r}}2\pi r dr, \tag{3.1}$$

where $dP(\Delta;t)/d\Delta$ denotes the probability density to delay Δ with the radial spread $\vec{r}$ integrated (the longitudinal distribution) and $dQ(r;\Delta)/d\vec{r}$ denotes the probability density for the spread $\vec{r}$ with the delay Δ fixed. So that the densities denoted as the "structure functions" are normalized as

$$\int_0^\infty \frac{dP(\Delta;t)}{d\Delta}d\Delta = 1, \quad \int_0^\infty \frac{dQ(r;\Delta)}{d\vec{r}}2\pi r dr = 1, \quad \text{and} \quad \int_0^\infty\int_0^\infty \frac{d^2\rho}{d\Delta d\vec{r}}d\Delta 2\pi r dr = 1. \tag{3.2}$$

3.1 The longitudinal distribution $dP(\Delta;t)/d\Delta$ of shower electrons

We derive the probability density of shower electrons to delay Δ, by analyzing MC showers of G41 series at $t = 5$, 10, 15, and 20 cu with the radial spread r integrated. We introduced a new variable

$$u \equiv 2\Delta/r_\mathrm{M}^2 \qquad \text{with} \tag{3.3}$$

$$r_\mathrm{M} \equiv E_\mathrm{s}/\varepsilon \tag{3.4}$$

of the Molière unit [6], and investigated the probability of electrons in 100 MC showers to fall into 80 bins of u, equally divided logarithmically from 10^{-6} to 10^2. The u-weighted probability density $u\,dP/du$ is indicated in Fig. 2 (dots).

In some analyses, the Γ distribution of $(a^{n+1}/\Gamma(n+1))t^n e^{-at}dt$ was assumed as a probability of the delay t for cascade-shower particles [7, 8]. The Γ distribution can reproduce the probability

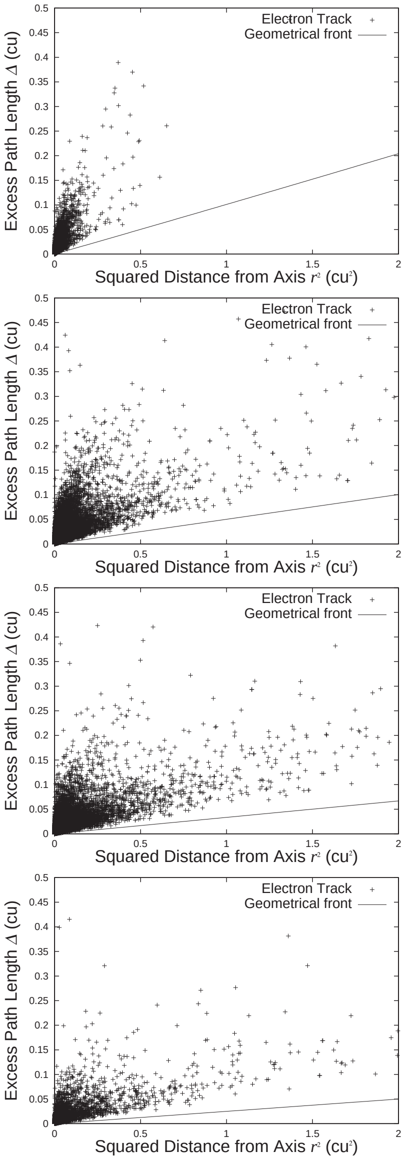

Figure 3: Scatter plot of electron tracks indicating Δ vs r^2 at $t = 5$, 10, 15, and 20, from top to bottom.

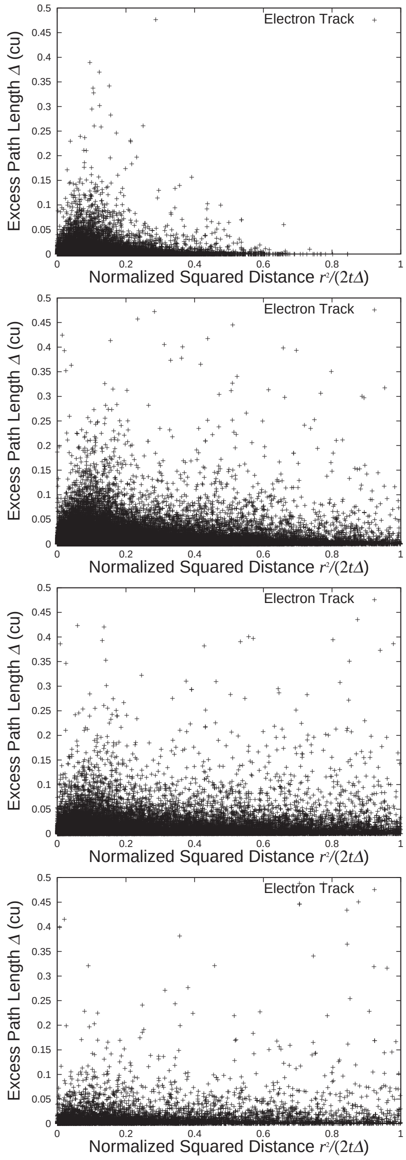

Figure 4: Scatter plot of electron tracks indicating Δ vs $r^2/(2t\Delta)$ at $t =$ 5, 10, 15, and 20, from top to bottom.

derived by MC in the early-stage of shower-development, though it falls too rapid in the tail-stage as analyzed in all-charged-particles investigations [8] and cannot reproduce the MC results of power-law decrease. So that, we assume the probability density as a Beta distribution of

$$dP = \frac{dP(\Delta;t)}{d\Delta}d\Delta \equiv \frac{dP(u;t)}{du}du = \frac{1}{B(a,b)}u^{a-1}(1+u)^{-a-b}du \tag{3.5}$$

with $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ of Beta function. Then they satisfy

$$\langle u \rangle = \frac{a}{b-1},\ \langle u^2 \rangle = \frac{a+1}{b-2}\langle u \rangle, \qquad \text{thus} \quad b = 2 + \frac{\langle u \rangle + 1}{\langle u^2 \rangle - \langle u \rangle^2}\langle u \rangle,\ a = (b-1)\langle u \rangle. \tag{3.6}$$

We show in Table 1 the values of a and b derived from 100 MC showers of G41 series, and show the probability density of the delay from (3.5) in Fig. 2 (lines) applying these a and b, which well reproduced the density derived by MC.

3.2 The radial distribution $dQ(r;\Delta)/d\vec{r}$ at fixed delays Δ of shower electrons

We investigate the $(1-r^2/2t\Delta)$-weighted probability density of $(1-r^2/2t\Delta)$ in Fig. 5 (dots) counting the shower electron to fall on the 100 bins of $(1-r^2/2t\Delta)$, equally divided logarithmically from 0.1 to 1, for respective 9 ranges of Δ (distinguished by different 9 dots), equally divided logarithmically from $10^{-5.0}$ to $10^{-0.5}$. We express the probability density of $(1-r^2/2t\Delta)$ by the power function of

$$dQ = \frac{dQ}{d(1-r^2/2t\Delta)} d\left(1-\frac{r^2}{2t\Delta}\right) = h\left(1-\frac{r^2}{2t\Delta}\right)^{h-1} d\left(1-\frac{r^2}{2t\Delta}\right), \tag{3.7}$$

where h is normalized as the total electron number to be 1. Then it satisfies

$$\left\langle 1-\frac{r^2}{2t\Delta}\right\rangle = \frac{h}{h+1}, \quad \text{thus} \quad h = 1/\left\{\left\langle 1-\frac{r^2}{2t\Delta}\right\rangle^{-1} - 1\right\}. \tag{3.8}$$

We drew the $(1-r^2/2t\Delta)$-weighted probability density $h(1-r^2/2t\Delta)^h$ derived from Eq. (3.7) in Fig. 5 (lines) applying the respective h at 9 ranges of Δ mentioned above derived from 100 MC showers of G41 series, which well reproduced the MC results. We also found that the power index of h does not depend on Δ when the number of electrons is much enough in the Δ range,

Thus we get the structure function for the probability density of radial spread r at the fixed delay Δ. Introducing a new variable

$$v = r^2/r_{\mathrm{M}}^2, \tag{3.9}$$

we have $r^2/(2t\Delta) = v/(tu)$. Then Eq. (3.7) gives

$$dQ = \frac{dQ}{d(v/tu)} d\left(\frac{v}{tu}\right) = h\left(1-\frac{v}{tu}\right)^{h-1} d\left(\frac{v}{tu}\right) \qquad \text{with} \quad 0 \le \frac{v}{tu} \le 1, \tag{3.10}$$

where it satisfies

$$\left\langle \frac{v}{tu}\right\rangle = \frac{1}{h+1}, \quad \text{thus} \quad h = \left\langle \frac{v}{tu}\right\rangle^{-1} - 1. \tag{3.11}$$

We show the power index h evaluated from 100 MC showers in Table 1. We drew the structure function for the radial distribution with the delay fixed, or the (v/tu)-weighted probability densities $(v/tu)\,d\rho/d(v/tu)$ determined in Eq. (3.10), in Fig. 6 (lines) with t separated by 5, 10, 15, and 20 and applying the value of h in the Table. The derived structure functions well reproduce the indicated radial distributions with the delay fixed (dots) appearing in the electrons in 100 MC showers.

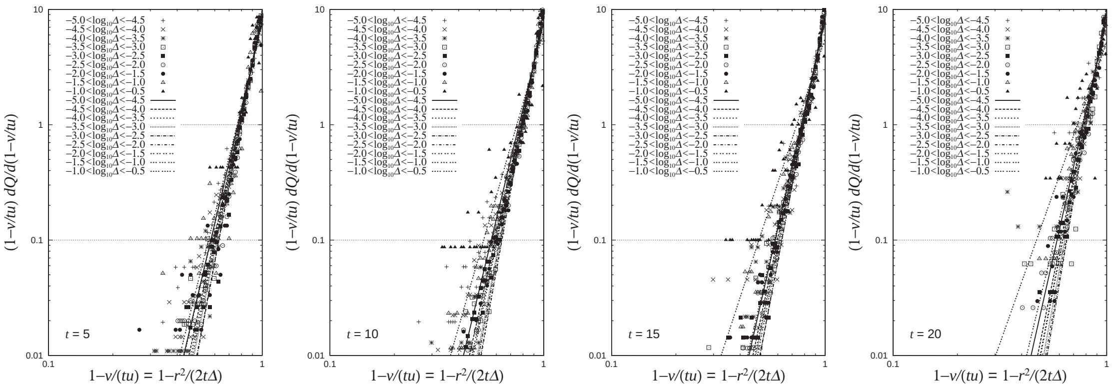

Figure 5: $1-v/tu = 1-r^2/2t\Delta$ distribution at various Δ regions derived from electrons in 100 MC showers, at t = 5, 10, 15, and 20. Δ is divided to 9 regions, [-5.0,-4.5], [-4.5,-4.0], $\cdots$, [-1.0,-0.5] in $\log_{10}$ values. $1-v/tu$ is distinguished in 100 bins from 0.1 to 1, equally divided logarithmically.

4. The shower-disc structure of electrons reproduced by the structure functions

4.1 Reproduction of the lateral distribution of the shower electrons

We derive the probability density of v/t from 100 MC showers of G41 series, by counting the shower electrons to fall on the 80 bins of v/t, equally divided logarithmically from 10^{-7} to 10. The results are indicated in Fig. 7 (dots) for t = 5, 10, 15, and 20, with v/t weighted.

On the other hand, we derive the lateral distribution from the assumed structure functions of (3.5) and (3.10);

$$d\rho = d\left(\frac{v}{t}\right)\int_{v/t}^{\infty}\frac{dQ}{d(v/tu)}\frac{dP(u;t)}{du}\frac{du}{u} = \frac{hd(v/t)}{B(a,b)}\int_{v/t}^{\infty}u^{a-2}(1+u)^{-a-b}\left(1-\frac{v/t}{u}\right)^{h-1}du. \quad (4.1)$$

Thus the (v/t)-weighted probability density is expressed as

$$\frac{(v/t)d\rho}{d(v/t)} = \frac{h(v/t)^a}{B(a,b)}\int_1^{\infty}\left(\frac{u}{v/t}\right)^{a-2}\left(1+\frac{v}{t}\frac{u}{v/t}\right)^{-a-b}\left(1-\frac{v/t}{u}\right)^{h-1}d\left(\frac{u}{v/t}\right). \quad (4.2)$$

The results derived by numerical integrations are indicated in Fig. 7 (lines), which well reproduced the MC results for t = 5, 10, 15, and 20.

The central properties ($v/t \to 0$) of the lateral distribution of (4.2) are described as

$$\frac{(v/t)d\rho}{d(v/t)} \to \frac{h(v/t)^a}{B(a,b)}\int_1^{\infty}\left(\frac{u}{v/t}\right)^{a-2}\left(1-\frac{v/t}{u}\right)^{h-1}d\left(\frac{u}{v/t}\right) = \frac{h(v/t)^a}{B(a,b)}B(1-a,h). \quad (4.3)$$

Table 1: Parameters a, b and h, derived from 100 MC showers.

t (CU)	5	10	15	20	25
a	0.28	0.50	0.69	0.83	1.02
b	2.81	2.76	2.62	2.54	2.66
h	8.59	9.46	9.19	9.65	8.92

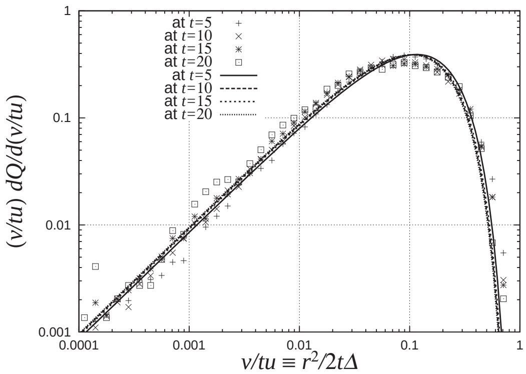

Figure 6: The (v/tu)-weighted probability densities $(v/tu)\,dQ/d(v/tu)$ for the normalized squared-radial-distance of $v/tu \equiv r^2/2t\Delta$ at $t = 5$, 10, 15, and 20.

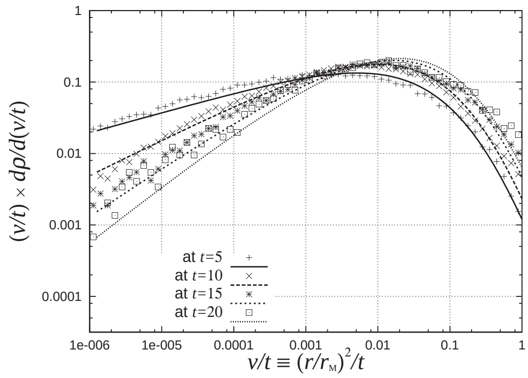

Figure 7: The radial distribution of shower electrons expressed by (v/t)-weighted probability density for the normalized squared-radial-distance of $v/t \equiv (r^2/r_{\rm M}^2)/t$ at $t = 5$, 10, 15, and 20.

We find the central power profiles of $(v/t)^{a-1}$ of the lateral distribution well agree with the starting power profiles of u^{a-1} at $u \to 0$ for the longitudinal distribution of (3.5).

4.2 Reproduction of the probability densities of delay at fixed radial distances

We count the shower electrons in 100 MC events to fall on the 60 bins of delay $u \equiv 2\Delta/r_{\rm M}^2$, equally divided logarithmically from 10^{-4} to 10, for respective 10 ranges of radial distance $\sqrt{v} = r/r_{\rm M} \simeq r/308$m, equally divided from 0 to 1. Then we get the probability densities $\delta^2\rho/\delta(v/t)/\delta u$ for $t = 5$, 10, 15, and 20 by dividing the counts with the total number of electrons in the respective ranges of $\sqrt{v}$, as indicated in Fig. 8 (dots) with u weighted, separating the ranges of v with the respective symbols of dot.

On the other hand, we evaluate the infinitesimal probability of shower electrons $d\rho$ to fall on the bin of du from the assumed structure functions of (3.5) and (3.10), for the respective finite ranges of $\delta(v/t) \equiv [v_0/t, v_1/t]$;

$$d\rho = du\int_{v_0/tu}^{v_1/tu}\frac{dP(u;t)}{du}\frac{dQ}{d(v/tu)}d\left(\frac{v}{tu}\right) = \frac{u^{a-1}(1+u)^{-a-b}du}{B(a,b)}\int_{(v_0/t)/u}^{(v_1/t)/u}h\left(1-\frac{v/t}{u}\right)^{h-1}d\frac{v/t}{u}. \quad (4.4)$$

So that we can derive the respective averaged electron densities in the finite ranges of $\delta(v/t)$ at the averaged radial locations of $(v_0+v_1)/2/t$;

$$u\frac{d}{du}\frac{\delta\rho}{\delta(v/t)} = \frac{u^a(1+u)^{-a-b}}{B(a,b)}\left\{\left(1-\frac{v_0/t}{u}\right)^h - \left(1-\frac{v_1/t}{u}\right)^h\right\}\Big/\frac{v_1-v_0}{t}. \quad (4.5)$$

The results for $t = 5$, 10, 15, and 20 are drawn in Fig. 8 (lines), which well reproduce the MC results.

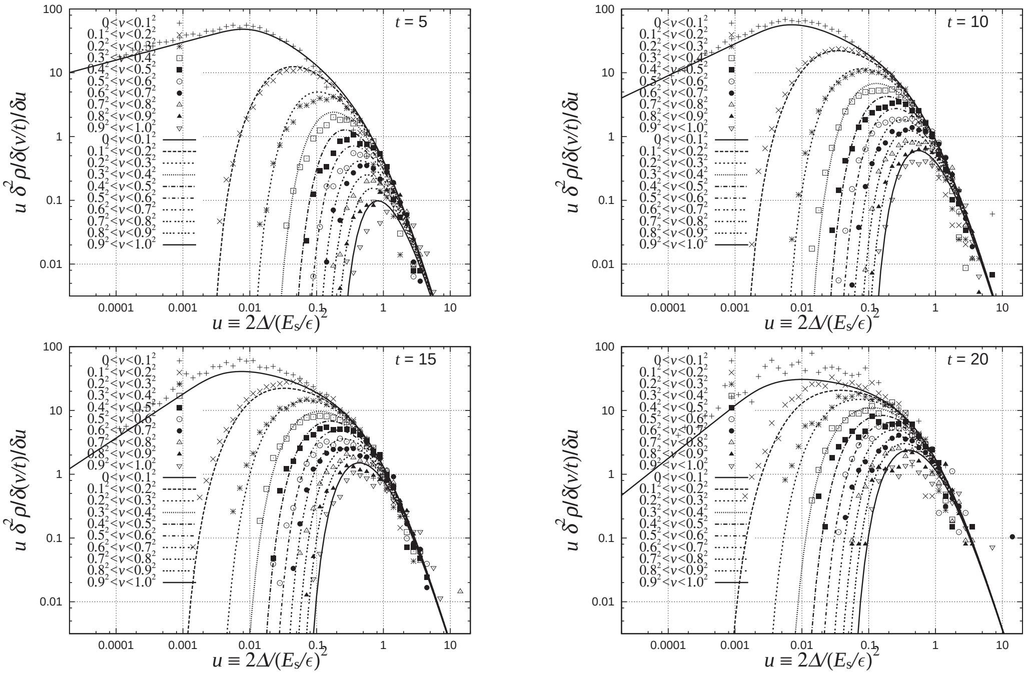

Figure 8: The delay $u \equiv 2\Delta/r_M^2$ distribution of shower electrons at fixed squared-radial-distances of $v \equiv r^2/r_M^2$, expressed by u-weighted probability densities $u\,\delta^2\rho/\delta(v/t)/\delta u$ at $t = 5$, 10, 15, and 20.

5. Conclusions and discussions

From 100 MC gamma-ray showers of G41 series ($W_0 = 10^4\varepsilon$ and $E > 10^{-1}\varepsilon$) in the uniform air medium, we got the structure functions of gamma-ray showers which are the probability density for the delay (1-dimensional) and the density for the lateral spread at fixed delays. We assumed Beta distributions for the both densities with the parameters a, b, and h determined by MC showers from the first and the second moments of the probability densities. We could reproduce the lateral distribution and the delay distribution at fixed radial spreads, which well reproduced the distributions derived from MC showers. This investigation started from the discussions between one of the authors (T.N.) and late John Linsley with mail, after La Jolla conference (1985).

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