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The Semi-Leptonic Weak Hamiltonian: Going Beyond Two-Loops

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The unitarity condition of the Cabibbo-Kobayashi-Maskawa provides an important precision test of the Standard Model. This test requires a good control of different theoretical contributions. In this talk, we discuss the short-distance corrections to the weak effective theory as well as the lattice-to-continuum matching for the semi-leptonic four-fermion operator. We compare different renormalisation schemes used in the computation of these radiative corrections, namely the W-Mass scheme and the $\overline{\mathrm{MS}}$ scheme. We also discuss the calculation of the two-loop $O(\alpha\alpha_s)$ electroweak corrections and the corresponding three-loop $O(\alpha\alpha_s^2)$ anomalous dimension for the effective theory Wilson coefficient. We also present numerical results for the $O(\alpha\alpha_s)$ conversion factor to the regularisation independent scheme, which is used in Lattice calculations.

The European Physical Society Conference on High Energy Physics (EPS-HEP2023)
21-25 August 2023
Hamburg, Germany

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1. Introduction

Leptonic and semi-leptonic decays of mesons and nuclear beta decays probe the CKM matrix and provide an electroweak precision test of the standard model (SM), see for example [1] and [2]. The precise measurements of Kaon decays [3] and nuclear beta decays [4] test Cabibbo-Kobayashi-Maskawa (CKM) matrix unitarity in the first row

$$\Delta_{CKM} = 1 - |V_{ud}^2| - |V_{us}^2| - O(|V_{ub}^2|) = 0. \tag{1}$$

Recent analyses and experimental results [5] have uncovered a tension of roughly 3σ with respect to the SM interpretation of these processes. Thus, an improvement in the precision of the theoretical predictions is required. In particular, the extraction of the CKM matrix elements relies on precise theoretical predictions of short distance QED and electroweak corrections, a determination of the relevant decay constants and form factors, which can be done in the framework of lattice QCD, see for example [6], and the treatment of isospin breaking corrections and long distance QED effects using a combination of both chiral perturbation theory and lattice field theory [7, 8]. In the future a first principle Lattice calculation of the QED corrections will become available and have to be matched to continuum perturbation theory.

In this talk, we describe the systematic inclusion of higher-order corrections to the short-distance contributions for the weak effective theory. In particular, we discuss the calculation of two-loop $O(\alpha\alpha_s)$ electroweak corrections at the high-scale, the three-loop $O(\alpha\alpha_s^2)$ Anomalous Dimension (ADM) and the matching to renormalsation schemes at the hadronic sale.

2. Effective Field Theory For Semi-leptonic Decays

The short-distance contribution to semi-leptonic processes can be approximated in the Standard Model, to an outstanding precision, by an effective Hamiltonian

$$\mathcal{H}(x) = \frac{4}{\sqrt{2}} G_F V_{ud}^* C_O O(x) \tag{2}$$

that involves only one charged current effective operator $O(x) = \left(\bar{d}(x)\gamma^\mu P_L u(x)\right)\left(\bar{\nu}_\ell(x)\gamma_\mu P_L \ell(x)\right)$, where $P_L = (1-\gamma^5)/2$, G_F is the Fermi constant extracted from the muon lifetime and C_O is the Wilson coefficient that is equal to one at leading order in the electroweak interactions. The full-theory description of the semi-leptonic processes is recovered by imposing a matching condition on S matrix elements at the electroweak scale $\mu_{EW} \sim M_W$, where M_W is the mass of the W boson; this allows for the extraction of the relevant Wilson coefficients that encodes the short-distance behaviour.

2.1 W-Mass Renormalisation Scheme

Higher-order corrections to the short-distance contribution are traditionally calculated in the so-called W-Mass scheme [9], which is still used in phenomenological analyses for the semi-leptonic decays [7, 8]. In this scheme, the amplitude is regularized and renormalised by splitting the photon propagator into two terms according to

$$\frac{1}{k^2} \longrightarrow \frac{1}{k^2 - M_W^2} - \frac{M_W^2}{k^2 - M_W^2}\frac{1}{k^2} = \gamma_> + \gamma_<, \tag{3}$$

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where k is the momentum carried by the photon. The first term of (3), i.e. $\gamma_>$ acts as a massive photon propagator and only diagrams that involve $\gamma_>$ are UV divergent after QCD renormalisation. At $O(G_F\alpha)$ all poles are absorbed in the renormalisation of the Fermi constant. The second term, that is $\gamma_<$, is UV finite, thanks to the W-boson mass M_W acting as a hard UV cut-off, but results in an IR contribution to the Fermi constant of $O(\alpha m_\mu^2/M_W^2)$. When G_F is used to normalize the weak Hamiltonian, such a contribution, while small, breaks the manifest separation of scales that is a main virtue of the effective field theory approach. Moreover, the presence of a hard UV cut-off, in this case the W-boson mass, does not allow for a straightforward re-summation of the large logarithms in renormalisation group improved perturbation theory. QED corrections for leptonic and semi-leptonic decays were calculated both in the current algebra approach [10], in chiral perturbation theory [11–13] or in a combined approach with chiral perturbation theory [7] where the electroweak box diagrams are calculated with lattice gauge theory [14].

The $\overline{\text{MS}}$ scheme is another possible scheme. It has been employed in Ref. [15, 16] for the calculation of higher-order QED corrections to the Fermi theory that determine G_F as defined in Ref. [17]. This scheme is also used for the calculation of electroweak corrections to the weak effective Hamiltonian [18], where the electroweak matching corrections and next-to-leading order anomalous dimensions for the operator O are given in Ref. [19]. Advantages of the $\overline{\text{MS}}$ scheme are the separation of the electroweak and hadronic scale, and the EFT description of the decays. Furthermore, this scale separation, together with the EFT approach, simplifies the new physics interpretation and allows for a systematic inclusion of higher-order radiative corrections [20, 21].

The Fermi Operator in its Fierz-rearranged form $(\bar{\mu}\gamma_\mu P_L e)(\bar{\nu}\gamma^\mu P_L \nu)$ comprises a conserved QED current for $m_\mu = m_e$. This symmetry restricts the scheme transformation between the $\overline{\text{MS}}$ and W-Mass scheme, since the decomposition (3) preserves the QED Ward identity. This is however not true for the semi-leptonic operator O.

Scheme Transformation

Practical calculations for semi-leptonic corrections in the W-Mass scheme only involve the UV-finite photon propagator $\gamma_<$, since the contribution of $\gamma_>$ is absorbed into the definition of G_F. Yet, this calculational procedure is not well suited for a systematic scheme transformation to the $\overline{\text{MS}}$ scheme, as the later scheme comprises the full range of momenta. To this end, we re-defined the W-Mass scheme as an "on-shell" renormalisation scheme, where we split up the amplitudes into $A_<$ and $A_>$ comprising $\gamma_<$ and $\gamma_>$ respectively. All amplitudes $A_>$ are cancelled by the introduction of a local counter-term, as shown in Fig. 1, including the finite part. The amplitude $A_<$ are finite after QCD subdivergencs have been subtracted. This defines the renormalisation condition in the W-Mass scheme both for the operators and the external fermion fields. Once the operator's renormalisation constant $Z_O^{W\text{-Mass}}$ is extracted, we can define the scheme transformation as

$$C^{\overline{\text{MS}}\to W\text{-Mass}} = Z_O^{\overline{\text{MS}}}\left(Z_O^{W-Mass}\right)^{-1}. \tag{4}$$

Here, both $Z_O^{\overline{\text{MS}}}$ and $Z_O^{W\text{-Mass}}$ are matrices, that involve also an evanescent operator, see Ref. [20] for details and specification of the notation. Using this definition of the scheme transformation, we compared our results in the $\overline{\text{MS}}$ with the results expressed in the W-Mass scheme [7], finding a total agreement.

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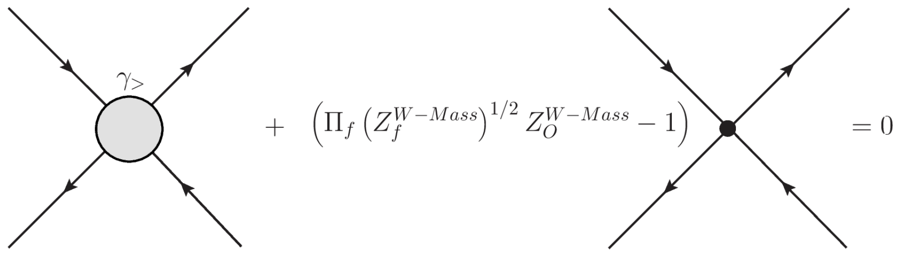

Figure 1: On-shell formulation of the W-Mass renormalisation conditions for the effective semi-leptonic operator. The renormalisation constants for the external fields are found by imposing similar conditions on the two-point Green's functions.

2.2 $\overline{\text{MS}}$ renormalisation Scheme

In Ref. [22] we calculate the electroweak corrections including next-to-leading logarithmic QCD corrections to the Wilson coefficient C_O. This calculation comprises the determination of the $O(\alpha\alpha_s)$ correction to C_O and the corresponding three-loop $O(\alpha\alpha_s^2)$ Anomalous Dimension (ADM) that governs its scale dependence. We employ the $\overline{\text{MS}}$ renormalisation scheme with Naive Dimensional Regularization (NDR), as no ambiguous traces involving an anti-commuting γ^5 appear. The calculation of the renormalisation constant is performed using Infra-Red re-arrangement [23]. This allows us to expand in external momenta and isolate the UV poles that determine the ADM. We explicitly check locality and gauge independence of our results. Here and also for the matching calculation we use the general R_ξ gauge for all gauge bosons.

The inclusion of the three-loop ADM improves the precision of the Renormalisation Group Equation (RGE) for C_O. The evolution kernel satisfies $\mu\frac{d}{d\mu}\mathcal{U} = \gamma_{OO}\mathcal{U}$ and its perturbative solution is given by

$$\begin{aligned}\mathcal{U}^{\overline{\text{MS}}}(\mu_1,\mu_2) = \left(\frac{\alpha(\mu_1)}{\alpha(\mu_2)}\right)^{\frac{\gamma^{(e)}_{OO}}{2\beta_0}} \left(\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)}\right)^{-\frac{\gamma^{(es)}_{OO}}{2\beta_{0,s}}\frac{\alpha(\mu_1)}{4\pi}} \Bigg(1 + \frac{\gamma^{(ee)}_{OO}}{2\beta_0}\left(\frac{\alpha(\mu_1)-\alpha(\mu_2)}{4\pi}\right) + \\ + \frac{\alpha(\mu 1)}{2\beta_{0,s}}\left(\gamma^{(es)}_{OO}\frac{\beta_{1,s}}{\beta_{0,s}} - \gamma^{(ess)}_{OO}\right)\left(\frac{\alpha_s(\mu_1)-\alpha_s(\mu_2)}{4\pi}\right)\Bigg),\end{aligned} \tag{5}$$

where γ^{ee}_{OO} and γ^{ess}_{OO} are the two-loop $O(\alpha^2)$ [21] and the three-loop $O(\alpha\alpha_s^2)$ ADM respectively.

The electroweak corrections at μ_{EW} and the ADM calculation are new results and were not included in the recent effective field theory determination of the V_{ud} CKM matrix element from neutron $\beta-$decay [21]. Their inclusion could potentially have an impact on the central value of this theoretical extraction of V_{ud} and hence impact the unitarity condition (1) of the CKM matrix.

2.3 Lattice Matching

The non perturbative hadronic matrix elements $\langle\pi(p)|\bar{d}\gamma^\mu P_L u|K(p')\rangle$ can be evaluated on the Lattice [24]. The operator O renormalises in the presence of QED corrections and the matrix elements has to be renormalised. To make contact to the $\overline{\text{MS}}$ continuum calculation, an intermediate scheme, such as Regularisation Independent (RI) [25, 26] scheme has to be introduced and a corresponding conversion factor has to be calculated. We derived the two-loop conversion factor

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in Ref. [20], that is needed at the NLL QCD accuracy. The definition of the lattice renormalisation scheme is sensitive to the choice of projectors. In particular, the projectors that has been used in the literature before lead to unnecessary $O(\alpha_s)$ contributions, which we could remove by a judicious choice for the projectors. The resulting improvement is shown in Fig. 2, where only a tiny residual scale dependence is present for the scheme defined with the improved projector.

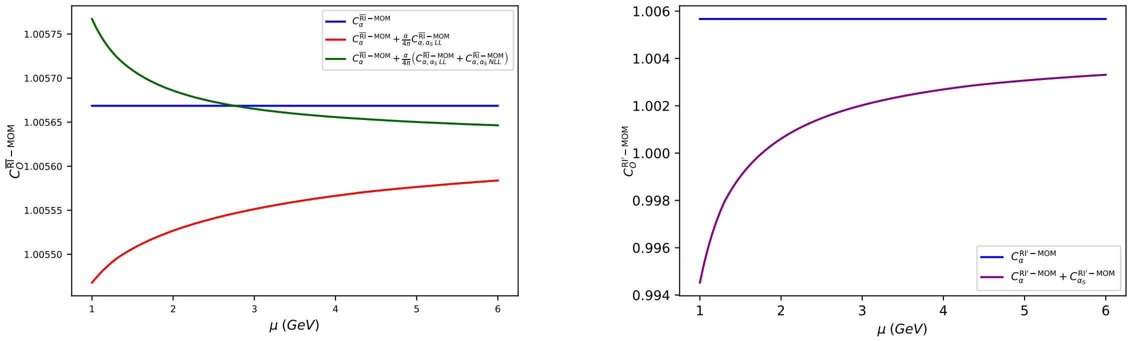

Figure 2: On the left: The Wilson coefficient in our expression of RI schemes. Scale dependence starts only at $O(\alpha\alpha_s)$. On the right: The Wilson coefficient in the traditional RI schemes. Scale dependence starts already at $O(\alpha_s)$.

3. Conclusions

In this proceeding we discussed higher order correction for the weak effective field theory of semi-leptonic processes. We explained the scheme conversions of the W-Mass and the $\overline{\text{MS}}$ renormalisation schemes at $O(\alpha\alpha_s)$, the two-loop $O(\alpha\alpha_s)$ electroweak matching and the three-loop $O(\alpha\alpha_s^2)$ anomalous dimension calculation. We plan [22] to use these results to improve upon the effective field theory analysis of the neutron $\beta-$decay, which could potentially impact the experimental test of CKM unitarity. In addition, we discussed our recent matching at $O(\alpha\alpha_s)$ between the non-perturbative lattice schemes and the $\overline{\text{MS}}$ schemes.

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