PROCEEDINGS OF SCIENCE

Formulae to predict the excess-path-length distribution of cascade-shower electrons

Talao Nakatsuka[*a,*] **and Kazuhide Okei**[b]

[a] *Okayama Shoka University, Okayama 700-8601, Japan*

[b] *Kawasaki Medical School, Kurashiki 701-0192, Japan*

E-mail: nakatuka@olive.plala.or.jp

Theoretical investigations of excess-path-length distribution for cascade-shower electrons are important to understand the arrival-time-distribution of shower electrons observed in the air shower experiment. We acquired the formulae to describe the excess-path-length distribution by solving the diffusion equation of the cascade process under Approximation A and B. Reliability of the formulae is examined by comparing them with the distribution derived by the Monte Carlo calculations.

The 38th International Cosmic Ray Conference (ICRC2023)
26 July – 3 August, 2023
Nagoya, Japan

[*] Retired now from the university.

[*] Speaker

https://pos.sissa.it/

PoS(ICRC2023)681

1. Introduction

Cascade-shower electrons show excess distribution of path-length due to the multiple Coulomb scattering with the matters of traverse, which is observed as the arrival-time-distribution of shower electrons in the air shower experiment. The distributions can be obtained from our Mellin transform of $\langle u^\kappa \rangle$, derived by solving the diffusion equation for the process [1].

Plain descriptions for the formulae $\langle u^\kappa \rangle$ under Approximation B are proposed, and the mean excess and the excess distribution of path-length averaged over shower electrons derived from our $\langle u^\kappa \rangle$ are indicated. The results are compared with those derived by a Monte Carlo (abbreviated by MC, hereafter) calculation [2]. The threshold-energy E dependence of the results, appearing in the MC results, are also discussed.

2. The mean k-th moment of the excess-path-length distribution for shower electrons

Let $\pi(E,\vec{\theta},\Delta,t)dEd\vec{\theta}d\Delta$ and $\gamma(E,\vec{\theta},\Delta,t)dEd\vec{\theta}d\Delta$ be the numbers of electron and photon of energy E, direction $\vec{\theta}$ and excess-path-length Δ within the infinitesimal ranges of dE, $d\vec{\theta}$ and $d\Delta$, at the traversed thickness of t in the unit of radiation length [3, 4]. Under the cascade process, $\pi(E,\vec{\theta},\Delta,t)$ and $\gamma(E,\vec{\theta},\Delta,t)$ satisfy the diffusion equation of

$$\frac{\partial}{\partial t}\begin{pmatrix}\pi(E,\theta,\Delta,t)\\ \gamma(E,\theta,\Delta,t)\end{pmatrix} = \begin{pmatrix}-A' & B'\\ C' & -\sigma_0\end{pmatrix}\begin{pmatrix}\pi\\ \gamma\end{pmatrix} - \frac{\theta^2}{2}\frac{\partial}{\partial\Delta}\begin{pmatrix}\pi\\ \gamma\end{pmatrix} + \frac{E_s^2}{4E^2}\nabla_\theta^2\begin{pmatrix}\pi\\ 0\end{pmatrix} + \frac{\varepsilon\partial}{\partial E}\begin{pmatrix}\pi\\ 0\end{pmatrix}, \quad (2.1)$$

where shower electrons lose their energies of εdt in each traverse of dt by ionization with the critical energies ε of 0 (Approximation A) or finite values (Approximation B). The operators A', B', C' and the constants σ_0, ε are indicated in Nishimura [4]. Note that the variable $\vec{\theta}$ in the densities are expressed by θ as $\pi(E,\theta,\Delta,t)$ and $\gamma(E,\theta,\Delta,t)$, as they are axially symmetric with $\vec{\theta}$.

We have the k-th moment of excess-path-length distribution for total shower electrons (with E from 0 to ∞) from the diffusion equation under Approximation B [1], as

$$\begin{aligned}\Pi_{\mathrm{B}}^{(k)}(E_0,0,t) &= \int_0^\infty dE\int_0^\infty 2\pi\theta d\theta J_0(\zeta\theta)\int_0^\infty \Delta^k\pi(E,\theta,\Delta,t)d\Delta\\ &\simeq \frac{(E_s^2/2\varepsilon^2)^k}{2\pi i}\int\frac{ds}{s}\left(\frac{E_0}{\varepsilon}\right)^s e^{\lambda_1(s)t}\frac{s}{s+2k}\frac{\{D_s\phi_0^{(k)}(s;\lambda)\}_{\lambda\to\lambda_1(s)}}{\lambda_1(s)-\lambda_2(s)}\left\{K_0^{(k)}(s,-s-2k)\right\}_{\lambda\to\lambda_1(s)}. \end{aligned} \quad (2.2)$$

Especially for $k=0$, we have the total number of shower electrons

$$\begin{aligned}\Pi_{\mathrm{B}}(E_0,0,t) &\simeq \frac{1}{2\pi i}\int\frac{ds}{s}\left(\frac{E_0}{\varepsilon}\right)^s e^{\lambda_1(s)t}\frac{\{D_s\phi_{00}(s;\lambda)\}_{\lambda\to\lambda_1(s)}}{\lambda_1(s)-\lambda_2(s)}\left\{K_0^{(0)}(s,-s)\right\}_{\lambda\to\lambda_1(s)}\\ &\simeq \Pi_{\mathrm{A}}(E_0,\varepsilon,t)\{K_0^{(0)}(\bar{s},-\bar{s})\}_{\lambda\to\lambda_1(\bar{s})} \qquad \text{with} \quad (2.3)\\ \ln\frac{E_0}{\varepsilon} &= -\lambda_1'(\bar{s})t+\frac{1}{\bar{s}}, \quad (2.4)\end{aligned}$$

indicated in the reviews of Rossi and Greisen, and Nishimura [3, 4], where $\Pi_{\mathrm{A}}(E_0,\varepsilon,t)$ denotes the number of shower electrons under Approximation A and $\bar{s}$ is called as the shower age. Thus we

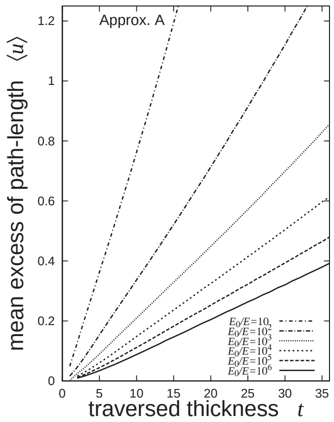

Figure 1: Mean excess $\langle u\rangle$ of path-length under Approximation A for shower electrons, with the threshold energy of E and $u \equiv 2E^2\Delta/E_s^2$ [1].

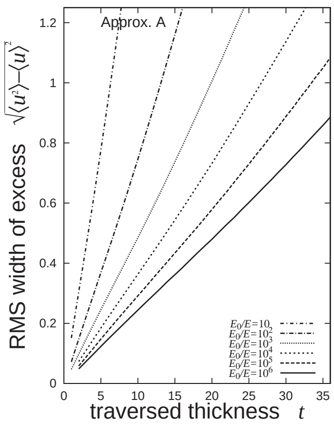

Figure 2: Root-Mean-Square excess $\sqrt{\langle u^2\rangle - \langle u\rangle^2}$ of path-length under Approximation A for shower electrons, with the threshold energy of E and $u \equiv 2E^2\Delta/E_s^2$ [1].

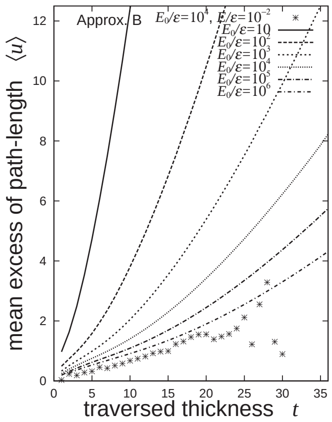

Figure 3: Mean excess $\langle u\rangle$ of path-length under approximation B for shower electrons, with the threshold energy E of 0 and $u \equiv 2\varepsilon^2\Delta/E_s^2$.

have the mean k-th moment of excess-path-length averaged over the total shower electrons, as

$$\langle\Delta^k\rangle = \frac{\Pi_{\rm B}^{(k)}(E_0,0,t)}{\Pi_{\rm B}(E_0,0,t)} \simeq \left(\frac{E_s^2}{2\varepsilon^2}\right)^k \frac{\bar{s}}{\bar{s}+2k}\left\{\frac{\phi_0^{(k)}(\bar{s};\lambda)}{\phi_{00}(\bar{s};\lambda))}\frac{K_0^{(k)}(\bar{s},-\bar{s}-2k)}{K_0^{(0)}(\bar{s},-\bar{s})}\right\}_{\lambda\to\lambda_1(\bar{s})} \quad \text{or} \tag{2.5}$$

$$\langle u^k\rangle \equiv \left\langle\left\{\frac{2\varepsilon^2\Delta}{E_s^2}\right\}^k\right\rangle \simeq \frac{\bar{s}}{\bar{s}+2k}\left\{\frac{\phi_0^{(k)}(\bar{s};\lambda)}{\phi_{00}(\bar{s};\lambda))}\frac{K_0^{(k)}(\bar{s},-\bar{s}-2k)}{K_0^{(0)}(\bar{s},-\bar{s})}\right\}_{\lambda\to\lambda_1(\bar{s})}, \tag{2.6}$$

where we introduced a new normalized variable of

$$u \equiv 2\varepsilon^2\Delta/E_s^2 \tag{2.7}$$

for the excess of path-length under Approximation B.

3. Plain descriptions of our Mellin transform $\langle u^\kappa\rangle$ under Approximation B

Let $dp_{\rm B}(u,\bar{s})/du$ be the probability density for electrons to show excess u of path-length in the shower of age $\bar{s}$. Mellin transform of the probability density is expressed as

$$\int_0^\infty u^\kappa \frac{dp_{\rm B}(u,\bar{s})}{du}du \equiv \langle u^\kappa\rangle, \tag{3.1}$$

which shows that the mean k-th moment $\langle u^k\rangle$ is the special value of the Mellin transform $\langle u^\kappa\rangle$ with κ at the integer k. So, we can obtain our Mellin transform $\langle u^\kappa\rangle$ by generalizing the mean k-th moment $\langle u^k\rangle$ from integer k to real κ with interpolation [1]. The results are described plainly as follows.

We express the functions of $\ln\{K_0^{(0)}(\bar{s},-\bar{s})\}_{\lambda\to\lambda_1(\bar{s})}$, $\ln\{K_0^{(1)}(\bar{s},-\bar{s}-2)\}_{\lambda\to\lambda_1(\bar{s})}$, and $\ln\{\Lambda(\bar{s})\}_{\lambda\to\lambda_1(\bar{s})}$ explicitly by quartic polynomials;

$$\ln\{K_0^{(0)}(\bar{s},-\bar{s})\}_{\lambda\to\lambda_1(\bar{s})} \simeq a_4\bar{s}^4 + a_3\bar{s}^3 + a_2\bar{s}^2 + a_1\bar{s} \qquad \text{with}$$
$$a_4 = -0.0130,\ a_3 = 0.144,\ a_2 = -0.522,\ a_1 = 1.20, \tag{3.2}$$
$$\ln\{K_0^{(1)}(\bar{s},-\bar{s}-2)\}_{\lambda\to\lambda_1(\bar{s})} \simeq b_4\bar{s}^4 + b_3\bar{s}^3 + b_2\bar{s}^2 + b_1\bar{s} + b_0 \qquad \text{with}$$
$$b_4 = 0.0176,\ b_3 = -0.239,\ b_2 = 1.10,\ b_1 = -1.11,\ b_0 = 3.24, \tag{3.3}$$
$$\ln\{\Lambda(\bar{s})\}_{\lambda\to\lambda_1(\bar{s})} \simeq c_4\bar{s}^4 + c_3\bar{s}^3 + c_2\bar{s}^2 + c_1\bar{s} + c_0 \qquad \text{with}$$
$$c_4 = -0.0101,\ c_3 = 0.155,\ c_2 = -0.984,\ c_1 = 4.09,\ c_0 = 2.54, \tag{3.4}$$

by interpolating the exact values of those at $\bar{s} = 1,\ 2,\ \cdots,$ and 5 derived through the recurrence equations, escaping from the converging ambiguities of infinite series for those at non-integer $\bar{s}$ [1].

Then we express $\ln\{\phi_0^{(k)}(\bar{s};\lambda)/\phi_{00}(\bar{s};\lambda)\}_{\lambda\to\lambda_1(\bar{s})}$ under Approximation B by quadratic function of κ;

$$\ln\left\{\frac{\phi_0^{(\kappa)}(\bar{s};\lambda)}{\phi_{00}(\bar{s};\lambda)}\right\}_{\lambda\to\lambda_1(\bar{s})} \simeq f_1\kappa + f_2\kappa^2 \equiv f(\kappa) \qquad \text{with} \tag{3.5}$$
$$f_1 = -\frac{1}{2}\ln\left\{\frac{\phi_0^{(2)}(\bar{s};\lambda)}{\phi_{00}(\bar{s};\lambda)}\right\}_{\lambda\to\lambda_1(\bar{s})} + 2\ln\left\{\frac{\phi_0^{(1)}(\bar{s};\lambda)}{\phi_{00}(\bar{s};\lambda)}\right\}_{\lambda\to\lambda_1(\bar{s})}, \quad f_2 = \ln\left\{\frac{\phi_0^{(1)}(\bar{s};\lambda)}{\phi_{00}(\bar{s};\lambda)}\right\}_{\lambda\to\lambda_1(\bar{s})} - f_1, \tag{3.6}$$

where they denote

$$\frac{\phi_0^{(1)}(s;\lambda)}{\phi_{00}(s;\lambda)} = \frac{\hat{v}^2 + (BC)_{s+2}}{D_{s+2}^2}, \tag{3.7}$$
$$\frac{\phi_0^{(2)}(s;\lambda)}{\phi_{00}(s;\lambda)} = \frac{2}{D_{s+2}D_{s+4}}\left[\frac{4\hat{v}^4 + 4\hat{v}\{2\hat{v} + (\lambda + A(s+4))\}(BC)_{s+4}}{D_{s+4}^2} + \frac{\hat{v}^2 + (BC)_{s+2}}{D_{s+2}}\frac{\hat{v}^2 + (BC)_{s+4}}{D_{s+4}}\right] \tag{3.8}$$

with $\hat{v} \equiv \lambda + \sigma_0$, $(BC)_s \equiv B(s)C(s)$, and $D_s \equiv (\lambda - \lambda_1(s))(\lambda - \lambda_2(s))$. On the other hand, as $K_0^{(\kappa)}(\bar{s},-\bar{s}-2\kappa)$ diverges at $\kappa = 2$ due to the pole of the second degree [1] we express $\ln\{(\kappa-2)^2K_0^{(\kappa)}(\bar{s},-\bar{s}-2\kappa)/(4K_0^{(0)}(\bar{s},-\bar{s}))\}_{\lambda\to\lambda_1(\bar{s})}$ by quadratic function of κ;

$$\ln\left\{\frac{(\kappa-2)^2K_0^{(\kappa)}(\bar{s},-\bar{s}-2\kappa)}{4K_0^{(0)}(\bar{s},-\bar{s})}\right\}_{\lambda\to\lambda_1(\bar{s})} \simeq g_1\kappa + g_2\kappa^2 \equiv g(\kappa) \qquad \text{with} \tag{3.9}$$
$$g_2 = \frac{1}{2}\ln\left\{\frac{\Lambda(\bar{s})}{K_0^{(0)}(\bar{s},-\bar{s})}\right\}_{\lambda\to\lambda_1(\bar{s})} - \ln\left\{\frac{K_0^{(1)}(\bar{s},-\bar{s}-2)}{K_0^{(0)}(\bar{s},-\bar{s})}\right\}_{\lambda\to\lambda_1(\bar{s})}, \quad g_1 = \ln\left\{\frac{K_0^{(1)}(\bar{s},-\bar{s}-2)}{4K_0^{(0)}(\bar{s},-\bar{s})}\right\}_{\lambda\to\lambda_1(\bar{s})} - g_2. \tag{3.10}$$

Thus we have our Mellin transform of $\langle u^\kappa\rangle$ as

$$\langle u^\kappa\rangle = \frac{\bar{s}/2}{\kappa+\bar{s}/2}\left\{\frac{\phi_0^{(\kappa)}(\bar{s};\lambda)}{\phi_{00}(\bar{s};\lambda)}\frac{K_0^{(\kappa)}(\bar{s},-\bar{s}-2\kappa)}{K_0^{(0)}(\bar{s},-\bar{s})}\right\}_{\lambda\to\lambda_1(\bar{s})} \simeq \frac{\bar{s}/2}{\kappa+\bar{s}/2}\frac{4}{(\kappa-2)^2}e^{f(\kappa)+g(\kappa)}. \tag{3.11}$$

Though our $\langle u^\kappa\rangle$ was generalized from $\langle u^k\rangle$ with interpolation within $0 < \kappa < 2$, we confirmed our $\langle u^\kappa\rangle$ is enough reliable up to the extended region of $-\bar{s}/2 \le \kappa$ [1].

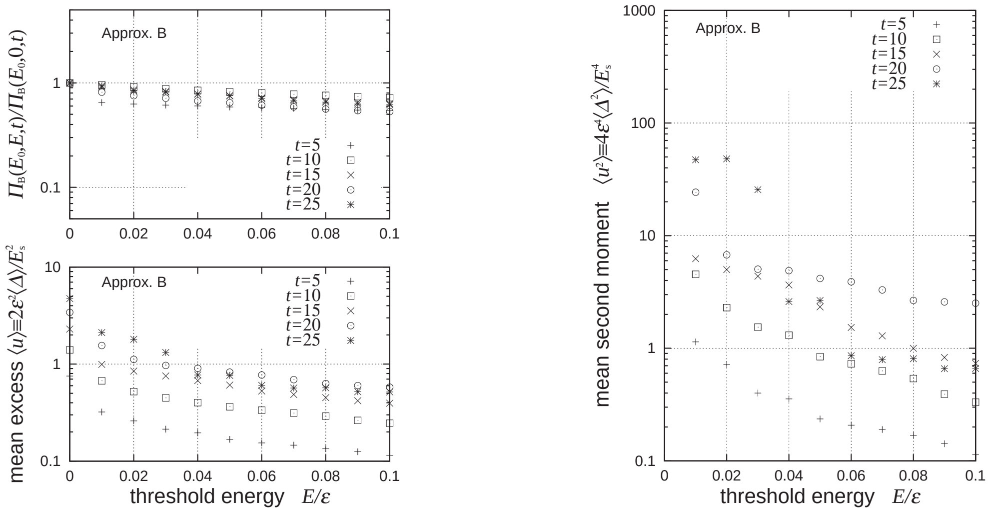

Figure 4: The threshold-energy E dependence of the number $\Pi_B(W_0, E, t)$ (top left), the mean first moment $\langle u \rangle$ (bottom left), and the mean second moment $\langle u^2 \rangle$ (right), appearing in the MC results with finite E 's. The results with $E = 0$ show those of analytically derived.

4. Mean moments of excess-path-length distribution for shower electrons

We indicate the analytical results [1] of mean excess $\langle u \rangle$ and root-mean-square excess $\sqrt{\langle u^2 \rangle - \langle u \rangle^2}$ of path-length for shower electrons under Approximation A in Figs. 1 and 2. We also indicate those of mean excess $\langle u \rangle$ under Approximation B for the total shower electrons with the threshold energy E of 0 in Fig. 3 (lines), which can be derived from the k-th moment of Eq. (2.6) with $k = 1$ and the age $\bar{s}$ determined by Eq. (2.4). Though, root-mean-square excess $\sqrt{\langle u^2 \rangle - \langle u \rangle^2}$ of path-length with the threshold energy E of 0 diverges under Approximation B, as $\langle u^2 \rangle$ determined by Eq. (2.6) with $k = 2$ diverges [1].

We compare the analytical results of mean excess $\langle u \rangle$ for shower electrons with the incident energy E_0 of $10^4\varepsilon$ and the threshold energy E of 0 under Approximation B (lines) with the MC results (dots) with E_0 of $10^4\varepsilon$ and E of 0.01 ε [2] in Fig. 3. We find the MC results show smaller values about a half compared with the analytical results, which disagreements come from the difference of the threshold energies E between the both.

We indicate in Fig. 4 the threshold-energy E dependence of the number $\Pi_B(W_0, E, t)$, the mean first moment $\langle u \rangle$, and the mean second moment $\langle u^2 \rangle$ of the shower electrons appearing in the MC results. The mean excesses $\langle u \rangle$ at E of 0.01ε appearing in the MC results also show about a half of those at E of 0 derived from the analytical $\langle u^\kappa \rangle$ of Eq. (2.6) with $k = 1$, as indicated in Fig 3. The mean second moments $\langle u^2 \rangle$ show strong dependence on the threshold energy E at finite energy regions, as indicated in Fig. 4. We have to take much care in evaluation of the threshold energy of E, in quantitative analyses of shower electrons relating to the root-mean-square width $\sqrt{\langle u^2 \rangle - \langle u \rangle^2}$ of shower electrons.

5. Excess-path-length distribution for shower electrons

We can derive the Δ- or u-weighted excess-path-length distribution under Approximation B [1], as

$$\Delta\frac{dP_{\rm B}(E_0,0,\Delta,t)}{d\Delta}=u\frac{dp_{\rm B}(u,\bar{s})}{du}\simeq\frac{1}{2\pi i}\int u^{-\kappa}\langle u^{\kappa}\rangle d\kappa \tag{5.1}$$

from our Mellin transform $\langle u^{\kappa}\rangle$ of Eq. (3.11), where $P_{\rm B}(E_0,0,\Delta,t)$ or $p_{\rm B}(u,\bar{s})$ denotes the probability for the total shower electrons (the threshold energy E of 0) to show their excess-path-lengths smaller than Δ or u. Thus we have

$$u\frac{dp_{\rm B}(u,\bar{s})}{du}\simeq\frac{2\bar{s}\bar{u}^{-\bar{\kappa}}e^{f(\bar{\kappa})+g(\bar{\kappa})}}{(\bar{\kappa}+\bar{s}/2)(2-\bar{\kappa})^2}\Big/\sqrt{2\pi\left\{f''(\bar{\kappa})+g''(\bar{\kappa})+\frac{1}{(\bar{\kappa}+\bar{s}/2)^2}+\frac{2}{(2-\bar{\kappa})^2}\right\}} \tag{5.2}$$

by the saddle point method, where the saddle point $\bar{\kappa}$ is taken at $-\bar{s}/2<\bar{\kappa}<2$ satisfying

$$\ln u=f'(\bar{\kappa})+g'(\bar{\kappa})-\frac{1}{\bar{\kappa}+\bar{s}/2}+\frac{2}{2-\bar{\kappa}}. \tag{5.3}$$

The results of excess-path-length distribution under Approximation A [1] and B are indicated in Figs. 5 and 6 (lines).

We find the probability density $dp_{\rm B}/du$ starts with $u^{\bar{s}/2-1}$ at the shower front of $u\ll 1$, due to the pole at $\kappa=-\bar{s}/2$ in our Mellin transform $\langle u^{\kappa}\rangle$ of Eq. (3.11). This fact is a characteristic property of the shower at the age $\bar{s}$, comparable with the fact that the lateral distribution decreases with $(\varepsilon^2r^2/E_{\rm s}^2)^{\bar{s}/2-1}$ near the shower axis of $r\ll 1$ [4]. The density also falls with $u^{-3}\ln u$ at $u\gg 1$, due to the pole of the second degree at $\kappa=2$ included in our $\langle u^{\kappa}\rangle$. Note that $dp_{\rm B}/du$ is function of only $\bar{s}$, and $dp_{\rm B}/du$ does not depend on the incident particle of electron or photon.

We compare our analytical results of excess-weighted probability density $u\,dp_{\rm B}/du$ for shower electrons with the incident energy E_0 of $10^4\varepsilon$ and the threshold energy E of 0 under Approximation B (lines) with those of the MC results [2] with E_0 of $10^4\varepsilon$ and E of 0.01 ε (dots) in Fig. 6. We find our analytical results of excess-path-length distribution agree fairly well with those derived by the MC method, in spite of the difference of the threshold energies E between the both.

6. Conclusions and discussions

Plain descriptions for our Mellin transform $\langle u^{\kappa}\rangle$ of excess-path-length distribution are proposed for shower electrons under Approximation B with the threshold energy E of 0 (Section 3).

The mean excesses $\langle u\rangle\equiv 2\varepsilon^2\langle\Delta\rangle/E_{\rm s}^2$ of path-length for shower electrons with the threshold energy E of 0 derived from Eq. (2.6) with $k=1$ are compared with those derived by the MC method with E of 0.01ε. The both increase similarly with the increase of traversed thickness, though show different values about twice due to difference of the threshold energies between the both (Fig. 3).

Threshold-energy E dependence of the mean k-th moment of the excess distribution is investigated for k of 0, 1, and 2 in the MC showers. The above difference in values of the mean excess $\langle u\rangle$ is indicated again. Strong dependence of the mean second moment $\langle u^2\rangle$ on the threshold energy is confirmed in finite regions of E (Fig. 4). Confirmation of the threshold energy will be important for

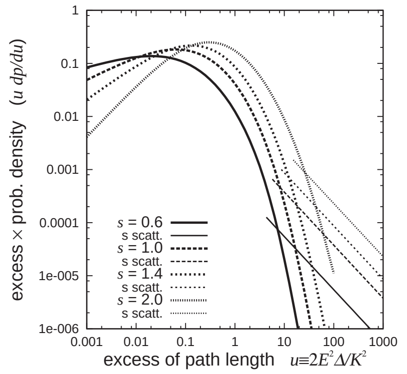

Figure 5: Probability densities of excess-path-length for shower electrons under Approximation A at $\bar{s}$ = 0.6, 1.0, 1.4, and 2.0 (thick lines), together with those determined by the single Rutherford scatterings (thin straight lines) [1].

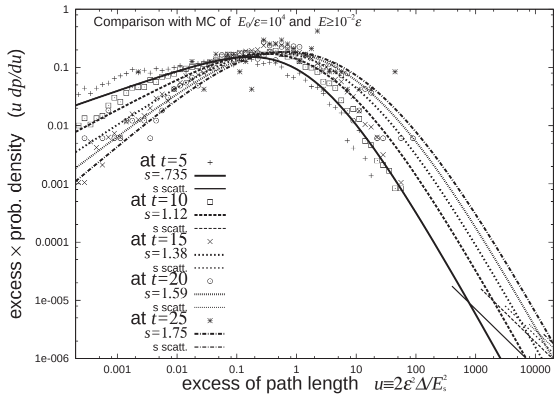

Figure 6: Probability densities of excess-path-length for shower electrons under Approximation B with the incident energy E_0 of $10^4\varepsilon$ and the threshold energy E of 0 at t = 5, 10, 15, 20, and 25 (thick lines) together with those determined by the single Rutherford scatterings (thin straight lines) [1], compared with those determined by the MC method with E_0 of $10^4\varepsilon$ and E of 0.01 ε (dots).

quantitative analyses of shower experiments relating to the root-mean-square width of the shower front (Fig. 4).

The excess-weighted probability densities $u\,dp/du$ for shower electrons with the threshold energy E of 0 derived analytically from our Mellin transform of $\langle u^\kappa \rangle$ are compared with those derived by the MC method with E of 0.01ε. The both agreed fairly well, in spite of the difference of the threshold energies between the both (Fig. 6).

7. Acknowledgments

The authors are very indebted to Prof. Jun Nishimura for his valuable advices and encouragements.

References

[1] T. Nakatsuka and K. Okei, to be submitted.

[2] T. Nakatsuka, K. Okei, H. Matsumoto, S. Tsuji, A. Iyono, S. Yamamoto, and J. Linsley, Proc. of *38th ICRC*, Nagoya, PoS(ICRC2023)571.

[3] B. Rossi and K. Greisen, Rev. Mod. Phys. **27**, 240(1941).

[4] J. Nishimura, in *Handbuch der Physik, Band 46*, edited by S. Flügge (Springer, Berlin, 1967), Teil **2**, p. 1.