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Bootstrapping perturbative and non-perturbative defect correlators

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Gabriel Bliard[a,b]

[a] *Laboratoire de Physique ,*
École Normale Supérieure,
Université PSL, CNRS, Sorbonne Université, Université Paris Cité,
24 rue Lhomond, F-75005 Paris, France

[b] *Institut für Physik, Humboldt-Universität zu Berlin and IRIS Adlershof, Zum Großen Windkanal 2, 12489 Berlin, Germany*

E-mail: gabriel.bliard@ens.fr

This proceeding covers results presented at the European network for Particle physics, Lattice field theory and Extreme computing (EuroPLEx2023) conference in Berlin in September 2023. Holographic Wilson lines and defects are studied and reviewed through the complimentary lenses of the analytical conformal bootstrap and the lattice formulation of the gauge-fixed string action. In the first approach, the constraints coming from superconformal symmetry along with minimal additional input are used to fix perturbative correlators of insertions on Wilson lines at strong coupling. In the second, the holographic dual of a Wilson line is considered. The worldsheet formulation of type IIB string theory is discretised in an attempt to describe the system non-perturbatively while preserving the most symmetry possible and matching perturbative computations in the continuum. The overlap and interplay of the methods are discussed.

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1. Introduction

Conformal field theories (CFTs) have enjoyed many successes in the past half-century, providing a better understanding of the renormalisation flow in quantum field theory (QFT) and critical phenomena [1], being dual to quantum gravitational systems via holography [2] and through the technical successes of the numerical bootstrap [3]. In recent years, this shift from a Lagrangian to a non-Lagrangian description has underlined the importance of conformal defects [4, 5]. These play a central role in holographic theories as their description includes Wilson lines in conformal theories [6]. The analysis of these provides a two-fold advantage: on one hand, the one-dimensional theory is simpler so one can focus on building techniques to then apply to higher-dimensional setups, on the other, the resulting information provides data on a larger array of non-local operators such as deformed Wilson lines [7].

Conformal symmetry is extremely constraining. The corresponding transformations preserve the angles of the space and comprise translations, Lorentz transformations, dilatations and special conformal transformations. The main consequence of this symmetry is the operator product expansion (OPE). Within a correlator, a product of two operators can be replaced by an infinite sum of single operators. As a consequence, all correlators are related to the CFT data; the 2- and 3-point functions between all operators of the theory.

The Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [8] is a relationship between a gravity theory in $d + 1$ dimensions and a CFT in d dimensions. This correspondence provides a powerful tool for studying strongly coupled CFTs using classical gravity theories. In particular, operators in strongly-coupled CFTs correspond to free fields in AdS (called generalised free fields or GFF) and their correlators can be computed using Witten diagrams [9].

The interplay of these two fundamental concepts will be explored in this proceeding. In section 2, the analytic conformal bootstrap will be presented as a tool to compute perturbations near GFF through two examples: a four-point correlator on the 1/2-BPS Wilson line in $\mathcal{N} = 3$ Chern-Simons theory with matter [10] and a five-point correlator on the 1/2-BPS Wilson line in $\mathcal{N} = 4$ super Yang-Mills theory [8]. In section 3, the direct holographic model is studied for the cusped Wilson line in $\mathcal{N} = 4$ SYM. In this context, the string worldsheet is discretised in a way which preserves more symmetry than previous attempts in an effort to have a regularisable theory.

The link between these two sections is clear from their complimentary approach. The first is analytic and provides exact results at each perturbative order giving access to CFT data. However, it relies on a single non-perturbative input to define the coupling constant. In supersymmetric theories, this is done through localisation and other exact results. The second section, the lattice description, could provide this information without the need for supersymmetry and therefore extend the power of the bootstrap approach.

As the conference overlapped with the defence of the author's thesis, many of the results presented here can be found in [11]. I invite the interested reader to refer to it for a deeper analysis, a more extensive review and a more exhaustive list of references.

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2. Successes of the Analytic bootstrap

The conformal bootstrap is a minimalist approach aimed at constraining CFTs using only symmetries and a minimal physical input. This method is particularly effective for extracting the conformal data of a theory, including the data related to defect theories [3]. The analytic bootstrap adds a concept well-known in the context of scattering amplitudes; an ansatz.

The analytic conformal bootstrap involves several key steps. First, it begins with the formulation of an ansatz for the perturbative expansion of correlators involving harmonic polylogarithms, where the transcendentality increases with each perturbative order [12]. This ansatz acts as a starting point for the analysis. Second, constraints are imposed based on the OPE. These constraints are enough to fix a minimal solution entirely which is then related to the full correlator with physical considerations; the physical input. From this solution, the perturbative CFT data is extracted. Finally, the recursive unmixed CFT data are inputted to the next perturbative order. This recursive aspect is vital for iteratively refining the solutions based on the previous order.

Despite the power of this method, its application becomes significantly more challenging in systems with fewer symmetries. This is because the constraints from the OPE are reduced, non-perturbative results are scarcer, and there is a greater degree of operator mixing. In one-dimensional systems, the mixing problem is particularly acute, as single and double-trace operators are indistinguishable, complicating the bootstrap process. The method's elegance lies in its systematic approach to understanding complex theories using fundamental symmetry principles and minimal assumptions. This process is explained in detail in [11, 13, 14] but will be illustrated for the case of the four-point function of the displacement operator inserted on the 1/2 BPS Wilson line in ABJM [10] and for the five-point function of the displacement operator inserted on the 1/2-BPS Wilson line in $\mathcal{N} = 4$ super Yang-Mills [8].

2.1 Bootstrapping ABJM dCFT

Let us illustrate this concept with the case of the 4-point correlator of the displacement supermultiplet on the 1/2-BPS Wilson line in $\mathcal{N} = 3$ Chern-Simons theory with matter (ABJM) studied in [11, 13]. The displacement supermultiplet can be written as a chiral/anti-chiral superfield, annihilated by half of the supercharges. Each element of it is related to a broken symmetry of the embedding theory, so that insertions of the highest-weight component on the line corresponds to acting with translations perpendicular to the line, therefore deforming the defect [4, 7]. From a purely $1d$ perspective, this is a superconformal theory where these insertions are the fundamental operators and the correlators describe the theory. In particular, the supercorrelator

$$\langle \Phi_{j_0}(y_1,\theta_1)\bar{\Phi}_{-j_0}(y_2,\bar{\theta}_2)\Phi_{j_0}(y_3,\theta_3)\bar{\Phi}_{-j_0}(y_4,\bar{\theta}_4)\rangle = \frac{C^2_{\Phi_{j_0}}}{\langle 1\bar{2}\rangle^{\frac{2j_0}{3}}\langle 3\bar{4}\rangle^{\frac{2j_0}{3}}} f(\mathcal{Z}), \tag{1}$$

describes all four-point functions of the supermultiplet, where

$$\mathcal{Z} = \frac{\langle 1\bar{2}\rangle\langle 3\bar{4}\rangle}{\langle 1\bar{4}\rangle\langle 3\bar{2}\rangle} \qquad\qquad \mathcal{X} = -\frac{\langle 1\bar{2}\rangle\langle 4\bar{3}\rangle}{\langle 1\bar{3}\rangle\langle 2\bar{4}\rangle} \tag{2}$$

are cross-ratios invariant under the superconformal transformations, and $\langle 1\bar{2}\rangle$ is the superdistance. Given that the 1-, 2- and 3- point functions are entirely fixed by the superconformal symmetry, this is the first correlator with dynamical information, through the CFT data of unprotected long operators. We can focus on the four-point function of the lowest-weight operator by setting

$$\theta_i = \bar{\theta}_i = 0. \tag{3}$$

This function admits an OPE expansion with constraints from the selection rules, the additional constraint from the integrated correlator in [15] allows us to fix the perturbative expansion from a generic ansatz of Harmonic Polylogarithms (HPL) with increasing transcendentality.

$$\begin{aligned}
f^{(0)}(\chi) &= 1 - \frac{\chi}{\chi-1} \qquad (4)\\
f^{(1)}(\chi) &= \epsilon\left(\left(\frac{1}{\chi}+2\right)\log(1-\chi) + \frac{-\chi+(3-2\chi)\chi\log(\chi)+1}{(\chi-1)^2}\right)\\
f^{(2)}(\chi) &= \epsilon^2\chi\left(\frac{(-2\chi^5+7\chi^4-9\chi^3)\log^2(\chi)}{2(\chi-1)^3\chi^3} - \frac{(1-2\chi)^2\log(1-\chi)\log(\chi)}{4(\chi-1)^2\chi^2}+\right.\\
&\left.+\frac{(-2\chi^5+7\chi^4-6\chi^3+\chi^2)\log(\chi)}{2(\chi-1)^3\chi^3} + \frac{1}{(\chi-1)\chi} + \chi \leftrightarrow 1-\chi\right).
\end{aligned}$$

Where ϵ is the perturbative expansion parameter, related to the 't Hooft coupling through [15] and χ is the space-time cross-ratio obtained by the bosonic component of $\mathcal{X}$ in 2. The third order as well as the relevant CFT data can be found in [11]. In determining these quantities, the integrated correlators established in [15] were invaluable as they provided non-perturbative input to fix ϵ in terms of the 't Hooft coupling as well as justified some of the constraints used in [13]. The information one can gather from these correlators is not only the CFT data, but also the features and functional form of the correlators. These can also be used to determine the first orders of the multipoint cases, analogously to [16–18] ultimately describing features of the deformed line.

2.2 Bootstrapping Multipoints in $\mathcal{N} = 4$

The same process was applied to the case of insertions on the 1/2-BPS Wilson line in $\mathcal{N} = 4$ Super Yang-Mills theory. The 4-point case was bootstrapped in [19]. However, the analytic bootstrap technique is not restricted to this case and can be applied for higher-point functions of recent interest [18, 20, 21]. In this context, we study the correlator

$$\langle \mathcal{D}_1\mathcal{D}_1\mathcal{D}_1\mathcal{D}_1\mathcal{D}_1\mathcal{D}_2\rangle = \frac{(u_1\cdot u_4)(u_2\cdot u_5)(u_3\cdot u_5)}{\tau_{14}^2\tau_{25}^2\tau_{35}^2}\mathcal{A}(\chi_1,\chi_2;r_1,r_2,s_1,s_2,t), \tag{5}$$

where $\mathcal{D}_k$ are the 1/2-BPS operators related to the displacement multiplet. In the limit where the fermionic coordinates are set to 0, the five-point correlator depends on 2 spacetime cross-ratios and

5 R-symmetry cross-ratios

$$\chi_1 = \frac{\tau_{12}\tau_{45}}{\tau_{14}\tau_{25}} \qquad \chi_2 = \frac{\tau_{13}\tau_{45}}{\tau_{14}\tau_{35}} \tag{6}$$

$$r_1 = \zeta_1\zeta_2 = \frac{y_{12}^2 y_{45}^2}{y_{14}^2 y_{25}^2}, \qquad s_1 = (1-\zeta_1)(1-\zeta_2) = \frac{y_{15}^2 y_{24}^2}{y_{14}^2 y_{25}^2}, \tag{7}$$

$$r_2 = \eta_1\eta_2 = \frac{y_{13}^2 y_{45}^2}{y_{14}^2 y_{35}^2}, \qquad s_2 = (1-\eta_1)(1-\eta_2) = \frac{y_{15}^2 y_{34}^2}{y_{14}^2 y_{35}^2}, \tag{8}$$

$$t = \frac{y_{15}^2 y_{23}^2 y_{45}^2}{y_{14}^2 y_{25}^2 y_{35}^2}, \tag{9}$$

The relation between these is constrained by the supersymmetry, most conveniently expressed in terms of superconformal Ward identities [18, 22–24]. As a result, the full correlator

$$\mathcal{A} = F_0 + \frac{r_1}{\chi_1^2}F_1 + \frac{s_1}{(1-\chi_1)^2}F_2 + \frac{r_2}{\chi_2^2}F_3 + \frac{s_2}{(1-\chi_2)^2}F_4 + \frac{t}{\chi_{12}^2}F_5\,, \tag{10}$$

depends on three functions related by braiding and crossing symmetry. Expanding this correlator at strong coupling, the first order can be found by performing Wick contractions

$$F_5^{(0)} = \sqrt{2}. \tag{11}$$

and the second order can be bootstrapped using the ansatz

$$\begin{aligned} -\frac{F_5^{(1)}(\chi_1,\chi_2)}{\chi_{12}^2} &= \\ p(\chi_1,\chi_2) + r_1(\chi_1,\chi_2)\log(\chi_1) + r_1(\chi_2,\chi_1)\log(\chi_2) + r_1(1-\chi_1, 1-\chi_2)\log(1-\chi_1) \\ + r_1(1-\chi_2, 1-\chi_1)\log(1-\chi_2) + r_5(\chi_1,\chi_2)\log(\chi_2-\chi_1). \end{aligned} \tag{12}$$

The same methods as above and the input from localisation for the topological operators [25], constrain it to

$$r_1(\chi_1) = \frac{\sqrt{2}}{\sqrt{\lambda}}\frac{\chi_1\left(\chi_1^2 - 3\chi_1\chi_2 + 4\chi_2^2\right)}{\chi_2^2(\chi_2-\chi_1)^3} \tag{13}$$

$$r_5(\chi_1-\chi_2) = \frac{\sqrt{2}}{\sqrt{\lambda}}\left(\frac{1}{\chi_1^2} + \frac{1}{(\chi_1-1)^2} + \frac{1}{\chi_2^2} + \frac{1}{(\chi_2-1)^2}\right) \tag{14}$$

$$p(\chi_1,\chi_2) = \frac{1}{2\sqrt{2}\sqrt{\lambda}}\left(\frac{4}{(\chi_1-1)(\chi_2-1)} + \frac{4}{\chi_1\chi_2} + \frac{19}{(\chi_1-\chi_2)^2}\right) \tag{15}$$

where λ is the 't Hooft coupling for $\mathcal{N}=4$ SYM. From this a number of conclusions can be drawn; the first as that the mixing of operators does not happenat this order, the second is the CFT data presented in [11, 26, 27]. One can also do this for the case for 6-point functions, which is being explored using many different methods such as the numerical bootstrap and explicit computations in the conformal gauge [20, 21].

2.3 Non-perturbative aspects of the correlators

One of the aspects of the perturbative expansion of the OPE is that the highest power of logarithms at each order is entirely fixed in terms of tree-level CFT data. There is a limit in which this is the leading contribution of the correlator; the double-scaling limit [28]. In this limit, all the perturbative orders of the leading behaviour can be fixed and resummed using the Borel transform. This matches computations done independently in [28] and constitutes some non-perturbative aspects of the correlator. Let us define the function related to (1) by

$$h(\chi) = f(\frac{\chi}{\chi - 1}) \tag{16}$$

Since the anomalous dimension was found $\gamma_\Delta^{(1)} = \Delta(\Delta + 2)$, the highest logarithmic power at each order can be found by resumming tree-level data. The double scaling limit is

$$\chi = \frac{2}{1 - i\sinh(t)} \qquad\qquad \kappa = \frac{\mathrm{e}^t}{4\sqrt{2\lambda}} \tag{17}$$

where λ goes to ∞ and t goes to ∞ such that κ is constant. In this sense, we are looking at the finite coupling but very near OPE limit. In this limit, the function h has the behaviour order by order

$$h_{\mathrm{DS}}(\kappa) = \sum_n (-\kappa)^n \Gamma(n+1) + O(\mathrm{e}^{-t}). \tag{18}$$

This series is Borel resummable where the Borel transform and resummation are respectively

$$\hat{h}_{\mathrm{DS}}(\zeta) = \frac{1}{1+\zeta} \qquad\qquad h_{\mathrm{DS}}(\mu) = \kappa^{-1} U\left(1, 1, \kappa^{-1}\right), \tag{19}$$

Here U is a confluent hypergeometric function of the second kind and one can expand it around $\kappa^{-1} = 0$ which gives a series in $\sqrt{\lambda}$ as opposed to $\frac{1}{\sqrt{\lambda}}$. This matches equation (3.20) in [28] if we identify $\Delta_V = \Delta_W = \frac{1}{2}$, corresponding to the conformal weight of the superconformal primary. This formalism in terms of conformal blocs is a reorganising of the perturbative series, where the (partially unmixed) conformal data is in reality contained in all perturbative orders as the leading powers of logarithms. In this way, one can access a limit of the non-perturbative correlator simply with first-order CFT data and exponentially suppressed terms with second-order data.

3. The need for non-perturbative data: contribution from lattice models

In the analytic bootstrap process presented in the previous section, there are two key elements which were needed. The first was conformal symmetry, which exists in many models of physical systems (such as the critical point of the Ising model). The second crucial aspect which was needed was a guiding line to map the bootstrap result to the physical theory. This was done in both cases above by having access to non-perturbative data. In the first, the integrated correlator constraints found in [15] related the bootstrap constant at each order to the Bremsstrahlung function, which is known to all order in λ [29]. In the second, the explicit conformal data in the topological sector can be used to fix the bootstrap constants since these are also known through localisation [25]. As we start studying systems with fewer supersymmetries, the need for non-perturbative data becomes a crucial aspect of the computation. One way to obtain this data is through lattice computations. Work has been done in [30–32] to discretise the worldsheet in the context of type IIB superstring theory in $\mathrm{AdS}_5 \times S^5$. This would allow for lattice simulations and the obtaining of non-perturbative data. Even though some of these quantities such as the cusp anomalous dimension are known at all orders [6], this paves the way for similar studies of non-perturbative worldsheet string theory.

3.1 Holography and the cusped Wilson Line

The cusped Wilson line studied in [33] is an interesting object. On the gravity side, it can be described by a type IIB superstring in $AdS_5 \times S^5$ with its boundary on the Wilson line. In particular, the worldsheet action can be gauge-fixed in the light-cone gauge giving the continuum action

$$\begin{aligned} S^{\text{cont}}_{\text{cusp}} = g \int dt ds \quad & \Big\{ \left|\partial_t x + \tfrac{m}{2} x\right|^2 + \tfrac{1}{z^4}\left|\partial_s x - \tfrac{m}{2} x\right|^2 + \left(\partial_t z^M + \tfrac{m}{2} z^M + \tfrac{i}{z^2} z_N \eta_i \left(\rho^{MN}\right)^i{}_j \eta^j\right)^2 \\ & + \tfrac{1}{z^4}\left(\partial_s z^M - \tfrac{m}{2} z^M\right)^2 + i\left(\theta^i \partial_t \theta_i + \eta^i \partial_t \eta_i + \theta_i \partial_t \theta^i + \eta_i \partial_t \eta^i\right) - \tfrac{1}{z^2}\left(\eta^i \eta_i\right)^2 \\ & + 2i\Big[\tfrac{1}{z^3} z^M \eta^i \left(\rho^M\right)_{ij}\left(\partial_s \theta^j - \tfrac{m}{2}\theta^j - \tfrac{i}{z}\eta^j\left(\partial_s x - \tfrac{m}{2} x\right)\right) \\ & \quad + \tfrac{1}{z^3} z^M \eta_i (\rho^{M\dagger})^{ij}\left(\partial_s \theta_j - \tfrac{m}{2}\theta_j + \tfrac{i}{z}\eta_j\left(\partial_s x - \tfrac{m}{2} x\right)^*\right)\Big]\Big\}, \end{aligned} \tag{20}$$

This action is interesting in several ways. First, it explicitly breaks the conformal invariance and part of the supersymmetry (though some of these are non-linearly realised and can be recovered e.g. in the conformal gauge). Second, it is a two-dimensional action which allows for lattice simulations of complexity L^2 where L is the length of the system. Thirdly, there are non-perturbative quantities which are known [6] and provide a guideline for this method. Finally, the fermions in question are grassmann-like objects, not full spinors, so are simpler to describe and simulate. As such, the study of this system provides a excellent starting point in the effort of describing worldsheet setups using lattice field theory.

3.2 Discretising the action and perturbative matching

The continuum version of the system was derived and studied in [33]. In this, the first perturbative correction to 0-, 1-and 2-point correlators we computed. In [34, 35], the discretised version of the action above was established and reviewed

$$\begin{aligned} S_{\text{cusp}} = g \sum_{s,t} a^2 \quad & \Big\{ \left|b_+ \hat{\partial}_t x + \tfrac{m}{2} x\right|^2 + \tfrac{1}{z^4}\left|b_- \hat{\partial}_s x - \tfrac{m}{2} x\right|^2 + \left(b_+ \hat{\partial}_t z^M + \tfrac{m}{2} z^M + \tfrac{i}{z^2} z^N \eta_i (\rho^{MN})^i{}_j \eta^j\right)^2 \\ & + \tfrac{1}{z^4}\left(\hat{\partial}_s z^M \hat{\partial}_s z^M + \tfrac{m^2}{4} z^2\right) + 2i\left(\theta^i \hat{\partial}_t \theta_i + \eta^i \hat{\partial}_t \eta_i\right) - \tfrac{1}{z^2}\left(\eta^i \eta_i\right)^2 \\ & + 2i\Big[\tfrac{1}{z^3} z^M \eta^i \left(\rho^M\right)_{ij}\left(b_+ \bar{\partial}_s \theta^j - \tfrac{m}{2}\theta^j - \tfrac{i}{z}\eta^j\left(b_- \hat{\partial}_s x - \tfrac{m}{2} x\right)\right) \\ & \quad + \tfrac{1}{z^3} z^M \eta_i (\rho^{M\dagger})^{ij}\left(b_+ \bar{\partial}_s \theta_j - \tfrac{m}{2}\theta_j + \tfrac{i}{z}\eta_j\left(b_- \hat{\partial}_s x^* - \tfrac{m}{2} x^*\right)\right)\Big]\Big\}. \end{aligned} \tag{21}$$

The guidelines for establishing this discretised action was the matching in the continuum while keeping as many symmetries as possible. In this context, some perturbative computations were made yielding results which could be made consistent with the continuum result. However, the system is naively non-renormalisable and the obtaining of finite results requires tuning. This conclusion indicates the need for additional constraints from symmetries to obtain finite quantities. As such, the future of this concept is in the exploration of simpler models in which the symmetry fixes the entire dependency of the fields on the lattice. In general, the action above is a specific case of a gauge-fixed non-linear-sigma-model (NLSM). The simplest supersymmetric NLSMs have a supersphere geometry which are known to be renormalisable. These are being studied in [36] and will provide a better understanding of the symmetry breaking when fixing the gauge or discretising the system and perhaps provide guidelines to study more complex systems such as the hyperbolic analog to the supersphere, which includes AdS geometries.

4. Conclusion

These two modern tools, the conformal bootstrap and lattice field theory, provide different approaches to study holographic Wilson lines. In the case of the analytic conformal bootstrap, perturbative correlators at strong-coupling can be fixed using superconformal symmetry, an ansatz and physical constraints. These provide insight into deformations of Wilson lines and describes the defect CFT through obtaining conformal data. However, the physical input needed at this stage is prohibitive for systems without supersymmetry or integrability. In the search for non-perturbative data and quantities, lattice field theory may provide a solution. While the current results cannot provide this data, the description of string worldsheets using a lattice attempts to answer a much more fundamental question: whether the non-perturbative definition of the the Green-Schwarz $AdS_5 \times S^5$ string can be accessed. While the initial studies in [34] indicate the importance of symmetry in such systems and underline the need for the study of simpler non-linear sigma models, they provide a starting point from which this technique can be expanded. Furthermore, the first successes in these descriptions will not only help the non-perturbative understanding of a few quantities, but could also be used as a non-perturbative guide to complement the bootstrap approach. While lattice field theory is very powerful at computing a few exact non-perturbative quantities, the bootstrap is extremely powerful at completing the spectrum through the consistency of the theory. These two approaches are complementary in the understanding of Wilson lines in holography.

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