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Mesonic decay constant and mass ratios and the conformal window

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Two particular ratios related to mesons are proposed for the study of the conformal window in $SU(3)$ gauge theory and fundamental fermions. Lattice and other studies indicate that the lower end, N_f^*, is at around 7 - 13 flavors which is a wide range without a clear consensus. Here we propose the decay constant to mass ratios of mesons, $f_{PS,V}/m_V$, as a proxy since below the conformal window lattice studies have shown that they are largely N_f-independent while at the upper end of the conformal window they are vanishing. The drop from the non-zero constant value to zero at $N_f = 16.5$ might be indicative of N_f^*. We compute f_V/m_V to N^{3}LO and f_{PS}/m_V to NNLO order in (p)NRQCD. The results are unambiguously reliable just below $N_f = 16.5$, hence the results are expanded á la Banks-Zaks in $\varepsilon = 16.5 - N_f$. The convergence properties of the series and matching with the non-perturbative infinite volume, continuum and chiral extrapolated lattice results at $N_f = 10$ suggest that the perturbative results might be reliable down to $N_f = 12$. A sudden drop is observed at $N_f = 12$ and $N_f = 13$ in f_V/m_V and f_{PS}/m_V, respectively.

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1. Introduction and summary

The lower end of the conformal window, usually denoted by N_f^*, for a given gauge group and fermion representation, has been an elusive object of study [1–20] . Naively, one would think the lattice approach would be an ideal way to study it because once the infinite volume, continuum and chiral limits are taken at each flavor N_f below the loss of asymptotic freedom, the results would be unambiguous with a similarly unambiguous conclusion about N_f^*. However it became clear that systematic effects close to the conformal window are significantly larger than at lower flavor numbers where the models are very similar to QCD. As a result there is not a clear consensus for $SU(3)$ and fundamental fermions, lattice results and non ab initio methods estimate N_f^* to be somewhere in the range $7 - 13$ which is rather broad.

In this contribution a new approach is proposed [21]: by matching the fully controlled lattice results obtained for low fermion numbers [22–24] and fully controlled perturbative results obtained close to but below $N_f = 16.5$ where asymptotic freedom is still present. The latter calculation will be the focus of our contribution. The former are available in the range $2 \le N_f \le 10$ and the task is then to study how far down the perturbative results can be trusted from $N_f = 16.5$ and if they can be matched to the last non-perturbatively obtained point at $N_f = 10$. The particular quantities to be investigated are dimensionless and finite ratios related to bound states; the decay constant to mass ratios of mesons. More precisely the ratios f_V/m_V and f_{PS}/m_V will be investigated in (p)NRQCD [25–29] which is the appropriate framework inside the conformal window. These are readily available in the range $2 \le N_f \le 10$ from past lattice studies either directly, as for f_{PS}/m_V, or indirectly by using the KSRF relations [30, 31] for f_V/m_V.

The setup of the perturbative calculation is as follows. We start from a CFT close to but below $N_f = 16.5$ which is weakly coupled as shown by Banks-Zaks [32]. All particles are of course massless and correlation functions fall off algebraically. A flavor singlet mass term is introduced which leads to bound states whose masses and decay constants are proportional to m^α with the same exponent $\alpha = 1/(1+\gamma)$ related to the mass anomalous dimension γ [33, 34]. The constant of proportionality can be computed perturbatively following the (p)NRQCD prescription. In the NRQCD language all N_f flavors are "heavy" and there are no "light" flavors and we need to keep the purely perturbative terms only. The ratio of decay constants and meson masses are then obtained as a series in the coupling with coefficients depending on N_f. In the final step both the coupling and any explicit N_f dependence is expanded in $\varepsilon = 16.5 - N_f$ leading to constant coefficients. The final series obtained in this way contains both powers of ε and its logarithm.

As always with a perturbative result its reliability or convergence properties are non-trivial. For the case of f_V/m_V we have N^{3}LO results and a comparison of the NNLO and N^{3}LO results show that it might be reliable down to $N_f = 12$. We assign a theoretical error by taking the difference between the last two available orders. The lattice result for f_V is not available directly, only for f_{PS} but we utilise the KSRF relation to estimate $f_V = \sqrt{2} f_{PS}$ on the range $2 \le N_F \le 10$. Curiously, the perturbative result at $N_f = 12$ is compatible with the last non-perturbative result at $N_f = 10$ within errors. Assuming a monotonous behavior the following picture emerges: f_V/m_V is constant outside the conformal window and drops sharply at around $N_f = 12$ finally reaching zero at $N_f = 16.5$. The sudden drop might be indicative of the lower end of the conformal window.

A similar analysis for the other ratio, f_{PS}/m_V, could not be fully carried out because the perturbative result is only available to NNLO order for f_{PS}. Nonetheless assuming the convergence properties are similar to f_V/m_V we are able to conclude similarly that the perturbative result might be reliable down to $N_f = 12$. A sudden drop in the ratio f_{PS}/m_V seems to occur at around $N_f = 13$.

The perturbative calculation can be viewed in one of two ways. First, as alluded to above, it may be thought of as perturbative (p)NRQCD without any terms containing Λ_{QCD} explicitly. Or, it is also instructive to view it as (p)NRQED with more diagrams due to the non-abelian nature of the interaction. If viewed this way the bound states in question are analogous to the positronium. This view is useful because it is easy to see that all decay constants and meson masses will be proportional to the fermion mass, just as the decay constants and masses of positronium are proportional to the electron mass. There is no other scale in QED than the electron mass, and there is no other scale than the fermion mass in our calculation either since we started from a CFT. The constant of proportionality in QED is well-known to contain powers of α as well as $\log(\alpha)$ which in our case will lead to ε and $\log(\varepsilon)$.

2. Perturbative results

As with any perturbative calculation a running scale μ is introduced and since we start from a CFT the natural scale is $\mu = m$, the mass of the fermions[1]. All results will be given in the $\overline{\mathrm{MS}}$ scheme and the renormalized coupling will be denoted by $g^2(\mu)/(16\pi^2) = g^2(m)/(16\pi^2) = a$. The decay constants and meson masses are expanded in a leading to [35–39]

$$\begin{aligned} f_{PS,V} &= b_0\, m\, a^{3/2} \left(1 + b_{10}a + b_{11}a \log a + b_{20}a^2 + b_{21}a^2 \log a + b_{22}a^2 \log^2 a + O(a^3)\right) \\ m_V &= c_0\, m \left(1 + c_{20}a^2 + c_{30}a^3 + c_{31}a^3 \log a + O(a^4)\right) \; . \end{aligned} \tag{1}$$

The leading term in the mass, $c_0 = 2$, just follows from having a free fermion and anti-fermion pair whereas the first correction $c_{20} < 0$ is familiar from the quantum mechanical binding energy in a Coulomb potential. Further radiative corrections are systematically obtained using (p)NRQCD. The leading expression, b_0, for the decay constants is proportional to the ground state wave function at the origin. The explicit form of the corrections, NNLO for f_{PS} and N^{3}LO for f_V, m_V can be found in [21].

In the ratio m drops out and the massless limit takes the running coupling to the fixed point $a(m) \to a_*$ which can be expanded in $\varepsilon = 16.5 - N_f$ and is known to 5-loops [40–45],

$$a_* = \varepsilon \left(e_0 + e_1\, \varepsilon + e_2\, \varepsilon^2 + e_3\, \varepsilon^3 + \ldots\right), \tag{2}$$

[1]One could choose any scale $\mu > m$

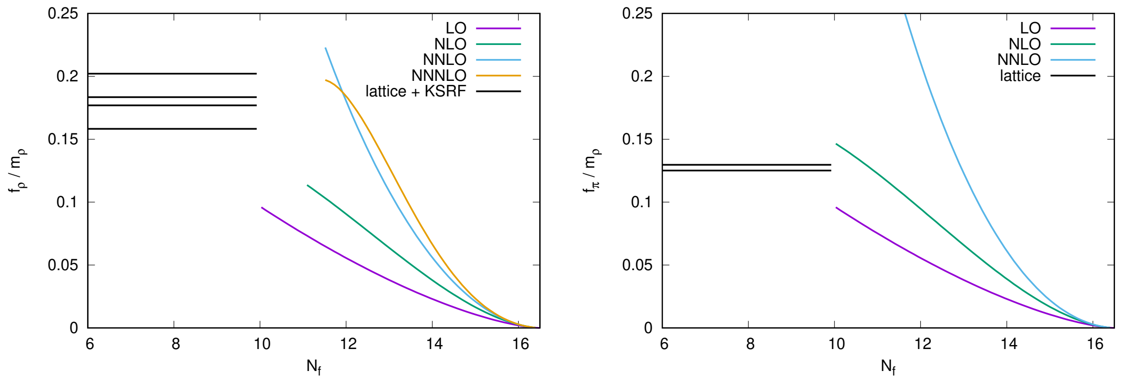

Figure 1: Left: the f_V/m_V ratio in increasing perturbative order. The non-perturbative result from combined lattice calculations [22–24] and the KSRF-relation is also shown. The smaller error band corresponds to the uncertainty of the lattice calculation, the wider one combines this with a conservative estimate of the uncertainty of the KSRF-relation itself. Right: The corresponding results for f_{PS}/m_V.

with some coefficients e_i. Combining this expansion with (1) leads to the final results,

$$
\begin{aligned}
\frac{f_V}{m_V} &= \varepsilon^{3/2} C_0 \left(1 + \sum_{n=1}^{3} \sum_{k=0}^{n} C_{nk}\, \varepsilon^n \log^k \varepsilon + O(\varepsilon^4) \right) \qquad (3) \\
C_0 &= 0.005826678 \\
C_{10} &= 0.4487893 \qquad C_{11} = -0.2056075 \\
C_{20} &= 0.2444502 \qquad C_{21} = -0.1624891 \quad C_{22} = 0.03522870 \\
C_{30} &= 0.10604(3) \qquad C_{31} = -0.1128420 \quad C_{32} = 0.03695458 \quad C_{33} = -0.005633665
\end{aligned}
$$

for the vector case and

$$
\begin{aligned}
\frac{f_{PS}}{m_V} &= \varepsilon^{3/2} C_0 \left(1 + \sum_{n=1}^{2} \sum_{k=0}^{n} D_{nk}\, \varepsilon^n \log^k \varepsilon + O(\varepsilon^3) \right) \qquad (4) \\
D_{10} &= 0.4654041 \quad D_{11} = -0.2056075 \\
D_{20} &= 0.2845697 \quad D_{21} = -0.1737620 \quad D_{22} = 0.03528692
\end{aligned}
$$

for the pseudo-scalar case.

Note that the coefficients in (3) and (4) are well-behaved as the order grows, in contrast to the generally factorially growing perturbative coefficients such as (1) and (2). Furthermore, since $f_{PS,V}/m_V$ are finite, RG-invariant, scheme independent physical quantities, all coefficients in (3) and (4) are scheme independent as well[2]. These coefficients are the main results of our work.

3. Matching low N_f and high N_f

The increasing orders for the two ratios are shown in figure 1. Clearly, the deviation between the NNLO and N^3LO results of f_V/m_V for $N_f \geq 12$ is not substantial. Quantitatively, in the range

[2]The coefficient C_{30} is only available in numerical form with some uncertainty at the moment.

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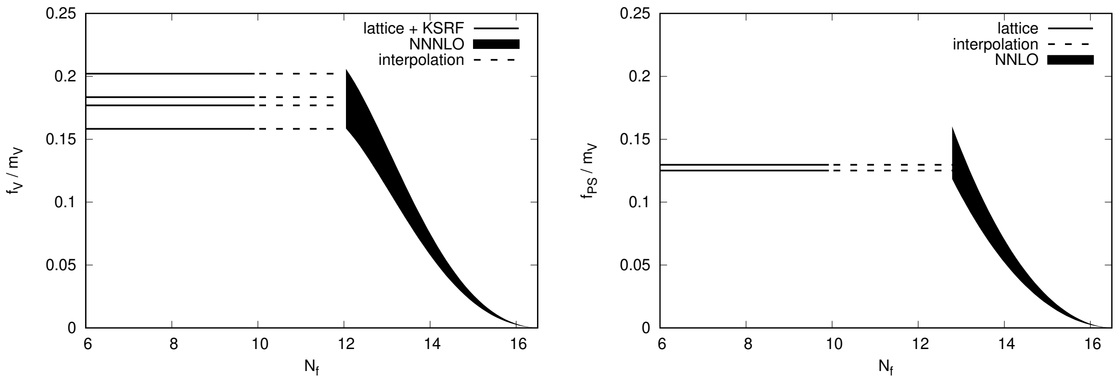

Figure 2: Non-perturbative lattice results in the range $2 \leq N_f \leq 10$ and the perturbative ones where they seem reliable. Assuming a monotonous behavior only a small range needs to be interpolated. Left: f_V/m_V, the wider error bands for $2 \leq N_f \leq 10$ includes the error from the usage of the KSRF relation. The error of the perturbative curve is estimated from the difference of the last two available orders. Right: f_{PS}/m_V, the error of the perturbative result is estimated from that of f_V/m_V.

$11.9 \leq N_f \leq 12.1$, the deviation between the NNLO and N^{3}LO results is at most 4%, or in the range $11.5 \leq N_f \leq 12.5$ at most 13%. We thus conclude that in the region of interest, $N_f \sim 12$, the N^{3}LO result is robust and reliable. We take as an estimate of the neglected higher orders the difference between the last two available orders.

In order to compare with the lattice results we would need f_V, however only f_{PS} was measured directly [22–24]. Here we use the KSRF relation $f_V = \sqrt{2} f_{PS}$ originating in vector meson universality to estimate f_V and assign a 12% uncertainty which holds in QCD. Somewhat unexpectedly the perturbative result at $N_f = 12$ matches the last non-perturbative lattice result almost exactly; see left panel of figure 2.

A similar analysis unfortunately cannot be completed for f_{PS}/m_V because f_{PS} is only available to NNLO order. One may nonetheless assume that the theoretical uncertainty is similar to that of f_V making a comparison with the lattice results at low N_f feasible; see right panel of 2.

In both cases a match between the low N_f and high N_f regions seems plausible. Assuming a monotonous behavior and trusting the lattice results below $N_f = 10$ and the perturbative ones above $N_f = 12$ or 13 leaves only a narrow range to be interpolated or more rigorously calculated in future lattice work. The following picture seems to emerge: a mostly N_f-independent flat curve drops sharply at around $N_f = 12$ and $N_f = 13$ for the two ratios, respectively. This sudden change in behavior might be indicative of N_f^*, the lower end of the conformal window.

4. Conclusion and outlook

In this contribution we presented a new approach to shed light on the emergence of conformal behavior from chirally broken dynamics as the flavor number increases which combines both perturbative and non-perturbative information. Well chosen dimensionless quantities were presented which can be easily measured in lattice calculations in the chiral limit and which can also be computed in perturbation theory, again in the massless limit. Current lattice calculations are able to provide unambiguous results for low N_f and the perturbative results are reliable at high $N_f \leq 16.5$. Curiously, the two approaches seem to match at around $N_f = 12, 13$ where a sudden change in behavior as a function of N_f is observed.

The results can be improved in a number of ways. First, direct lattice results at $N_f = 12$ would be very useful. The difficulty is controlling all 3 sources of systematic effects, finite volume, finite lattice spacing and finite mass which certainly would lead to very costly calculations. Second, the dominant source of uncertainty of f_V/m_V was the use of the KSRF relation, which in principle could be eliminated once f_V is measured directly on the lattice. Third, currently the highest (p)NRQCD order for f_{PS} is NNLO which in principle could be extended to N^{3}LO, similarly to f_V. However going beyond N^{3}LO order for any quantity does not seem feasible in the near future since the 6-loop β-function would be needed for that.

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