Formulae to predict the excess-path-length distribution of cascade-shower electrons

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Theoretical investigations of excess-path-length distribution for cascade-shower electrons are important to understand the arrival-time-distribution of shower electrons observed in the air shower experiment. We acquired the formulae to describe the excess-path-length distribution by solving the diffusion equation of the cascade process under Approximation A and B. Reliability of the formulae is examined by comparing them with the distribution derived by the Monte Carlo calculations.
1. Introduction

Cascade-shower electrons show excess distribution of path-length due to the multiple Coulomb scattering with the matters of traverse, which is observed as the arrival-time-distribution of shower electrons in the air shower experiment. The distributions can be obtained from our Mellin transform of \( \langle u^k \rangle \), derived by solving the diffusion equation for the process [1].

Plain descriptions for the formulæ \( \langle u^k \rangle \) under Approximation B are proposed, and the mean excess and the excess distribution of path-length averaged over shower electrons derived from our \( \langle u^k \rangle \) are indicated. The results are compared with those derived by a Monte Carlo (abbreviated by MC, hereafter) calculation [2]. The threshold-energy \( E \) dependence of the results, appearing in the MC results, are also discussed.

2. The mean \( k \)-th moment of the excess-path-length distribution for shower electrons

Let \( \pi(E, \tilde{\theta}, \Delta, t)dEd\tilde{\theta}d\Delta \) and \( \gamma(E, \tilde{\theta}, \Delta, t)dEd\tilde{\theta}d\Delta \) be the numbers of electron and photon of energy \( E \), direction \( \tilde{\theta} \) and excess-path-length \( \Delta \) within the infinitesimal ranges of \( dE \), \( d\tilde{\theta} \) and \( d\Delta \), at the traversed thickness of \( t \) in the unit of radiation length \([3,4]\). Under the cascade process, \( \pi(E, \tilde{\theta}, \Delta, t) \) and \( \gamma(E, \tilde{\theta}, \Delta, t) \) satisfy the diffusion equation of

\[
\frac{\partial}{\partial t} \begin{pmatrix} \pi(E, \theta, \Delta, t) \\ \gamma(E, \theta, \Delta, t) \end{pmatrix} = \begin{pmatrix} -A' & B' \\ C' & -\sigma_0 \end{pmatrix} \begin{pmatrix} \pi \\ \gamma \end{pmatrix} - \frac{\theta^2}{2} \frac{\partial}{\partial \Delta} \begin{pmatrix} \pi \\ \gamma \end{pmatrix} + \frac{E^2}{4E^2} \nabla^2 \tilde{\theta} \begin{pmatrix} \pi \\ 0 \end{pmatrix} + \frac{e}{\partial E} \begin{pmatrix} \pi \\ 0 \end{pmatrix},
\]

(2.1)

where shower electrons lose their energies of \( \varepsilon dt \) in each traverse of \( dt \) by ionization with the critical energies \( \varepsilon \) of 0 (Approximation A) or finite values (Approximation B). The operators \( A' \), \( B' \), \( C' \) and the constants \( \sigma_0 \), \( \varepsilon \) are indicated in Nishimura [4]. Note that the variable \( \tilde{\theta} \) in the densities are expressed by \( \theta \) as \( \pi(E, \theta, \Delta, t) \) and \( \gamma(E, \theta, \Delta, t) \), as they are axially symmetric with \( \tilde{\theta} \).

We have the \( k \)-th moment of excess-path-length distribution for total shower electrons (with \( E \) from 0 to \( \infty \)) from the diffusion equation under Approximation B [1], as

\[
\Pi_B^{(k)}(E_0, 0, t) = \int_0^\infty dE \int_0^\infty 2\pi d\theta J_0(\zeta \theta) \int_0^\infty \Delta^2 \pi(E, \theta, \Delta, t) d\Delta
\]

\[
\simeq \frac{(E_0^2/2e^2)^k}{2\pi i} \int \frac{ds}{s} \left( \frac{E_0}{e} \right)^s e^{\lambda_1(s) t} \frac{s}{s + 2k} \frac{D_{\lambda_0}(s; \lambda)}{\lambda_1(s) - \lambda_2(s)} \left\{ K_0^{(k)}(s, -s - 2k) \right\}_{\lambda \rightarrow \lambda_1(s)}.
\]

(2.2)

Especially for \( k = 0 \), we have the total number of shower electrons

\[
\Pi_B(E_0, 0, t) \simeq \frac{1}{2\pi i} \int \frac{ds}{s} \left( \frac{E_0}{e} \right)^s e^{\lambda_1(s) t} \frac{D_{\lambda_0}(s; \lambda)}{\lambda_1(s) - \lambda_2(s)} \left\{ K_0^{(0)}(s, -s) \right\}_{\lambda \rightarrow \lambda_1(s)}
\]

\[
\simeq \Pi_A(E_0, \varepsilon, t) \left\{ K_0^{(0)}(\bar{s}, -\bar{s}) \right\}_{\lambda \rightarrow \lambda_1(\bar{s})}
\]

\[
\ln \frac{E_0}{\varepsilon} = -\lambda_1'(\bar{s}) t + \frac{1}{\bar{s}},
\]

(2.3)

(2.4)

indicated in the reviews of Rossi and Greisen, and Nishimura [3,4], where \( \Pi_A(E_0, \varepsilon, t) \) denotes the number of shower electrons under Approximation A and \( \bar{s} \) is called as the shower age. Thus we
have the mean $k$-th moment of excess-path-length averaged over the total shower electrons, as

$$
\langle \Delta^k \rangle = \frac{\Pi_B^{(k)}(E_0,0,t)}{\Pi_B(E_0,0,t)} \approx \left( \frac{E_s^2}{2E^2} \right)^k \frac{\bar{s}}{\bar{s}+2k} \left\{ \frac{\phi_0^{(k)}(\bar{s},\lambda)}{\phi_0(\bar{s},\lambda)} \right\} \text{ or } (2.5)
$$

$$
\langle \phi^k \rangle \equiv \left\{ \frac{2E^2\Delta}{E_s^2} \right\}^k \approx \frac{\bar{s}}{\bar{s}+2k} \left\{ \frac{\phi_0^{(k)}(\bar{s},\lambda)}{\phi_0(\bar{s},\lambda)} \right\} \lambda_{1}(\bar{s}) \text{ or } (2.6)
$$

where we introduced a new normalized variable of

$$
u \equiv 2E^2\Delta/E_s^2 \quad (2.7)
$$

for the excess of path-length under Approximation B.

3. Plain descriptions of our Mellin transform $\langle \phi^k \rangle$ under Approximation B

Let $dp_B(u,\bar{s})/du$ be the probability density for electrons to show excess $u$ of path-length in the shower of age $\bar{s}$. Mellin transform of the probability density is expressed as

$$
\int_0^\infty u^\kappa \frac{dp_B(u,\bar{s})}{du} du \equiv \langle \phi^\kappa \rangle, \quad (3.1)
$$

which shows that the mean $k$-th moment $\langle \phi^k \rangle$ is the special value of the Mellin transform $\langle \phi^\kappa \rangle$ with $\kappa$ at the integer $k$. So, we can obtain our Mellin transform $\langle \phi^k \rangle$ by generalizing the mean $k$-th moment $\langle \phi^k \rangle$ from integer $k$ to real $\kappa$ with interpolation [1]. The results are described plainly as follows.
We express the functions of \( \ln(K_0^{(0)}(\hat{s}, -\hat{s})) \) \( \lambda \rightarrow \lambda_1(\hat{s}) \), \( \ln(K_0^{(1)}(\hat{s}, -\hat{s} - 2)) \) \( \lambda \rightarrow \lambda_1(\hat{s}) \), and \( \ln(\Lambda(\hat{s})) \) \( \lambda \rightarrow \lambda_1(\hat{s}) \) explicitly by quartic polynomials;

\[
\begin{align*}
\ln(K_0^{(0)}(\hat{s}, -\hat{s})) & \sim a_4 \hat{s}^3 + a_3 \hat{s}^2 + a_2 \hat{s} + a_1 \hat{s} \quad \text{with} \\
& \quad a_4 = -0.0130, a_3 = 0.144, a_2 = -0.522, a_1 = 1.20, \quad (3.2) \\
\ln(K_0^{(1)}(\hat{s}, -\hat{s} - 2)) & \sim b_4 \hat{s}^4 + b_3 \hat{s}^3 + b_2 \hat{s}^2 + b_1 \hat{s} + b_0 \quad \text{with} \\
& \quad b_4 = 0.0176, b_3 = -0.239, b_2 = 1.10, b_1 = -1.11, b_0 = 3.24, \quad (3.3) \\
\ln(\Lambda(\hat{s})) & \sim c_4 \hat{s}^4 + c_3 \hat{s}^3 + c_2 \hat{s}^2 + c_1 \hat{s} + c_0 \quad \text{with} \\
& \quad c_4 = -0.0101, c_3 = 0.155, c_2 = -0.984, c_1 = 4.09, c_0 = 2.54, \quad (3.4)
\end{align*}
\]

by interpolating the exact values of those at \( \hat{s} = 1, 2, \cdots, 5 \) derived through the recurrence equations, escaping from the converging ambiguities of infinite series for those at non-integer \( \hat{s} \) [1].

Then we express \( \ln(\phi_0^{(1)}(\hat{s}; \lambda)/\phi_0(\hat{s}; \lambda)) \) \( \lambda \rightarrow \lambda_1(\hat{s}) \) under Approximation B by quadratic function of \( \kappa \);

\[
\begin{align*}
\ln \left\{ \frac{\phi_0^{(2)}(\hat{s}; \lambda)}{\phi_0(\hat{s}; \lambda)} \right\} \lambda \rightarrow \lambda_1(\hat{s}) & \approx f_1 \kappa + f_2 \kappa^2 \equiv f(\kappa) \quad \text{with} \\
& \quad f_1 = -\frac{1}{2} \ln \left\{ \frac{\phi_0^{(2)}(\hat{s}; \lambda)}{\phi_0(\hat{s}; \lambda)} \right\} \lambda \rightarrow \lambda_1(\hat{s}) + 2 \ln \left\{ \frac{\phi_0(\hat{s}; \lambda)}{\phi_0^{(1)}(\hat{s}; \lambda)} \right\} \lambda \rightarrow \lambda_1(\hat{s}) - f_1, \quad (3.5)
\end{align*}
\]

where they denote

\[
\begin{align*}
\phi_0^{(1)}(\hat{s}; \lambda) &= \hat{\nu}^2 + (BC)_{s+2} \\
\phi_0^{(2)}(\hat{s}; \lambda) &= \frac{2}{D_{s+2}D_{s+4}} \left[ 4\hat{\nu}^4 + 4\hat{\nu}\{2\hat{\nu} + (\lambda + A(s + 4))\}(BC)_{s+4} + \hat{\nu}^2 + (BC)_{s+2}\hat{\nu}^2 + (BC)_{s+4} \right] \quad (3.8)
\end{align*}
\]

with \( \hat{\nu} \equiv \nu + \sigma_0, (BC)_s \equiv B(s)C(s) \), and \( D_s \equiv (\lambda - \lambda_1(s))(\lambda - \lambda_2(s)) \). On the other hand, as \( K_0^{(0)}(\hat{s}, -\hat{s} - 2\kappa) \) diverges at \( \kappa = 2 \) due to the pole of the second degree [1], we express \( \ln\{ (\kappa - 2)^2K_0^{(0)}(\hat{s}, -\hat{s} - 2\kappa)/(4K_0^{(0)}(\hat{s}, -\hat{s})) \} \lambda \rightarrow \lambda_1(\hat{s}) \) by quadratic function of \( \kappa \);

\[
\begin{align*}
\ln \left\{ \frac{\nu(\kappa)(\hat{s}, \hat{s})}{K_0^{(0)}(\hat{s}, -\hat{s})} \right\} \lambda \rightarrow \lambda_1(\hat{s}) & \approx g_1 \kappa + g_2 \kappa^2 \equiv g(\kappa) \quad \text{with} \\
& \quad g_2 = \frac{1}{2} \ln \left\{ \frac{\Lambda(\hat{s})}{K_0^{(0)}(\hat{s}, -\hat{s})} \right\} \lambda \rightarrow \lambda_1(\hat{s}) + \ln \left\{ \frac{K_0^{(1)}(\hat{s}, -\hat{s} - 2)}{K_0^{(0)}(\hat{s}, -\hat{s})} \right\} \lambda \rightarrow \lambda_1(\hat{s}) - g_2. \quad (3.10)
\end{align*}
\]

Thus we have our Mellin transform of \( \langle u^k \rangle \) as

\[
\langle u^k \rangle = \frac{\langle \hat{s} \rangle}{\kappa + \langle \hat{s} \rangle} \left\{ \frac{\phi_0^{(1)}(\hat{s}; \lambda)}{\phi_0(\hat{s}; \lambda)} \right\} K_0^{(0)}(\hat{s}, -\hat{s} - 2\kappa) \quad \text{with} \\
& \quad \kappa = \langle \hat{s} \rangle + \frac{4\langle \hat{s} \rangle^2 - \kappa^2}{\langle \hat{s} \rangle^2} e^{\langle k \rangle + g(\kappa)}. \quad (3.11)
\]

Though our \( \langle u^k \rangle \) was generalized from \( \langle u^k \rangle \) with interpolation within \( 0 < \kappa < 2 \), we confirmed our \( \langle u^k \rangle \) is enough reliable up to the extended region of \( -\langle \hat{s} \rangle \leq \kappa \leq \langle \hat{s} \rangle \) [1].
4. Mean moments of excess-path-length distribution for shower electrons

We indicate the analytical results [1] of mean excess $\langle u \rangle$ and root-mean-square excess $\sqrt{\langle u^2 \rangle - \langle u \rangle^2}$ of path-length for shower electrons under Approximation A in Figs. 1 and 2. We also indicate those of mean excess $\langle u \rangle$ under Approximation B for the total shower electrons with the threshold energy $E$ of 0 in Fig. 3 (lines), which can be derived from the $k$-th moment of Eq. (2.6) with $k = 1$ and the age $\bar{s}$ determined by Eq. (2.4). Though, root-mean-square excess $\sqrt{\langle u^2 \rangle - \langle u \rangle^2}$ of path-length with the threshold energy $E$ of 0 diverges under Approximation B, as $\langle u^2 \rangle$ determined by Eq. (2.6) with $k = 2$ diverges [1].

We compare the analytical results of mean excess $\langle u \rangle$ for shower electrons with the incident energy $E_0$ of $10^4 \varepsilon$ and the threshold energy $E$ of 0 under Approximation B (lines) with the MC results (dots) with $E_0$ of $10^4 \varepsilon$ and $E$ of 0.01 $\varepsilon$ [2] in Fig. 3. We find the MC results show smaller values about a half compared with the analytical results, which disagreements come from the difference of the threshold energies $E$ between the both.

We indicate in Fig. 4 the threshold-energy $E$ dependence of the number $\Pi_B(W_0, E, t)$, the mean first moment $\langle u \rangle$, and the mean second moment $\langle u^2 \rangle$ of the shower electrons appearing in the MC results. The mean excesses $\langle u \rangle$ at $E$ of 0.01 $\varepsilon$ appearing in the MC results also show about a half of those at $E$ of 0 derived from the analytical $\langle u \kappa \rangle$ of Eq. (2.6) with $k = 1$, as indicated in Fig 3. The mean second moments $\langle u^2 \rangle$ show strong dependence on the threshold energy $E$ at finite energy regions, as indicated in Fig. 4. We have to take much care in evaluation of the threshold energy of $E$, in quantitative analyses of shower electrons relating to the root-mean-square width $\sqrt{\langle u^2 \rangle - \langle u \rangle^2}$ of shower electrons.
5. Excess-path-length distribution for shower electrons

We can derive the $\Delta$- or $u$-weighted excess-path-length distribution under Approximation B [1], as

$$\frac{dP_B(E_0,0,\Delta,t)}{d\Delta} = u \frac{dP_B(U_s)}{du} \approx \frac{1}{2\pi i} \int u^{-\kappa} \langle u^\kappa \rangle d\kappa$$  \hspace{1cm} (5.1)$$

from our Mellin transform $\langle u^\kappa \rangle$ of Eq. (3.11), where $P_B(E_0,0,\Delta,t)$ or $P_B(U_s)$ denotes the probability for the total shower electrons (the threshold energy $E_0$ of 0) to show their excess-path-lengths smaller than $\Delta$ or $u$. Thus we have

$$u \frac{dP_B(U_s)}{du} \approx \frac{2\bar{s}u^{-\kappa}e^{f(\bar{k})+g(\bar{k})}}{(\bar{k}+\bar{s}/2)(2-\bar{k})^2/\sqrt{2\pi} \int 2\bar{\kappa} \left\{ f''(\bar{k}) + g''(\bar{k}) + \frac{1}{(\bar{k}+\bar{s}/2)^2 + \frac{2}{(2-\bar{k})^2}} \right\}}$$  \hspace{1cm} (5.2)$$

by the saddle point method, where the saddle point $\bar{k}$ is taken at $-\bar{s}/2 < \bar{k} < 2$ satisfying

$$\ln u = f'(\bar{k}) + g'(\bar{k}) - \frac{1}{\bar{k}+\bar{s}/2} + \frac{2}{2-\bar{k}}.$$  \hspace{1cm} (5.3)$$

The results of excess-path-length distribution under Approximation A [1] and B are indicated in Figs. 5 and 6 (lines).

We find the probability density $dP_B/du$ starts with $u^{1/2-1}$ at the shower front of $u \ll 1$, due to the pole at $\kappa = -\bar{s}/2$ in our Mellin transform $\langle u^\kappa \rangle$ of Eq. (3.11). This fact is a characteristic property of the shower at the age $\bar{s}$, comparable with the fact that the lateral distribution decreases with $(\varepsilon^2 r^2 / E_0^2)^{1/2-1}$ near the shower axis of $r \ll 1$ [4]. The density also falls with $u^{-3} \ln u$ at $u \gg 1$, due to the pole of the second degree at $\kappa = 2$ included in our $\langle u^\kappa \rangle$. Note that $dP_B/du$ is function of only $\bar{s}$, and $dP_B/du$ does not depend on the incident particle of electron or photon.

We compare our analytical results of excess-weighted probability density $u dP_B/du$ for shower electrons with the incident energy $E_0$ of $10^4 \varepsilon$ and the threshold energy $E$ of 0 under Approximation B (lines) with those of the MC results [2] with $E_0$ of $10^4 \varepsilon$ and $E$ of 0.01 $\varepsilon$ (dots) in Fig. 6. We find our analytical results of excess-path-length distribution agree fairly well with those derived by the MC method, in spite of the difference of the threshold energies $E$ between the both.

6. Conclusions and discussions

Plain descriptions for our Mellin transform $\langle u^\kappa \rangle$ of excess-path-length distribution are proposed for shower electrons under Approximation B with the threshold energy $E$ of 0 (Section 3).

The mean excesses $\langle u \rangle \equiv 2\varepsilon^2 \langle \Delta \rangle / E_0^2$ of path-length for shower electrons with the threshold energy $E$ of 0 derived from Eq. (2.6) with $k = 1$ are compared with those derived by the MC method with $E$ of 0.01 $\varepsilon$. The both increase similarly with the increase of traversed thickness, though show different values about twice due to difference of the threshold energies between the both (Fig. 3).

Threshold-energy $E$ dependence of the mean $k$-th moment of the excess distribution is investigated for $k$ of 0, 1, and 2 in the MC showers. The above difference in values of the mean excess $\langle u \rangle$ is indicated again. Strong dependence of the mean second moment $\langle u^2 \rangle$ on the threshold energy is confirmed in finite regions of $E$ (Fig. 4). Confirmation of the threshold energy will be important for

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Figure 5: Probability densities of excess-path-length for shower electrons under Approximation A at \( \bar{s} = 0.6, 1.0, 1.4 \), and 2.0 (thick lines), together with those determined by the single Rutherford scatterings (thin straight lines) [1].

Figure 6: Probability densities of excess-path-length for shower electrons under Approximation B with the incident energy \( E_0 \) of \( 10^4 \varepsilon \) and the threshold energy \( E \) of 0 at \( t = 5, 10, 15, 20, \) and 25 (thick lines) together with those determined by the single Rutherford scatterings (thin straight lines) [1], compared with those determined by the MC method with \( E_0 \) of \( 10^4 \varepsilon \) and \( E \) of 0.01 \( \varepsilon \) (dots).

quantitative analyses of shower experiments relating to the root-mean-square width of the shower front (Fig. 4).

The excess-weighted probability densities \( \mu \frac{d\mu}{d\mu} \) for shower electrons with the threshold energy \( E \) of 0 derived analytically from our Mellin transform of \( \langle \mu^k \rangle \) are compared with those derived by the MC method with \( E \) of 0.01 \( \varepsilon \). The both agreed fairly well, in spite of the difference of the threshold energies between the both (Fig. 6).

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References


