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A Pythagoras-like theorem for the Jarlskog invariant of CP violation

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The $\nu_\mu \to \nu_e$ oscillation probability is fully determined by the CP-conserving quantities $\mathcal{R}_{ij} \equiv \mathrm{Re}(U_{\mu i} U_{ej} U^*_{\mu j} U^*_{ei})$ and the Jarlskog invariant of CP violation $\mathcal{J}_\nu \equiv (-1)^{i+j}\,\mathrm{Im}(U_{\mu i} U_{ej} U^*_{\mu j} U^*_{ei})$ (for $i, j = 1, 2, 3$ and $i < j$), where U is the unitary PMNS lepton flavor mixing matrix. We find that a Pythagoras-like relation $\mathcal{J}_\nu^2 = \mathcal{R}_{12}\mathcal{R}_{13} + \mathcal{R}_{12}\mathcal{R}_{23} + \mathcal{R}_{13}\mathcal{R}_{23}$ holds, and it may hopefully offer a novel cross-check of the result of $\mathcal{J}_\nu$ that will be directly measured in the next-generation long-baseline neutrino oscillation experiments. Terrestrial matter effects are briefly discussed.

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1. Introduction

The experimental discoveries of atmospheric, solar, reactor and accelerator neutrino oscillations have constituted the most remarkable evidence for new physics beyond the standard model (SM) of particle physics [1]. The relevant data can be well described by the 3×3 Pontecorvo-Maki-Nakagawa-Sakata (PMNS) lepton flavor mixing matrix U, together with the neutrino mass-squared differences $\Delta_{ij} \equiv m_i^2 - m_j^2$ (for $i, j = 1, 2, 3$ and $i \neq j$). Although the unitarity of U has been tested at the level of $O\left(\lesssim 10^{-2}\right)$ [2], whether it is exactly unitary or not remains an open question. That is why the next-generation neutrino oscillation experiments aim to convincingly establish leptonic CP violation and accurately measure all the flavor mixing parameters that are accessible in both the *appearance* (i.e., $\nu_\alpha \to \nu_\beta$ and $\overline{\nu}_\alpha \to \overline{\nu}_\beta$ with $\beta \neq \alpha$) and *disappearance* (i.e., $\nu_\alpha \to \nu_\alpha$ and $\overline{\nu}_\alpha \to \overline{\nu}_\alpha$ for $\alpha = e, \mu, \tau$) oscillation channels. Among them, $\nu_\mu \to \nu_e$ and $\overline{\nu}_\mu \to \overline{\nu}_e$ oscillations are the main focus for a realistic experimental exploration of CP violation in the neutrino sector. A preliminary but encouraging evidence for CP violation in these two channels has recently be reported by the T2K Collaboration at the 2σ level [3, 4].

Given unitarity of the 3×3 PMNS matrix U, the probabilities of $\nu_\mu \to \nu_e$ and $\overline{\nu}_\mu \to \overline{\nu}_e$ oscillations in vacuum can be compactly expressed as

$$
\begin{aligned}
P(\nu_\mu \to \nu_e) &= -4 \sum_{i<j} \left(\mathcal{R}_{ij} \sin^2 \frac{\Delta_{ji} L}{4E} \right) - 8 \mathcal{J}_\nu \prod_{i<j} \sin \frac{\Delta_{ji} L}{4E} \ , \\
P(\overline{\nu}_\mu \to \overline{\nu}_e) &= -4 \sum_{i<j} \left(\mathcal{R}_{ij} \sin^2 \frac{\Delta_{ji} L}{4E} \right) + 8 \mathcal{J}_\nu \prod_{i<j} \sin \frac{\Delta_{ji} L}{4E} \ ,
\end{aligned} \tag{1}
$$

where L is the baseline length, E denotes the neutrino beam energy, $\mathcal{R}_{ij} \equiv \mathrm{Re}(U_{\mu i} U_{ej} U^*_{\mu j} U^*_{ei})$ are CP-conserving, and $\mathcal{J}_\nu = (-1)^{i+j} \ \mathrm{Im}(U_{\mu i} U_{ej} U^*_{\mu j} U^*_{ei})$ is the Jarlskog invariant of leptonic CP violation (for $i, j = 1, 2, 3$ and $i < j$) [5–7]. It is expected that $\mathcal{R}_{ij}$ and $\mathcal{J}_\nu$ will all be determined from a precision measurement of the dependence of $P(\nu_\mu \to \nu_e)$ or $P(\overline{\nu}_\mu \to \overline{\nu}_e)$ on L/E.

In this talk we are going to show that the Jarlskog invariant $\mathcal{J}_\nu$ is related to the CP-conserving quantities $\mathcal{R}_{12}$, $\mathcal{R}_{13}$ and $\mathcal{R}_{23}$ as follows: $\mathcal{J}_\nu^2 = \mathcal{R}_{12}\mathcal{R}_{13} + \mathcal{R}_{12}\mathcal{R}_{23} + \mathcal{R}_{13}\mathcal{R}_{23}$, which will subsequently be referred to as a Pythagoras-like theorem for CP violation in neutrino oscillations. Such a novel relation allows us to *calculate* the size of $\mathcal{J}_\nu$ from the experimental values of $\mathcal{R}_{ij}$, and thus it may hopefully provide a useful cross-check of the result of $\mathcal{J}_\nu$ that will be *directly measured* in the next-generation $\nu_\mu \to \nu_e$ and $\overline{\nu}_\mu \to \overline{\nu}_e$ oscillation experiments (e.g., Hyper-K and DUNE). Terrestrial matter effects on $\mathcal{J}_\nu$ and $\mathcal{R}_{ij}$ will be briefly discussed.

2. The theorem in vacuum

Let us begin with a particular orthogonality relation of the 3×3 PMNS lepton flavor mixing matrix U, namely,

$$
U_{e1} U^*_{\mu 1} + U_{e2} U^*_{\mu 2} + U_{e3} U^*_{\mu 3} = 0 \ , \tag{2}
$$

which defines a triangle in the complex plane. This leptonic unitarity triangle is referred to as $\triangle_\tau$ [8], and it is closely associated with $\nu_\mu \to \nu_e$ and $\overline{\nu}_\mu \to \overline{\nu}_e$ oscillations. There are three straightforward

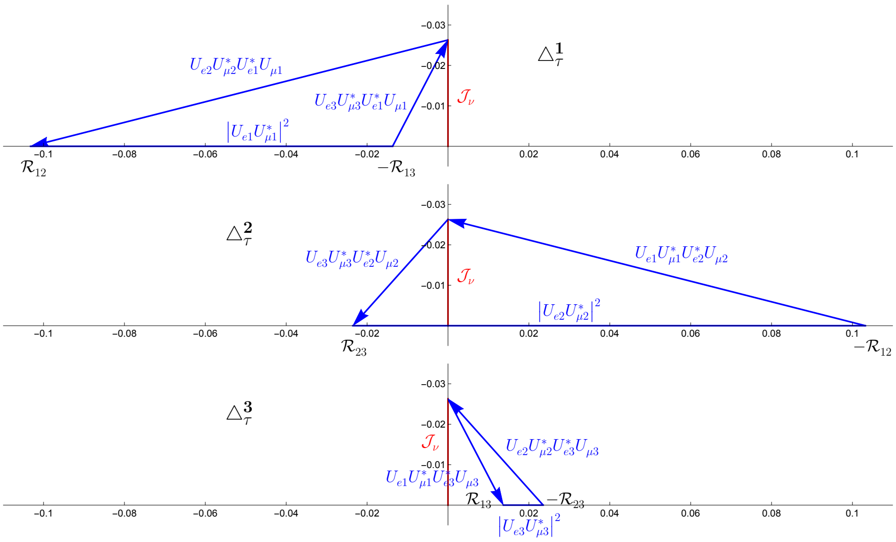

Figure 1: A numerical illustration of the three rescaled unitarity triangles originating from $\triangle_\tau$ in the complex plane for the **normal** neutrino mass ordering, where the best-fit values of the relevant flavor mixing and CP-violating parameters [9] have been input.

ways to rescale $\triangle_\tau$, leading to three fully rephasing-invariant triangles $\triangle_\tau^{\mathbf{n}}$ (for $\mathbf{n} = \mathbf{1}, \mathbf{2}, \mathbf{3}$) whose real and positive sides are

$$\begin{aligned}
\triangle_\tau^{\mathbf{1}} : \quad \left|U_{e1}U^*_{\mu1}\right|^2 &= -U_{\mu1}U_{e2}U^*_{\mu2}U^*_{e1} - U_{\mu1}U_{e3}U^*_{\mu3}U^*_{e1} \; , \\
\triangle_\tau^{\mathbf{2}} : \quad \left|U_{e2}U^*_{\mu2}\right|^2 &= -U_{\mu2}U_{e3}U^*_{\mu3}U^*_{e2} - U_{\mu2}U_{e1}U^*_{\mu1}U^*_{e2} \; , \\
\triangle_\tau^{\mathbf{3}} : \quad \left|U_{e3}U^*_{\mu3}\right|^2 &= -U_{\mu3}U_{e1}U^*_{\mu1}U^*_{e3} - U_{\mu3}U_{e2}U^*_{\mu2}U^*_{e3} \; ,
\end{aligned} \tag{3}$$

respectively. In Fig. 1 (or Fig. 2) we have numerically illustrated the sizes and shapes of $\triangle_\tau^{\mathbf{n}}$ by using the best-fit values of the flavor mixing parameters of U obtained from a recent global analysis of current neutrino oscillation data [9]. One can see that the heights of triangles $\triangle_\tau^{\mathbf{1}}$, $\triangle_\tau^{\mathbf{2}}$ and $\triangle_\tau^{\mathbf{3}}$ corresponding to the respective bases $|U_{e1}U^*_{\mu1}|^2$, $|U_{e2}U^*_{\mu2}|^2$ and $|U_{e3}U^*_{\mu3}|^2$ are all equal to $\mathcal{J}_\nu$, a salient feature of the unitarity of U. Numerically, we find $\mathcal{J}_\nu \simeq -2.63 \times 10^{-2}$ (or -3.30×10^{-2}) in the normal (or inverted) neutrino mass ordering cases.

Note that triangle $\triangle_\tau^{\mathbf{n}}$ is apparently associated with two right triangles which share the common side characterized by $\mathcal{J}_\nu$ (for $\mathbf{n} = \mathbf{1}, \mathbf{2}, \mathbf{3}$), as shown in Fig. 1 or Fig. 2. Given the magic relations among the sides of these right triangles as dictated by the famous Pythagorean theorem, we immediately arrive at [10]

$$\mathcal{J}_\nu^2 = \mathcal{R}_{12}\mathcal{R}_{13} + \mathcal{R}_{12}\mathcal{R}_{23} + \mathcal{R}_{13}\mathcal{R}_{23} \; . \tag{4}$$

In other words, the CP-violating quantity $\mathcal{J}_\nu$ and the CP-conserving quantities $\mathcal{R}_{ij}$ satisfy a Pythagoras-like theorem in the three-flavor space. Note that $\mathcal{J}_\nu$, $\mathcal{R}_{12}$, $\mathcal{R}_{13}$ and $\mathcal{R}_{23}$ can all be extracted from a *single* neutrino oscillation channel $\nu_\mu \to \nu_e$ or its CP-conjugated process $\overline{\nu}_\mu \to \overline{\nu}_e$, at least in principle. This nontrivial observation makes it possible to cross check the magnitude of $\mathcal{J}_\nu$ that will be directly measured in the upcoming long-baseline experiments by calculating it with the help of the Pythagoras-like relation in Eq. (4).

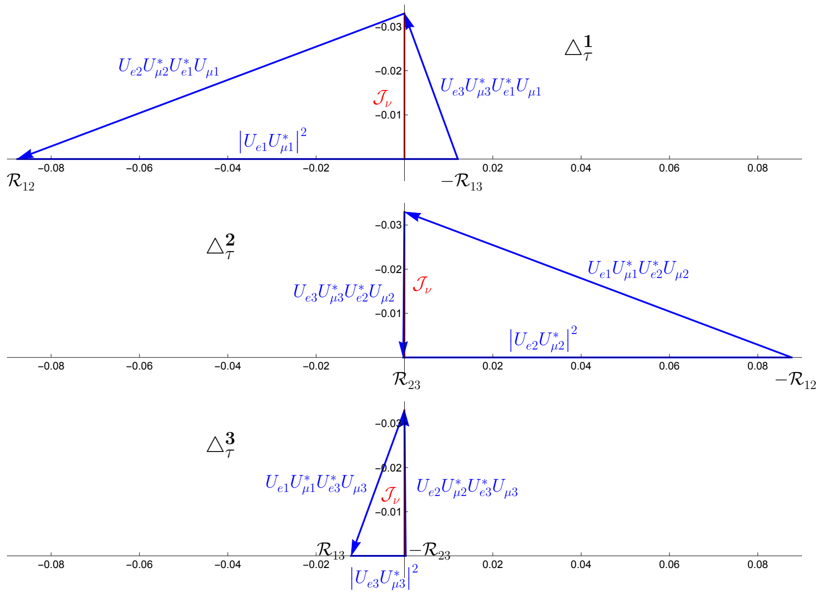

Figure 2: A numerical illustration of the three rescaled unitarity triangles originating from $\triangle_\tau$ in the complex plane for the **inverted** neutrino mass ordering, where the best-fit values of the relevant flavor mixing and CP-violating parameters [9] have been input.

3. Terrestrial matter effects

It is inevitable for the long-baseline experiments to take into account the terrestrial matter effect. Given a constant matter profile, the neutrino oscillation probabilities in matter can be written in the same way as Eq. (1) in vacuum by simply replacing Δ_{ji}, $\mathcal{R}_{ij}$ and $\mathcal{J}_\nu$ with the corresponding effective parameters $\widetilde{\Delta}_{ji}$, $\widetilde{\mathcal{R}}_{ij}$ and $\widetilde{\mathcal{J}}_\nu$ in matter. After a lengthy but straightforward calculation, we find that $\widetilde{\mathcal{R}}_{ij}$ and $\mathcal{R}_{ij}$ are related to each other through [10]

$$
\begin{aligned}
\widetilde{\mathcal{R}}_{12} &= \frac{\Delta^2_{21}\left(\Delta_{31}-\Delta^\prime_{11}\right)\left(\Delta_{32}-\Delta^\prime_{22}\right)}{\widetilde{\Delta}^2_{21}\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}\mathcal{R}_{12} - \frac{\Delta^2_{31}\Delta^\prime_{22}\left(\Delta_{21}-\Delta^\prime_{11}\right)}{\widetilde{\Delta}^2_{21}\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}\mathcal{R}_{13} + \frac{\Delta^2_{32}\Delta^\prime_{11}\left(\Delta_{21}+\Delta^\prime_{22}\right)}{\widetilde{\Delta}^2_{21}\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}\mathcal{R}_{23} \; , \\
\widetilde{\mathcal{R}}_{13} &= \frac{\Delta^2_{31}\left(\Delta_{21}-\Delta^\prime_{11}\right)\left(\Delta_{32}+\Delta^\prime_{33}\right)}{\widetilde{\Delta}^2_{31}\widetilde{\Delta}_{21}\widetilde{\Delta}_{32}}\mathcal{R}_{13} + \frac{\Delta^2_{21}\Delta^\prime_{33}\left(\Delta_{31}-\Delta^\prime_{11}\right)}{\widetilde{\Delta}^2_{31}\widetilde{\Delta}_{21}\widetilde{\Delta}_{32}}\mathcal{R}_{12} - \frac{\Delta^2_{32}\Delta^\prime_{11}\left(\Delta_{31}+\Delta^\prime_{33}\right)}{\widetilde{\Delta}^2_{31}\widetilde{\Delta}_{21}\widetilde{\Delta}_{32}}\mathcal{R}_{23} \; , \\
\widetilde{\mathcal{R}}_{23} &= \frac{\Delta^2_{32}\left(\Delta_{21}+\Delta^\prime_{22}\right)\left(\Delta_{31}+\Delta^\prime_{33}\right)}{\widetilde{\Delta}^2_{32}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}}\mathcal{R}_{23} - \frac{\Delta^2_{21}\Delta^\prime_{33}\left(\Delta_{32}-\Delta^\prime_{22}\right)}{\widetilde{\Delta}^2_{32}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}}\mathcal{R}_{12} + \frac{\Delta^2_{31}\Delta^\prime_{22}\left(\Delta_{32}+\Delta^\prime_{33}\right)}{\widetilde{\Delta}^2_{32}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}}\mathcal{R}_{13} \; , \qquad (5)
\end{aligned}
$$

where the explicit expressions of $\widetilde{\Delta}_{ji}$ and $\Delta^\prime_{ii} \equiv \widetilde{m}^2_i - m^2_i$ have already been given in Refs. [11–13]. On the other hand, $\widetilde{\mathcal{J}}_\nu$ and $\mathcal{J}_\nu$ satisfy the well-known Naumov relation [14]

$$
\widetilde{\mathcal{J}}_\nu = \frac{\Delta_{21}\Delta_{31}\Delta_{32}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}\mathcal{J}_\nu \; . \qquad (6)
$$

The unitarity of $\widetilde{U}$ assures that a similar Pythagoras-like relation $\widetilde{\mathcal{J}}^2_\nu = \widetilde{\mathcal{R}}_{12}\widetilde{\mathcal{R}}_{13} + \widetilde{\mathcal{R}}_{12}\widetilde{\mathcal{R}}_{23} + \widetilde{\mathcal{R}}_{13}\widetilde{\mathcal{R}}_{23}$ holds. A general observation is that both $\mathcal{R}_{ij}$ and $\mathcal{J}_\nu$ are quite sensitive to the matter-induced corrections (see Ref. [10] for a detailed numerical analysis).

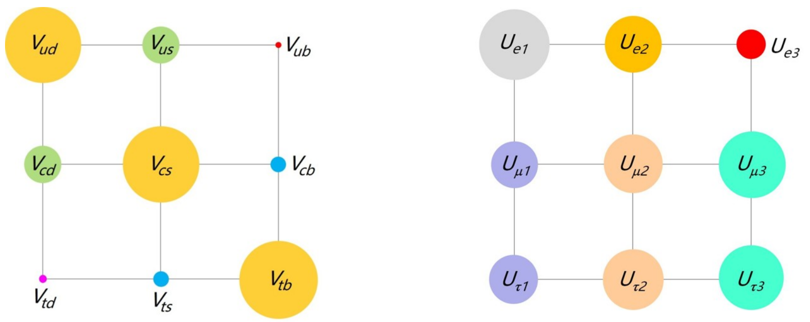

Figure 3: The size (represented by the area of the circle) of the moduli of nine elements of the CKM (left) and PMNS (right) matrices.

4. Discussions and remarks

In the standard three-family scheme the phenomena of flavor mixing and CP violation are described by 3×3 Cabibbo-Kobayashi-Maskawa (CKM) matrix V in the quark sector and the 3×3 PMNS matrix U in the lepton sector, respectively. Both the CKM and PMNS parameters are fundamentally important and thus should be measured as precisely as possible so as to understand the underlying flavor dynamics beyond the SM. It is obvious that the lepton flavor mixing pattern is quite different from the quark flavor mixing pattern, and a possible reason for this difference should be more or less related to the mechanism responsible for the origin of tiny neutrino masses. When taking into account the unitarity requirement of the CKM matrix V, we may easily derive a remarkable ordering for the nine elements of V, as first observed in Ref. [15]: $|V_{tb}| > |V_{ud}| > |V_{cs}| \gg |V_{us}| > |V_{cd}| \gg |V_{cb}| > |V_{ts}| \gg |V_{td}| > |V_{ub}|$. In comparison, the PMNS matrix U does not have such a structural hierarchy. As illustrated in Fig. 3, U actually exhibits an approximate μ-τ permutation symmetry (i.e., $|U_{\mu i}| \sim |U_{\tau i}|$ for $i = 1, 2, 3$).

The uniqueness of the Jarlskog invariant as a rephasing-independent measure of weak CP violation is a key feature of the 3×3 unitary flavor mixing matrix. In this talk we have reported a Pythagoras-like theorem for the expression of $\mathcal{J}_\nu$ in terms of the CP-conserving quantities $\mathcal{R}_{12}$, $\mathcal{R}_{13}$ and $\mathcal{R}_{23}$, which all appear in the probabilities of $\nu_\mu \to \nu_e$ and $\overline{\nu}_\mu \to \overline{\nu}_e$ oscillations. Some discussions based on the novel analytical results can be found in Ref. [10] about how Eq. (4) will be modified if the unitarity condition of U in Eq. (2) is slightly violated. What is more, such a Pythagoras-like theorem can be simply extended to the studies of $\nu_\mu \to \nu_\tau$ and $\overline{\nu}_\mu \to \overline{\nu}_\tau$ or $\nu_e \to \nu_\tau$ and $\overline{\nu}_e \to \overline{\nu}_\tau$ oscillations. A similar idea is also applicable to the quark sector.

It is well known that the Jarlskog invariant can be expressed in terms of any four independent moduli of the nine quark or lepton flavor mixing matrix elements, as guaranteed by the relevant unitarity conditions. We have found that, *for the first time*, the correct size of the Jarlskog invariant of CP violation in the quark sector can be numerically calculated from eight different sets of well measured moduli of the CKM matrix elements without making any special assumption [16]. This encouraging observation implies that the unitarity of the CKM matrix deserves a further test in the upcoming precision measurement era of quark flavor physics characterized by the High-Luminosity Large Hadron Collider. We expect that the same is true in the lepton sector in the near future.

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