Endpoint divergences in inclusive $\bar{B} \to X_s \gamma$

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The subleading so-called resolved contributions represent the largest uncertainty in the inclusive decay mode $\bar{B} \to X_s \gamma$. However there had been no complete proof of factorization of these subleading contributions. This failure of factorisation can be traced back to endpoint divergences and cured by recently proposed refactorisation techniques.

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1. Introduction

The inclusive decay modes $\bar{B} \to X_{s,d}\gamma$ and $\bar{B} \to X_{s,d}\ell^+\ell^-$ are golden modes of the indirect search for new physics at the Belle-II experiment [1–3]. The Belle II experiment will collect two orders of magnitude larger data samples than the B factories [4] and will measure the inclusive decays with the highest precision. Therefore, subleading effects have become now relevant on the theory side.

Within the heavy mass expansion (HME), these inclusive so-called penguin modes are dominated by the partonic contributions, which can be calculated perturbatively, and sub-leading contributions start at the quadratic level, $(\Lambda/m_b)^2$ only. However, it is well known that this operator product expansion breaks down in these inclusive modes if one considers operators beyond the leading ones. This breakdown manifests in nonlocal power corrections, also called resolved contributions. They are characterised by containing subprocesses in which the photon couples to light partons instead of connecting directly to the effective weak-interaction vertex [5].

These resolved contributions can be systematically calculated using soft-collinear effective theory (SCET). In the case of the inclusive $\bar{B} \to X_s \gamma$ decay, all resolved contributions to $O(1/m_b)$ have been calculated some time ago [6]. Also, the analogous contributions to the inclusive $\bar{B} \to X_{s,d}\ell^+\ell^-$ decays have been analysed to $O(1/m_b)$ [7, 8].

Recently, the uncertainty due to the resolved contribution was reduced with the help of a new hadronic input [9, 10]. But these resolved contributions still represent the largest uncertainty of order $4 - 5\%$ in the prediction of the inclusive decay rate $\bar{B} \to X_s \gamma$ [12] and of the low-q^2 observables of $\bar{B} \to X_{s,d}\ell^+\ell^-$ [13–15].

Moreover, a large scale dependence and also a large charm mass dependence were identified in the lowest order result of the resolved contribution, which calls for a systematic calculation of α_s corrections and renormalisation group (RG) summation [10, 16]. A mandatory prerequisite for this task is an all-order in the strong coupling constant α_s factorisation formula for the subleading power corrections.

One finds a factorisation formula for the various contributions to the inclusive penguin decays [6], where the symbol $\otimes$ denotes the convolution of the soft and jet functions.

$$d\Gamma(\bar{B} \to X_s \gamma, \ell^+\ell^-) = \sum_{n=0}^{\infty} \frac{1}{m_b^n} \sum_i H_i^{(n)} J_i^{(n)} \otimes S_i^{(n)}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{m_b^n} \left[\sum_i H_i^{(n)} J_i^{(n)} \otimes S_i^{(n)} \otimes \bar{J}_i^{(n)} + \sum_i H_i^{(n)} J_i^{(n)} \otimes S_i^{(n)} \otimes \bar{J}_i^{(n)} \otimes \bar{J}_i^{(n)} \right]. \quad (1)$$

The first line describes the so-called direct contributions, while the second line contains the resolved contributions. The latter appear first only at the order $1/m_b$ in the heavy-quark expansion. Here hard functions $H_i^{(n)}$ describe physics at the high scale m_b. $J_i^{(n)}$ are the so-called jet functions which represent the physics of the hadronic final state X_s at the intermediate hard-collinear scale $\sqrt{m_b \Lambda_{\text{QCD}}}$. The soft functions $S_i^{(n)}$, the so-called shape functions, parameterise the hadronic physics at the scale Λ_{QCD}. Within the resolved contributions, we have a new ingredient in the factorisation formula, the so-called anti-hardcollinear jet functions $\bar{J}_i^{(n)}$ due to the coupling of virtual photons with virtualities of order $\sqrt{m_b \Lambda_{\text{QCD}}}$ to light partons instead of the weak vertex directly. They are

not represented by cut propagators as the usual jet functions but as full propagator functions dressed by Wilson lines.

However, the specific resolved $O_{8g} - O_{8g}$ contribution does not factorise because the convolution integral is ill-defined. The authors of Ref. [6] claimed that there is an essential difference between divergent convolution integrals in power-suppressed contributions of exclusive B decays and the divergent convolution integral in the present case, while the former were of IR origin, the latter divergence were of UV nature. Nevertheless, using a hard cut-off in the resolved contribution, the sum of direct and resolved $O_{8g} - O_{8g}$ contributions was shown to be scale and scheme independent at the lowest order. However the failure of factorisation did not allow for a consistent resummation of large logarithms. In a recent paper, the divergences in the resolved and in the direct contributions were identified as endpoint divergences. It was shown that also the divergence in the direct contribution can be traced back to a divergent convolution integral [17].

Recently new techniques [18–21] were presented in specific collider applications, which allow for an operator-level reshuffling of terms within the factorisation formula so that all endpoint divergences cancel out. This idea of refactorisation was now implemented in this flavour example of the resolved contributions, which includes nonperturbative soft functions, and the subleading shape functions, not present in collider applications [17]. This is the first QCD application of these new refactorsiation techniques in flavour physics.

In the following, we present the various steps of this analysis. This analysis leads to a renormalised factorisation theorem on the operator level for these resolved contributions to all orders in the strong coupling constant. This result establishes the validity of the general factorisation theorem, given in Eq. 1, - also for the $O_{8g} - O_{8g}$ contributions. This theorem now allows for higher-order calculations of the resolved contributions and consistent summation of large logarithms [17].

2. Matching on SCET and degeneracies in the EFT

The first step in the derivation of a factorisation theorem is hard matching. We have to match the electroweak operator onto SCET. The operator under consideration, O_8 [1]:

$$O_{8g} = -\frac{g_s}{8\pi^2} m_b \, \bar{s} \sigma_{\mu\nu} (1 + \gamma_5) G^{\mu\nu} b \,, \tag{2}$$

matches onto two possible SCET operators. The direct contribution is represented by a next-to-leading power (NLP) B-type current in SCET, i.e. a power-suppressed current composed of two collinear building blocks (see left figure of Figure 2):

$$O_{8g}^{B1} (u) = \int \frac{dt}{2\pi} e^{-i u m_b t} \overline{\chi}_{hc} (t\bar{n}) \, \gamma_{\nu\perp} \, Q_s \, \mathcal{B}^{\nu}{}_{\overline{hc}\perp} (0) \, \gamma_{\mu\perp} \, \mathcal{A}^{\mu}{}_{hc\perp} (0) \, (1 + \gamma_5) \, h (0) \,, \tag{3}$$

with Q_s as the electric charge of the strange quark in units of e and the electromagnetic gauge-invariant transverse photon field

$$\mathcal{B}^{\nu}{}_{\overline{hc}\perp} = e \left(A^{\nu}_{\perp} - \frac{\partial^{\nu}_{\perp}}{n\partial} nA \right). \tag{4}$$

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[1] For the various conventions we refer the reader to Ref. [17]
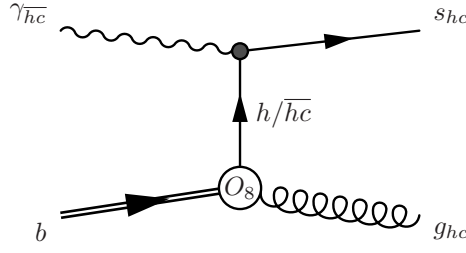
Figure 1: QCD diagram at LO, see text.
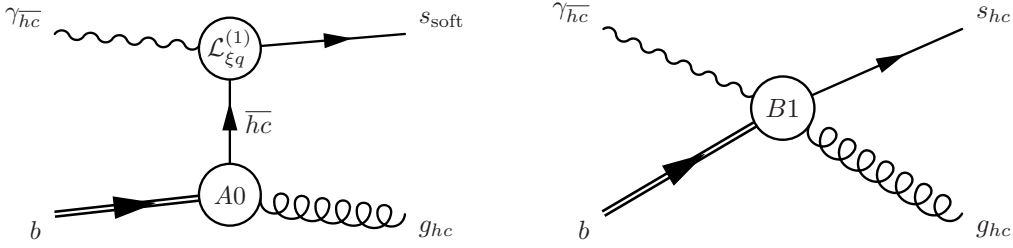
Figure 2: SCET diagrams: direct (left) and resolved (right) contributions, see text.

The second operator in SCET on which the electroweak operator O_{8g} matches is a leading-power (LP) *A*-type current, i.e. a leading-power current including one collinear building block, :

$$O_{8g}^{A0}(0) = \overline{\chi_{\overline{hc}}}(0) \frac{\slashed{n}}{2} \gamma_{\mu\perp} \mathcal{A}^{\mu}_{hc\perp}(0)(1+\gamma_5) h(0) \, , \tag{5}$$

with the gauge invariant gluon field $\mathcal{A}^{\mu}_{hc\perp} = W^{\dagger}_{hc}\left[D^{\mu}_{hc\perp}W_{hc}\right] = \mathcal{A}^{a\mu}_{hc\perp}t^a$. Then the resolved contribution is represented by a time-ordered product of this leading-power (LP) *A*-type current, with a subleading $\mathcal{L}^{(1)}_{\xi q}$ Lagrangian, see right figure of Fig. 2.

We can compare the different kinematics of the diagrams with *A*- and *B*-type currents in Figure 2. The external s-quark carries hardcollinear momentum. Therefore the intermediate propagator is hard. This situation is represented in SCET by the *B*-type current. When the momentum of the external s-quark tends to zero, the propagator becomes anti-hardcollinear and cannot be integrated out – it must be reproduced by a dynamical field in the low energy EFT. This situation is represented in SCET by the time-order product of subleading Lagrangian and the *A*-type current. The degeneracy in the EFT description is the reason why the SCET develops divergencies in the convolution integrals.

For the explicit matching of the two currents at leading order (LO) we find

$$\mathbf{C^{B1}_{LO}(m_b, u)} = (-1)\frac{\overline{u}}{u}\frac{m_b^2}{4\pi^2}\frac{G_F}{\sqrt{2}}\lambda_t \, C_{8g} = (-1)\frac{\overline{u}}{u}\mathbf{C^{A0}_{LO}(m_b)} \, , \tag{6}$$

where we use the hardcollinear momentum fraction $u = \frac{\bar{n}p_s}{m_b}$ and $\overline{u} = 1 - u = \frac{\bar{n}p_r}{m_b}$ with the hardcollinear momenta of the strange quark and the gluon, $\bar{n}p_s$ and $\bar{n}p_r$ in the direct contribution.
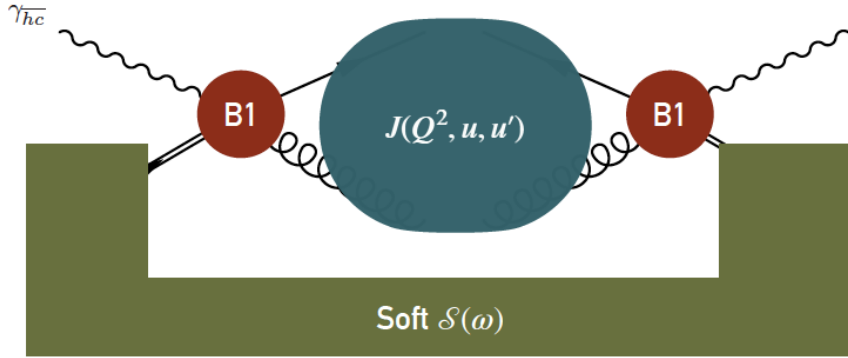
Figure 3: Schematic description of the factorisation of the direct contribution, see text.

3. Bare factorisation theorem and endpoint divergences

The derivation of the factorisation theorem follows the standard approach [22–24]. We first perform the soft decoupling transformation [25], but we do not use a new notation for the hardcollinear fields after decoupling. The decay rate is obtained from the imaginary part of the current-current correlator. The states factorise and thus allow taking matrix elements separately in hardcollinear, anti-hardcollinear and soft sectors. The hardcollinear matrix elements lead to jet functions. Integration of the anti-hardcollinear fields at the amplitude level leads to antihardcollinear (radiative) jet functions. We can then simplify the Dirac and colour structure, and the remaining soft matrix element defines the shape functions.

For the direct contribution we find the bare factorisation theorem on the operator level (see Figure 3 for a schematic description):

$$\frac{d\Gamma}{dE_\gamma} = \mathcal{N}_B \int_0^1 du \, \mathbf{C^{B1}}(\mathbf{m_b, u}) \int_0^1 du' \mathbf{C^{B1*}}(\mathbf{m_b, u'}) \int_{-p_+}^{\overline{\Lambda}} d\omega \, \mathbf{J}(\mathbf{M_B}(\mathbf{p_+} + \omega), \mathbf{u, u'})\mathcal{S}(\omega) \quad (7)$$

$\mathcal{N}_B$ is a prefactor [2]. The hard function $\mathbf{C^{B1}}$ is given at LO in the last section as a result of the matching of QCD on SCET. The hardcollinear jet function is a genuine next-to-leading (NLP) object. We define it as a vacuum matrix element of a product of hardcollinear fields:

$$\mathbf{J}\left(\mathbf{p^2, u, u'}\right) = \frac{(-1)}{2N_c} \frac{1}{2\pi} \int \frac{dt dt'}{(2\pi)^2} \, d^4x \, e^{-im_b(ut - u't') + ipx} \quad (8)$$

$$\text{Disc}\left[\langle 0| \, tr\left[\frac{1 + \slashed{v}}{2}(1 - \gamma_5)\mathcal{A}_{hc\perp}(x) \gamma_\perp^\nu \chi_{hc}(t'\bar{n} + x) \overline{\chi}_{hc}(t\bar{n}) \gamma_{\nu\perp} \mathcal{A}_{hc\perp}(0)(1 + \gamma_5)\right]|0\rangle\right] .$$

The soft function - the leading power (LP) shape function - is defined as [26]

$$\mathcal{S}(\omega) = \frac{1}{2m_B} \int \frac{dt}{2\pi} e^{-i\omega t} \langle B| \, h(tn) \, S_n(tn) \, S_n^\dagger(0) \, h(0) \, |B\rangle . \quad (9)$$

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[2] $\mathcal{N}_B = e^2 Q_s^2 \frac{E_\gamma}{2\pi}$

with soft Wilson lines denoted by S.

There is an endpoint divergence in the convolution of the hard matching coefficients and the jet function within the direct contribution. At LO, one finds explicitly with the hard matching coefficient $\mathbf{C_{LO}^{B1}\,(m_b, u)}$ given in Eq. 6 and the LO jet function given by [3]

$$\mathbf{J}\left(\mathbf{p^2}, \mathbf{u}, \mathbf{u'}\right) = C_F \frac{\alpha_s}{4\pi\, m_b} \theta(p^2)\, A(\epsilon)\, \delta(u - u')u^{1-\epsilon}(1 - u)^{-\epsilon}\left(\frac{p^2}{\mu^2}\right)^{-\epsilon} \tag{10}$$

that the convolution of both quantities. diverges in the $u \to 0$ limit:

$$\int_0^1 \frac{du}{u} \int_0^1 \frac{du'}{u'} \delta(u - u')u^{1-\epsilon} \approx \int_0^1 \frac{du}{u^{1+\epsilon}} \ . \tag{11}$$

Consequently, due to endpoint divergences, the bare factorisation formula of the direct contribution is already invalid for the $d \to 4$ limit at LO.

For the resolved contribution we find the following factorisation theorem on the operator level (for a schematic description of the factorisation of the resolved contribution see Figure 4):

$$\frac{d\Gamma}{dE_\gamma} = \mathcal{N}_A \left|\mathbf{C^{A0}\,(m_b)}\right|^2 \int_{-p_+}^{\overline{\Lambda}} d\omega\, \mathbf{J_g}\left(m_b\,(p_+ + \omega)\right) \int d\omega_1 \int d\omega_2\, \overline{\mathbf{J}}\,(\omega_1)\, \overline{\mathbf{J}}^*\,(\omega_2)\, S\,(\omega, \omega_1, \omega_2)\ , \tag{12}$$

with the prefactor $\mathcal{N}_A = \mathcal{N}_B \equiv \mathcal{N}$. The hard function $\mathbf{C^A}$ is given at LO in the last section as a result of the matching of QCD on SCET again. In the hardcollinear sector, there are only gluon fields, so the standard LO gluon jet function $\mathbf{J_g(p^2)}$ appears. The anti-hardcollinear jet function is defined on the amplitude level because there is no energetic particle emitted in the anti-hardcollinear directions besides the photons:

$$O_{T\xi q} = i \int d^d x T \left[\mathcal{L}_{\xi q}\,(x)\,, O_{8g}^{A0}\,(0) \right]$$

$$= \int d\omega \int \frac{dt}{2\pi} e^{-it\omega} \left[\overline{q_s}\right]_\alpha (tn) \left[\overline{\mathbf{J}}\,(\omega)\right]_{\alpha\beta}^{\mathbf{a}\,\nu\mu} Q_s\, \mathcal{B}^\nu_{\overline{hc\perp}}\,(0)\, \mathcal{A}_{hc\perp}^{\mu\,a}\,(0)\, [h\,(0)]_\beta\ . \tag{13}$$

One can prove that the antihardcollinear jet function has a unique structure to all orders in α_s:

$$\left[\overline{\mathbf{J}}\,(\omega)\right]_{\alpha\beta}^{\mathbf{a}\,\nu\mu} = \overline{J}\,(\omega)\, t^a \left[\gamma_\perp^\nu \gamma_\perp^\mu \frac{\slashed{\bar{n}}\slashed{n}}{4}\right]_{\alpha\beta}\ , \tag{14}$$

Finally, the soft function is given by

$$S\,(\mathbf{u}, \mathbf{t}, \mathbf{t'}) = (d - 2)^2 g_s^2 \langle B|\, \overline{h}\,(un)\,(1 - \gamma_5) \left[S_n\,(un)\, t^a S_n^\dagger\,(un)\right] S_{\bar{n}}\,(un)\, S_{\bar{n}}^\dagger\,(t'\bar{n} + un) \tag{15}$$

$$\frac{\slashed{n}\slashed{\bar{n}}}{4} q_s\,(t'\bar{n} + un)\, \overline{q}_s\,(t\bar{n})\, \frac{\slashed{\bar{n}}\slashed{n}}{4} S_{\bar{n}}\,(t\bar{n})\, S_{\bar{n}}^\dagger\,(0) \left[S_n\,(0)\, t^a S_n^\dagger\,(0)\right] (1 + \gamma_5)\, h\,(0)\, |B\rangle\, /\,(2m_B)\ .$$

The Fourier transform is defined as

$$S\,(\omega, \omega_1, \omega_2) = \int \frac{du}{2\pi} e^{-iu\omega} \int \frac{dt}{2\pi} e^{-it\omega_1} \int \frac{dt'}{2\pi} e^{it'\omega_2} S\,(\mathbf{u}, \mathbf{t}, \mathbf{t'})\ . \tag{16}$$

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[3]$A(\epsilon)$ is a function which is finite for $\epsilon \to 0$.

$$\mathcal{S}\left(\omega, \omega_1, \omega_2\right) = \int \frac{du}{2\pi} e^{-iu\omega} \int \frac{dt}{2\pi} e^{-it\omega_1} \int \frac{dt'}{2\pi} e^{it'\omega_2} \mathcal{S}\left(u, t, t'\right)$$
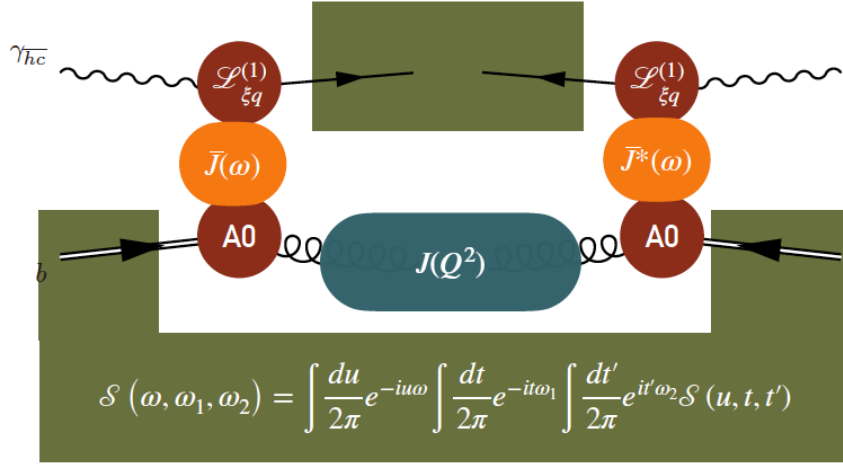
Figure 4: Schematic description of the factorisation of the resolved contribution, see text.

At LO the factorization formula is well behaved as long as $\omega_1, \omega_2 \sim \omega$:

$$\frac{d\Gamma}{dE_\gamma} = 2\mathcal{N}_A \left|\mathbf{C_{LO}^{A0}(m_b)}\right|^2 \int_{-p_+}^{\overline{\Lambda}} d\omega \; \delta\left(\mathbf{m_b(p_+ + \omega)}\right) \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\omega_1} d\omega_2 \frac{1}{(\omega_1 - i\epsilon)} \frac{1}{(\omega_2 + i\epsilon)} \mathcal{S}(\omega, \omega_1, \omega_2). \tag{17}$$

However, for $\omega_{1,2} \gg \omega$, the soft function can be shown to be asymptotically constant, which leads to endpoint divergence in the convolution integrals of jet and shape functions for large $\omega_{1,2}$.

In this limit, the soft strange quarks become hardcollinear and can be decoupled from the soft gluons. This way, the soft function reduces to the LP shape function, which also occurs in the direct contribution. This can be shown explicitly taking into account that the $\omega_{1,2} \to \infty$ limit corresponds to the $t, t' \to 0$ limit in Eq. 15. In fact, the soft function in the limit $\omega_{1,2}$, denoted by $\widetilde{S}$, can be matched on the LP shape function with a perturbative kernel K:

$$\widetilde{S}\left(\omega, \omega_1, \omega_2\right) = \int d\omega' K(\omega, \omega', \omega_1, \omega_2) S(\omega') . \tag{18}$$

At LO, we find:

$$\widetilde{S}\left(\omega, \omega_1, \omega_2\right) = C_F A(\epsilon) \frac{\alpha_s}{(4\pi)} \omega_1^{1-\epsilon} \delta(\omega_1 - \omega_2) \int_\omega^{\overline{\Lambda}} d\omega' \, S(\omega') \left(\frac{(\omega' - \omega)}{\mu^2}\right)^{-\epsilon} , \tag{19}$$

which includes the leading power shape function $S(\omega)$.

This makes the endpoint divergence in the convolution integral of jet and shape function within the resolved contribution in the asymptotic limit $\omega_{1,2} \to \infty$ manifest. In this limit, the resolved contribution at LO can be written as:

$$\frac{d\Gamma}{dE_\gamma}\Big|_A^{\text{asy}} = 2\mathcal{N} |\mathbf{C_{LO}^{A0}(m_b)}|^2 \int_{-p_+}^{\overline{\Lambda}} d\omega \mathbf{J_g^{LO}(m_b(p_+ + \omega))} \int_{m_b}^{\infty} d\omega_1 \overline{\mathbf{J}}_{\mathbf{LO}}(\omega_1) \int_0^{\omega_1} d\omega_2 \overline{\mathbf{J}}_{\mathbf{LO}}^*(\omega_2) \, \widetilde{S}(\omega, \omega_1, \omega_2)$$

$$= \mathcal{N} |\mathbf{C_{LO}^{A0}(m_b)}|^2 \frac{\alpha_s C_F}{(4\pi) m_b} \frac{1}{\epsilon} A(\epsilon) \int_{-p_+}^{\overline{\Lambda}} d\omega \, S_{\mathbf{LO}}(\omega') \left(\frac{m_b(\omega + p_+)}{\mu^2}\right)^{-\epsilon} . \tag{20}$$

7

4. Refactorisation to all orders in α_s

Evaluating the direct contribution in the asymptotic limit $u, u' \rightarrow 0$ (before integration)

$$\frac{d\Gamma}{dE_\gamma}|_B^{u,u'\rightarrow 0} = -\mathcal{N} \left|\mathbf{C_{LO}^{A0}(m_b)}\right|^2 \frac{\alpha_s C_F}{(4\pi) m_b} \frac{1}{\epsilon} A(\epsilon) \int_{-p_+}^{\overline{\Lambda}} d\omega\, \mathcal{S}_{LO}(\omega) \left(\frac{m_b(\omega + p_+)}{\mu^2}\right)^{-\epsilon}, \quad (21)$$

and comparing with the asymptotic limit ($\omega_1, \omega_2 \rightarrow \infty$) of the resolved contribution given in Eq. 20 one verifies at LO

$$\frac{d\Gamma}{dE_\gamma}|_A^{\text{asy}} = (-1)\frac{d\Gamma}{dE_\gamma}|_B^{u,u'\rightarrow 0}. \quad (22)$$

Thus, the sum of the two terms is finite and equal to zero, and the endpoint divergences cancel. This reflects the fact that in the limits $u \rightarrow 0$ and $\omega_1 \sim \omega_2 \gg \omega$ the two terms of the subleading $O_8 - O_8$ contribution have the same structure. This LO result is a special case of the all-order relation which can be formulated at the operator level.

One can derive three refactorisation conditions which reflect this fact that in the limits $u \sim u' \ll 1$ and $\omega_1 \sim \omega_2 \gg \omega$ the resolved and the direct contribution have the same structure to all orders in α_s. The refactorisation relations are operatorial relations that then guarantee the cancellation of endpoint divergences between the two terms to all orders in α_s as we will see below.

- We find that in the limit $u \rightarrow 0$, the matching coefficient can be further factorised

$$\left[\!\left[\mathbf{C^{B1}(m_b, u)}\right]\!\right] = (-1)\mathbf{C^{A0}(m_b)}\, m_b\, \overline{\mathbf{J}}(\mathbf{um_b}), \quad (23)$$

where $\left[\!\left[g(u)\right]\!\right]$ only denotes the leading term of a function $g(u)$ in the limit $u \rightarrow 0$ and without any higher power corrections in $u \ll 1$.

- We find the new soft function $\widetilde{S}(\omega, \omega_1, \omega_2)$ which corresponds to the function $S(\omega, \omega_1, \omega_2)$ in the limit $\omega_1 \sim \omega_2 \gg \omega$. In this limit, we can consider the light soft quarks to be hard-collinear. In this function $\widetilde{S}(\omega, \omega_1, \omega_2)$ higher power corrections in $\omega/\omega_{1,2}$ are neglected.

- We find that the jet function $J(m_b(p_+ + \omega), u, u')$ fulfills the following relation in the limit $u \rightarrow 0$ and $u' \rightarrow 0$:

$$\int_{-p_+}^{\overline{\Lambda}} d\omega \left[\!\left[\mathbf{J}(\mathbf{m_b}(\mathbf{p_+} + \omega), \mathbf{u}, \mathbf{u'})\mathcal{S}(\omega)\right]\!\right] = \int_{-p_+}^{\overline{\Lambda}} d\omega \mathbf{J_g}(\mathbf{m_b}(\mathbf{p_+}+\omega))\widetilde{S}(\omega, \mathbf{m_b u}, \mathbf{m_b u'}), \quad (24)$$

where the brackets indicate that the $u \rightarrow 0$ *and* $u' \rightarrow 0$ limits have to be taken.

Using these all-orders refactorisation conditions we then can cast the two asymptotic subtraction terms into the following form:

$$0 = 2\mathcal{N}\left|\mathbf{C^{A0}}(m_b)\right|^2 \int_{-p_+}^{\overline{\Lambda}} d\omega \mathbf{J_g}(\mathbf{m_b}(\mathbf{p_+} + \omega)) \int_{m_b}^{\infty} d\omega_1 \overline{\mathbf{J}}(\omega_1) \int_0^{\omega_1} d\omega_2 \overline{\mathbf{J}}^*(\omega_2)\widetilde{S}(\omega, \omega_1, \omega_2)$$

$$+ 2\mathcal{N}\int_0^1 du \left[\!\left[\mathbf{C^{B1}(m_b, u)}\right]\!\right] \int_u^1 du' \left[\!\left[\mathbf{C^{B1*}(m_b, u')}\right]\!\right] \int_{-p_+}^{\overline{\Lambda}} d\omega \left[\!\left[\mathbf{J}(\mathbf{m_b}(\mathbf{p_+} + \omega), \mathbf{u}, \mathbf{u'})\mathcal{S}(\omega)\right]\!\right]. \quad (25)$$

We then subtract these asymptotic terms from the sum of the two all-order bare factorisation theorems for resolved and direct contribution derived in the last section (see Eqs. 7 and 12) and obtain the endpoint finite factorisation theorem.

$$
\frac{d\Gamma}{dE_\gamma}\Big|_{A+B} = 2\mathcal{N} \int_{-p_+}^{\overline{\Lambda}} d\omega \Bigg\{ \mathbf{J_g}(\mathbf{m_b}(\mathbf{p_+} + \omega)) \left|\mathbf{C^{A0}\left(m_b\right)}\right|^2
$$

$$
\times \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\omega_1} d\omega_2 \overline{\mathbf{J}}(\omega_1)\,\overline{\mathbf{J}}^*(\omega_2) \left[\mathcal{S}\left(\omega, \omega_1, \omega_2\right) - \theta(\omega_1 - m_b)\theta(\omega_2)\widetilde{\mathcal{S}}(\omega, \omega_1, \omega_2) \right]
$$

$$
+ \int_0^1 du \int_u^1 du' \left[\mathbf{C^{B1}\left(m_b, u\right)}\ \mathbf{C^{B1*}\left(m_b, u'\right)}\ \mathbf{J\left(m_b\left(p_+ + \omega\right), u, u'\right)}\mathcal{S}\left(\omega\right) \right.
$$

$$
\left. - \left[\!\left[\mathbf{C^{B1}\left(m_b, u\right)}\right]\!\right] \left[\!\left[\mathbf{C^{B1*}\left(m_b, u'\right)}\right]\!\right] \left[\!\left[\mathbf{J\left(m_b\left(p_+ + \omega\right), u, u'\right)}\mathcal{S}(\omega)\right]\!\right] \right] \Bigg\}, \tag{26}
$$

At this point, the convolutions integrals in the A- and B-type contributions are no longer divergent, and we can renormalise the functions entering the factorisation theorem and take the limit $d \to 4$. In a final step, we are able to show that endpoint reshuffling and renormalisation effectively commute [17].

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