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On Elementary and Composite Particles: the Case of Exotic Hadrons

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By elementary hadron we mean a compact bound state of quarks as opposite to a composite hadron, which is made up by other elementary hadrons. To decide the structure of exotic hadrons one may either look for patterns in spectroscopy, as the quark model suggests, or resort to the physics of shallow bound states, virtual states, cusps etc. In recent times, following an experimental study by the LHCb collaboration, a discussion has been raised on the role of the scattering effective range r_0 as a discriminating parameter between the elementary and composite hypotheses for the $X(3872)$. I will present a brief introduction to this topic.

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1. Introduction

The wealth of data on exotic hadron resonances has stimulated an interesting discussion about how to distinguish between elementary and composite unstable particles. Deuteron is stable, and form factors, telling about its composite structure, can be measured. In a paper dating back to 1965 [1], Weinberg found a criterion to decide if the deuteron is elementary (a compact six-quark state) or molecular (a proton-neutron nucleus) on the basis of *proton-neutron scattering* only. His arguments, leading to evidence for a composite deuteron, turned out recently to be of great interest for discussing the nature of the exotic resonances like the $X(3872)$, which could be a mesonic-deuteron made by a D and a $\bar{D}^*$, with a binding energy estimated at about 100 KeV. In a publication a few years before Weinberg's, Landau [2] presented an argument relating the binding energy of a composite state to the coupling to its constituents: this can be found also in [1], but with a notable difference. In [1] the theoretical possibility is considered of revealing an elementary deuteron, excluded by construction in [2]. As commented in [3], a molecular X should simply respect the relation determined in [2] between binding energy B and coupling g to $D\bar{D}^*$.

According to [1], the scattering parameter which carries the information on compositeness is the *effective range* r_0, provided that a positive scattering length a is also measured. In the following I will review the relation between r_0 and compositeness using a derivation half-way between [1] and [2]. Experimental data on the lineshape of the X can be used to extract r_0 and to determine its nature. The interpretation of the results obtained is still somewhat controversial.

2. Elementary versus composite, the role of the effective range r_0

Assume that the X is either a elementary state $\mathfrak{X}$ with a superposition to the continuum $D\bar{D}^*(\boldsymbol{k})$ or, possibly, a superposition of a compact state $\mathfrak{X}$ and a bound state of D and $\bar{D}^*$ — in the latter case there should be two particles in the spectrum. In both cases [1]

$$|X\rangle = \sqrt{Z}|\mathfrak{X}\rangle + \int_{\boldsymbol{k}} C_{\boldsymbol{k}} \underbrace{|D\bar{D}^*(\boldsymbol{k})\rangle}_{|\alpha\rangle} \tag{1}$$

but with different coefficients $C_{\boldsymbol{k}}$ and $\langle\mathfrak{X}|\alpha\rangle = 0$. Only one state has been resolved to the date of this writing, so we might conclude that X is either a purely molecular state, $Z = 0$, or an elementary state $0 < Z < 1$, strongly coupled to the continuum. The case $Z = 1$ is singular in that it corresponds to an elementary $\mathfrak{X}$ not interacting with D or $\bar{D}^*$ — this corresponds to the $Z = 1$ normalization of the free particle propagator in the quantum field theory language[1].

The completeness relation $1 = |\mathfrak{X}\rangle\langle\mathfrak{X}| + \int |\alpha\rangle\langle\alpha|$ substituted in $\langle X|1|X\rangle$ gives

$$1 - Z = \int |\langle\alpha|X\rangle|^2 d\alpha = \int \frac{|\langle\alpha|V|X\rangle|^2}{(E(\alpha)+B)^2} d\alpha \equiv \int \frac{g_W^2}{(E(\alpha)+B)^2} d\alpha \tag{2}$$

[1] If the bare field Φ_μ of the X can annihilate a one-particle state (of mass m) with amplitude $Ne_\mu/\sqrt{2E}$, then the complete propagator of Φ_μ has a residue proportional to $Z = |N|^2$ at the pole $-m^2$, where $Z = 1-$[coupling to the continuum of states $|\boldsymbol{k}_1, \boldsymbol{k}_2\rangle, |\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3\rangle, \ldots$]. The latter multi-particles states are not introduced as bound states of some potential.

provided that $\langle X|X\rangle = 1$ (eigenstate of $H = H_0 + V$) and that the interaction V is defined by

$$(H_0 + V)|X\rangle = -B|X\rangle \qquad\qquad H_0|\alpha\rangle = E(\alpha)|\alpha\rangle \tag{3}$$

where B is the binding energy or distance from threshold. The (bare) elementary $\mathfrak{X}$ particle corresponds to a discrete eigenstate of the free-particle Hamiltonian H_0, $H_0|\mathfrak{X}\rangle = E_{\mathfrak{X}}|\mathfrak{X}\rangle$. Notice that in the non-relativistic formalism

$$d\alpha = \frac{d^3k}{(2\pi)^3} = \frac{k^2dk}{2\pi^2} = \frac{1}{2\pi^2}(\sqrt{2mE})^2\frac{d\sqrt{2mE}}{dE}dE = \frac{1}{4\pi^2}(2m)^{3/2}\sqrt{E}dE \tag{4}$$

allows to compute the integral in (2)

$$\int_0^\infty \frac{\sqrt{E}}{(E+B)^2}dE = \frac{\pi}{2\sqrt{B}} \tag{5}$$

and therefore to solve (2) in terms of the coupling g_W

$$g_W^2 = \frac{2\pi\sqrt{2mB}}{m^2}(1-Z) \equiv \frac{2\pi\varkappa}{m^2}(1-Z) \tag{6}$$

Notice that $g_W = 0$ for $Z = 1$; in the case of the deuteron this would mean zero coupling to proton and neutron, as opposite to the expectations for an elementary deuteron. Indeed from (1) and (3) we see that $Z = 1$ requires $V = 0$ since, from (1) and (2)

$$1 - Z = \int_{\boldsymbol{k}} |C_{\boldsymbol{k}}|^2 \tag{7}$$

so $|X\rangle = |\mathfrak{X}\rangle$ and $E_{\mathfrak{X}} = -B$.

The elementary particle is found for *any* value of Z in the range $0 < Z < 1$.

Let $\alpha = \beta = D\bar{D}^*(\boldsymbol{k})$. Assume[2] that the S-matrix element $M_{\beta\alpha}$ is dominated by the propagation of the X so that the scattering amplitude can be written in the polar form [4]

$$\begin{aligned} f(\alpha\to\beta) &= -\frac{1}{2\pi E}\sqrt{\frac{k'E_1'E_2'E_1E_2}{k}}\,M_{\beta\alpha} = -\frac{1}{8\pi E}(2m_D)(2m_{D^*})\,M_{\beta\alpha} \\ &= -\frac{1}{8\pi E}(2m_D)(2m_{D^*})(2m_X)\,M_{\beta X}\frac{1}{p^2+m_X^2-i\epsilon}M_{X\alpha} \\ &\simeq -\frac{1}{8\pi E}8mm_X^2\frac{|\langle D\bar{D}^*|V|X\rangle|^2}{p^2+m_X^2-i\epsilon} \equiv -\frac{1}{8\pi E}8mm_X^2\frac{g_W^2}{p^2+m_X^2-i\epsilon} \end{aligned} \tag{8}$$

where m is the reduced mass $m = m_D m_{D^*}/(m_D + m_{D^*})$ and $m_X \simeq m_D + m_{D^*}$. This pole structure would be found in a field theory in which the Lagrangian contains the elementary Φ_μ field of the X or in the case of a bound state X made up by the elementary fields appearing in the Lagrangian.

Adopt the polar expression (8) in the form

$$f(\alpha\to\beta) = -\frac{1}{8\pi E}\frac{g^2}{p^2+m_X^2-i\epsilon} \tag{9}$$

[2]This assumption follows in the presence of a non-vanishing amplitude $\langle D\bar{D}^*|X\rangle$: non-vanishing matrix element of the one-particle state of mass m_X and $D^\dagger(\bar{D}^*)^\dagger\Psi_0$, the latter being the vacuum. The factor $\sum_{\rm pol}|e_{(X)}\cdot e_{(D^*)}|^2 \simeq 3$ is incorporated in the definition of g_W.

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with

$$g^2 \equiv (8mm_X^2) \times g_W^2 \tag{10}$$

Consider the case of low energy scattering. Neglecting terms of order B^2 and $T^2(T = k^2/2m)$ we find (see Eqs. (3-6) in [3])

$$f(\alpha \to \beta) = -\frac{1}{8\pi m_X}\frac{g^2}{(p_D + p_{D^*})^2 + m_X^2 - i\epsilon} \simeq -\frac{1}{16\pi m_X^2}\frac{g^2}{B+T} \tag{11}$$

Equation (11) has to be compared with the general formula for the scattering amplitude in the effective range expansion

$$f = \frac{1}{-\varkappa_0 + \frac{1}{2}r_0 k^2 - ik} \tag{12}$$

where $\varkappa_0 = 1/a$ and r_0 is the effective range. If $k = i\sqrt{2mB} \equiv i\varkappa$, matching the shallow bound state, we require the pole condition [5]

$$\left(-\varkappa_0 + \frac{1}{2}r_0 k^2 - ik\right)_{k=i\varkappa} = 0 \tag{13}$$

which implies

$$-\varkappa_0 = -\varkappa + \frac{1}{2}r_0\varkappa^2 \tag{14}$$

to be substituted back in (12), finding

$$f = \frac{1}{\frac{r_0}{2}(k^2+\varkappa^2) - (\varkappa + ik)} = \frac{1}{\frac{r_0}{2}(k^2+\varkappa^2) - \frac{(\varkappa+ik)(\varkappa-ik)}{(\varkappa-ik)}} = \tag{15}$$

$$= -\frac{1}{\frac{r_0}{2}(k^2+\varkappa^2) - \frac{1}{2\varkappa}(k^2+\varkappa^2)} = -\frac{\varkappa}{m(1-r_0\varkappa)}\frac{1}{B+T} \tag{16}$$

We have reduced the general expression for the scattering formula in a form amenable to comparison with the polar expression found in (11). From (11) and (16), and making use of (6) and (10), we find the main result: r_0 depends on Z as in

$$r_0 = -\frac{Z}{1-Z}R + O\left(\frac{1}{\Lambda}\right) \tag{17}$$

where

$$R = \frac{1}{\varkappa} = \frac{1}{\sqrt{2mB}} \tag{18}$$

and $1/\Lambda = 1/m_\pi$, in the case of the deuteron, obtained by integrating out the contribution of the pion since $m_n - m_p \ll m_\pi$. In the case of the X, however, $m_{D^*} - m_D \simeq m_\pi$: the pion cannot be integrated out and the scale Λ (see next section) is found to be $\Lambda \approx 40$ MeV.

For $Z = 0$ (the molecule) we would expect $r_0 \approx 1/|\Lambda|$ fm, in the case of a purely attractive potential — the sign of r_0 has to be positive in that case, according to some general results reviewed in [5]. As we will see in Section 4, including the pion interactions between the molecular components does not generate a purely attractive potential and a special treatment is required.

For $0 < Z < 1$ (the elementary particle) since R is large, B being very small, there is a large negative pull towards the negative range of r_0 values. If we had large enough positive $O(1/\Lambda)$

corrections (that is not the case, see Section 4) we might also have $0 < Z < 1$ and $r_0 > 0$. A negative experimental value of r_0 is the token for the elementary state, as long as $O(1/\Lambda)$ corrections are positive.

The scattering length a is obtained by plugging the expression (17) for r_0 into

$$\left(-\varkappa_0 + \frac{1}{2} r_0 k^2 - ik\right)_{k=i\varkappa} = 0 \tag{19}$$

giving a positive value, up to corrections $O(1/\Lambda)$

$$a = \frac{2(1-Z)}{2-Z} R + O\left(\frac{1}{\Lambda}\right) \tag{20}$$

Let us consider now the argument discussed in [2]. The potential scattering of two very slow particles a, b, with V featuring a shallow bound state at $-B$, has a universal scattering amplitude which does not depend on the details of V, and is given by

$$f(\alpha \to \beta) = \frac{1}{k(\cot\delta_0 - i)} = -\frac{1}{\sqrt{2m}} \frac{\sqrt{B} - i\sqrt{T}}{B + T} \tag{21}$$

where $\alpha = \beta = ab$. This formula is obtained by the *shallow bound state condition* [6] summarized in $\cot\delta_0 = -\sqrt{B/T}$. In this treatment elementary states are excluded.

A comparison of (21), at $k = i\varkappa$, with our the pole formula in (16) can be done. From

$$-\frac{1}{\sqrt{2m}} \frac{2\sqrt{B}}{B+T} = \left(-\frac{\varkappa}{m(1 - r_0\varkappa)} \frac{1}{B+T}\right)_{r_0 \to 0} \tag{22}$$

we get (see Eq. (10)) the Landau coupling

$$g^2 = 16\pi\sqrt{2mB}\,\frac{m_X^2}{m} = 8mm_X^2 \times (g_W)_{Z=0} \tag{23}$$

where m is the reduced mass of a, b particles and $m_X \simeq (m_a + m_b)$. Therefore the coupling of a *bona fide* molecule to its constituent hadrons should simply obey the relation (23)[3].

3. The effective range of $X(3872)$ from data

In the case of the deuteron it is found experimentally that

$$r_0 \simeq +1.74 \text{ fm} \tag{24}$$

which is perfectly compatible with $Z = 0$, given that the $O(1/m_\pi)$ corrections are expected to be positive for an attractive potential [5]. In the case of the X, LHCb finds [7] (see the analysis in [5])

$$r_0 = -\frac{Z}{1-Z} R = -5.34 \text{ fm} \tag{25}$$

$$a = \frac{2(1-Z)}{2-Z} R = 28 \text{ fm} \tag{26}$$

[3]The total width of the X times its branching fraction into $D\bar{D}^*$ is proportional to the coupling squared, i.e. to the binding energy. The measured branching fraction, lifetime and binding energy should satisfy the Landau relation in the case of a molecular X [3].

giving

$$Z = 0.15 \qquad \text{and} \qquad B = 20 \text{ KeV} \tag{27}$$

This is not the expression of a "small quota" of the compact component: it simply means that the X is a compact state, with a significant coupling to the continuum [4]. If we naively estimate the $O(1/\Lambda)$ corrections from pion interactions to be approximately ±5 fm we would have [5]

$$r_0 \simeq -\frac{Z}{1-Z} R \pm 5 \text{ fm} = -5.34 \text{ fm} \tag{28}$$

In the positive sign case this means $0.2 < Z < 0.5$ for reasonable values of R. In the negative sign case, however, we could have $Z \simeq 0$ at $r_0 \simeq -5.3$ fm. We need to know the size and sign of the correction. This has been studied in [8] and [9] using the diagrammatic methods in a non-relativistic effective field theory for the X. When including pion interactions it is found that the correction to $r_0 = 0$ is indeed negative, but small (~ 0.2, rather than 5 fm!). We confirm this using the method of the Distorted-Wave-Born-Approximation in non-relativistic quantum mechanics [10], allowing a simpler calculation.

In a very recent paper by the BESIII collaboration [11], the results of LHCb are somewhat confirmed, although with sizeable uncertainties

$$\begin{aligned} r_0 &= -4.1^{+0.9+2.8}_{-3.3-4.4} \text{ fm} \\ a &= +16.5^{+27.6+27.7}_{-7.0-5.6} \text{ fm} \end{aligned} \tag{29}$$

More precise experimental determinations of r_0 for the X and possibly for all other exotic resonances are needed. In [12] the T^+_{cc} resonance has been studied finding $-16.9 < r_0 < 0$ (fm).

4. Pion interactions in the molecular picture

The molecular X is the expression of a bound state in the δ-function potential

$$U_s(r) = -\lambda_0 \delta^3(\boldsymbol{r}) \tag{30}$$

This is a good approximation to the real (unknown) potential given that the wavefunction is expected to be broader than the range R_0 of the potential: $R = 1/\sqrt{2mB} \gg R_0$.

The pion exchange potential to be added to the strong $\delta^3(r)$ potential could, in principle, contribute to r_0 with the negative sign. Let us compute the potential as the Fourier transform of the pion exchange diagram (actually is a u-channel) in the non-relativistic approximation. The mass scale $\mu^2 = (m_{D^*} - m_D)^2 - m_\pi^2$ appears in

$$U_w = -\frac{g^2}{2f_\pi^2} \boldsymbol{e}_i^{(\lambda)} \bar{\boldsymbol{e}}_j^{(\lambda')} \int \frac{q_i q_j \, e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 - \mu^2 - i\epsilon} \frac{d^3 q}{(2\pi)^3} = -\frac{g^2}{6 f_\pi^2} \left(\delta^3(r) + \mu^2 \frac{e^{i\mu r}}{4\pi r} \right) \boldsymbol{e}^{(\lambda)} \cdot \bar{\boldsymbol{e}}^{(\lambda')} \tag{31}$$

where $\boldsymbol{e}_i^{(\lambda)}$ and $\bar{\boldsymbol{e}}_j^{(\lambda')}$ are the polarizations vectors of the initial and final D^* mesons respectively, and we used the following S−wave relation to solve the integral

$$\langle n_i n_j \rangle = \frac{1}{3} \delta_{ij} \tag{32}$$

[4] Or there are two states in the spectrum: the two orthogonal combinations of elementary and molecular bound state.
[5] Since we estimate $\Lambda \simeq 40$ MeV, and do not know the sign and size of the corrections — see Section 4.

In the non-relativistic limit of D and D^* at rest, $\boldsymbol{e}^{(\lambda)} \cdot \bar{\boldsymbol{e}}^{(\lambda')} = \delta_{\lambda\lambda'}$, the longitudinal polarization (along z) being $e^L \simeq (0,0,1,0)$.

The total potential $V = U_s + U_w$ contains therefore an infinite range complex term[6]: a Yukawa complex potential which reflects the fact that a real pion can be produced in the decay of the unstable D^* and still mediate the $D^*\bar{D} \to D\bar{D}^*$ interaction. An estimate of the α constant might be reassuring on that U_w could not possibly spoil the Weinberg analysis described in the previous sections. Still a calculation of the contribution to r_0 is needed because inverse powers of μ can occur. The full potential is

$$V = V_s + V_w = -(\lambda_0 + 4\pi\alpha)\,\delta^3(\boldsymbol{r}) - \alpha\mu^2 \frac{e^{i\mu r}}{r} \tag{33}$$

where $\alpha = g^2/(24\pi f_\pi^2) = 5 \times 10^{-4}/\mu^2$, V_s includes the contribution of both the 'strong' Dirac-δ potential and that from pion interactions and V_w is the complex Yukawa potential. The general formula for the scattering amplitude has two terms, f_s, with $r_0 = 0$ by definition, and the perturbative correction f_w due to V_w

$$f \equiv \frac{1}{k\cot\delta(k) - ik} = f_s + f_w = \frac{1}{-\frac{1}{a} - ik} + f_w \tag{34}$$

where

$$f_w = -\frac{2m}{4k^2}\int V_w(r)\,\chi_s^2(r)\,dr \tag{35}$$

and $\chi_s(r)$ are the scattering wavefunctions in the $\delta^3(r)$ potential while m is the reduced mass of the $D\bar{D}^*$ pair. A derivation of f_w can be found in the Appendix.

Thus we conclude that r_0 due to pion interactions in the molecular picture of the X is determined by the k^2 coefficient in the double expansion around $r_0 = 0$ and $\alpha = 0$ of

$$f^{-1} = \left(\frac{1}{-\frac{1}{a} - ik} - \frac{2m}{4k^2}\int V_w(r)\,\chi_s^2(r)\,dr\right)^{-1} \tag{36}$$

To compute the integral in f_w we might proceed along the following steps: *i*) Use $e^{-\mu r}$ in place of $e^{i\mu r}$ and set $\mu \to i\mu$ in the final results; *ii*) Replace $\chi_s^I(r) = 2kr\left(\frac{e^{i\delta}\sin(kr+\delta)}{kr} - \frac{e^{i\delta}\sin\delta}{kr}\right)$ for $0 < r < \lambda$ and $\chi_s^{II}(r) = 2kr\left(\frac{e^{i\delta}\sin(kr+\delta)}{kr}\right)$ for $r > \lambda$. The subtraction in the origin is done to implement the relation (3.16) in [13]. Finally one sets $\lambda \to 0$. The integral turns out to be finite; *iii*) Replace $\delta = \cot^{-1}(-1/ka_s)$ [13]; *iv*) Double-expand the result around $k = 0$ and $\alpha = 0$; *v*) Take the $\lambda \to 0$ limit and set $\mu \to i\mu$ eventually. The same integral has been computed removing the divergence at $r = 0$ with other regularizations, as discussed in [10], finding the same results.

From formula (36) one gets

$$r_0 = 2m\alpha\left(\frac{2}{\mu^2 a^2} + \frac{8i}{3\mu a} - 1\right) \tag{37}$$

[6]with oscillating real and imaginary parts: not a purely attractive potential.

This formula agrees analytically with what found in [9], in the limit $m_\pi/m_D \ll 1$ [10]. The numerical values found reassure on the applicability of the Weinberg criterion

$$-0.20\ \text{fm} \lesssim \ \text{Re}\, r_0 \lesssim -0.15\ \text{fm} \tag{38}$$

and

$$0\ \text{fm} \lesssim \ \text{Im}\, r_0 \lesssim 0.17\ \text{fm} \tag{39}$$

An experimental value $r_0 \simeq -5.3$ fm is not compatible with what expected in the molecule picture ($r_0 = 0$), also after including pion interactions.

5. Conclusions

To investigate the structure of the exotic unstable hadrons one may either look for patterns in spectroscopy, as the quark model suggests [14], or resort to the physics of shallow bound states [15, 16]. The absence so far of charged partners for the $X(3872)$, the most studied among exotic hadrons, is a problem for the elementary interpretation. The very notion of a compact tetraquark was also challenged in a well known discussion on QCD in a large number of colors [17] but reconsidered more recently in [18]. For the time being, the scattering effective range r_0 deduced from the lineshape of the X, as measured by LHCb and BESIII, speaks in favor of a compact state, which would also help to explain the production of X at hadron colliders [19]. A general consensus on the method used to extract r_0 from data should be found.

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Appendix

To obtain formula (35) we start from the Born formula for the scattering amplitude

$$f = -\frac{m}{2\pi} \int V(r)\, e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot \boldsymbol{r}}\, d^3 r \tag{40}$$

valid when the scattering field can be regarded as a perturbation[7] and all the scattering phases δ_ℓ are small. The plane wave factors can be expanded as in

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{\ell=0}^{\infty} i^\ell j_\ell(kr)(2\ell+1) P_\ell(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}}) \tag{41}$$

[7]The condition $|V| \ll 1/mR_0^2$ in our case can be written (taking $|V| = |V(R_0)|$) as

$$\alpha\mu^2 \frac{\cos\mu R_0}{R_0} \ll \frac{1}{mR_0^2}$$

which works as long as $R_0 \lesssim 20$ fm.

and

$$e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}} = \sum_{\ell=0}^{\infty} i^{\ell} j_{\ell}(k'r)(2\ell+1)(-1)^{\ell} P_{\ell}(\hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{r}}) \tag{42}$$

In the $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}}$ product, $i^{2\ell}(-1)^{\ell} = +1$ and using the relation

$$\int P_{\ell}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)P_{\ell'}(\boldsymbol{n}_1\cdot\boldsymbol{n}_3)d\Omega_1 = \delta_{\ell\ell'}\frac{4\pi}{(2\ell+1)}P_{\ell}(\boldsymbol{n}_2\cdot\boldsymbol{n}_3) \tag{43}$$

we get

$$f = -2m\sum_{\ell=0}^{\infty}(2\ell+1)P_{\ell}(\cos\theta)\int V(r)(j_{\ell}(kr))^2 r^2 dr \tag{44}$$

to be compared with

$$f = \sum_{\ell=0}^{\infty}(2\ell+1)P_{\ell}(\cos\theta)\frac{e^{i\delta_{\ell}}\sin\delta_{\ell}}{k} \tag{45}$$

giving for the partial wave amplitude

$$f_{\ell} = \frac{e^{i\delta_{\ell}}\sin\delta_{\ell}}{k} = -2m\int V(r)(j_{\ell}(kr))^2 r^2 dr \tag{46}$$

Using the definition of the free particle reduced wave function

$$\chi_{\ell}^{(0)}(r) = 2kr\, j_{\ell}(kr) \tag{47}$$

we finally have

$$f_{\ell} = -\frac{2m}{4k^2}\int V(r)\,(\chi_{\ell}^{(0)}(r))^2\, dr \tag{48}$$

The f_w in (34) corresponds to the $\ell = 0$ partial amplitude

$$f_w = -\frac{2m}{4k^2}\int_0^{\infty} V_w(r)\,(\chi_s(r))^2\, dr \tag{49}$$

where the Distorted-Wave-Born-Approximation consists here in substituting the free particle wave functions with the scattering wave functions for the V_s potential

$$\chi_{\ell=0}^{(0)}(r) \to \chi_s(r) \tag{50}$$

The $\chi_s(r)$ are determined in [13].

References

[1] S. Weinberg, Phys. Rev. **137**, B672-B678 (1965) doi:10.1103/PhysRev.137.B672

[2] L.D. Landau, JETP **39**, 1865 (1960)

[3] A. D. Polosa, Phys. Lett. B **746**, 248-250 (2015) doi:10.1016/j.physletb.2015.05.017 [arXiv:1505.03083 [hep-ph]].

[4] S. Weinberg, Cambridge University Press, 2005, ISBN 978-0-521-67053-1. See Eq. (3.6.7), divided by $(2\pi)^3$.

[5] A. Esposito, L. Maiani, A. Pilloni, A. D. Polosa and V. Riquer, Phys. Rev. D **105**, no.3, L031503 (2022) doi:10.1103/PhysRevD.105.L031503 [arXiv:2108.11413 [hep-ph]].

[6] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics vol. 3*, Academic Press; see Eq. (133.7).

[7] R. Aaij *et al.* [LHCb], Phys. Rev. D **102**, no.9, 092005 (2020) doi:10.1103/PhysRevD.102.092005 [arXiv:2005.13419 [hep-ex]].

[8] E. Braaten, Phys. Rev. D **91**, no.11, 114007 (2015) doi:10.1103/PhysRevD.91.114007 [arXiv:1503.04791 [hep-ph]];

[9] E. Braaten, L. P. He and J. Jiang, Phys. Rev. D **103**, no.3, 036014 (2021) doi:10.1103/PhysRevD.103.036014 [arXiv:2010.05801 [hep-ph]].

[10] A. Esposito, D. Germani, A. Glioti, A. D. Polosa, R. Rattazzi and M. Tarquini, Phys. Lett. B **847**, 138285 (2023) doi:10.1016/j.physletb.2023.138285 [arXiv:2307.11400 [hep-ph]].

[11] M. Ablikim *et al.* [BESIII], [arXiv:2309.01502 [hep-ex]].

[12] M. Mikhasenko, [arXiv:2203.04622 [hep-ph]].

[13] R. Jackiw, 'Delta Function Potentials in two- and three- dimensional quantum mechanics' in Diverse Topics in Theoretical and Mathematical Physics, World Scientific. See also Gosdzynsky, Tarrach (https://doi.org/10.1119/1.16691).

[14] L. Maiani, F. Piccinini, A. D. Polosa and V. Riquer, Phys. Rev. D **71**, 014028 (2005) doi:10.1103/PhysRevD.71.014028 [arXiv:hep-ph/0412098 [hep-ph]]. L. Maiani, F. Piccinini, A. D. Polosa and V. Riquer, Phys. Rev. D **89**, 114010 (2014) doi:10.1103/PhysRevD.89.114010 [arXiv:1405.1551 [hep-ph]]; L. Maiani, A. D. Polosa and V. Riquer, Phys. Lett. B **778**, 247-251 (2018) doi:10.1016/j.physletb.2018.01.039 [arXiv:1712.05296 [hep-ph]].

[15] M. B. Voloshin, Phys. Lett. B **579**, 316-320 (2004) doi:10.1016/j.physletb.2003.11.014 [arXiv:hep-ph/0309307 [hep-ph]].

[16] E. Braaten and M. Kusunoki, Phys. Rev. D **69**, 074005 (2004) doi:10.1103/PhysRevD.69.074005 [arXiv:hep-ph/0311147 [hep-ph]].

[17] E. Witten, Nucl. Phys. B **160**, 57-115 (1979) doi:10.1016/0550-3213(79)90232-3

[18] S. Weinberg, Phys. Rev. Lett. **110**, 261601 (2013) doi:10.1103/PhysRevLett.110.261601 [arXiv:1303.0342 [hep-ph]].

[19] C. Bignamini, B. Grinstein, F. Piccinini, A. D. Polosa and C. Sabelli, Phys. Rev. Lett. **103**, 162001 (2009) doi:10.1103/PhysRevLett.103.162001 [arXiv:0906.0882 [hep-ph]]; A. Esposito, A. L. Guerrieri, L. Maiani, F. Piccinini, A. Pilloni, A. D. Polosa and V. Riquer, Phys. Rev. D **92**, no.3, 034028 (2015) doi:10.1103/PhysRevD.92.034028 [arXiv:1508.00295 [hep-ph]]; A. Esposito, E. G. Ferreiro, A. Pilloni, A. D. Polosa and C. A. Salgado, Eur. Phys. J. C **81**, no.7, 669 (2021) doi:10.1140/epjc/s10052-021-09425-w [arXiv:2006.15044 [hep-ph]].

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