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Overview of hadron structure from lattice QCD

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We review the latest progress in lattice QCD calculations of the hadron structure. There have been improved calculations of form factors and gravitational form factors by removing the lattice artifacts. Additionally, we highlight the rapid development in lattice calculations of x-dependent parton distributions. This area of study has expanded from various one-dimensional parton distribution functions (PDFs) to three-dimensional distributions like generalized parton distribution functions (GPDs) and transverse-momentum-dependent PDFs (TMDs). These advancements are being achieved within the large momentum effective theory (LaMET) framework and similar methodologies, with considerable progress in controlling lattice artifacts as well as enhancing theoretical accuracy.

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1. Introduction

The intricate structure of hadrons, governed by the strong force and Quantum Chromodynamics (QCD), remains a fundamental yet challenging aspect in understanding the subatomic world. Over several decades, a variety of high-energy experiments have delivered cutting-edge insights into nucleon structures, especially their one-dimensional structure [1–3], and have begun to reveal the 3D and spin-dependent structures. The future measurements from high-energy experiments like the the 12 GeV upgrade of Jefferson's Lab CEBAF accelerator [4, 5] and planned Electron-Ion Collider (EIC) at Brookhaven National Laboratory [6, 21] will continue the endeavor with unprecedented precision.

As we anticipate these high-energy experiments to further our understanding of hadron structure, the role of lattice QCD simulations on large-scale supercomputers becomes increasingly relevant, as it could provide complementary knowledge and potential guidance. However, computing parton distributions in lattice QCD, despite being a non-perturbative technique, is not straightforward. This is due to parton distributions typically being defined from light-cone correlations, which pose a challenge for the Euclidean lattice. Historically, what could be computed were moments derived from the matrix elements of local operators. The lattice determination of these moments, especially the form factors and gravitational form factors, is rapidly advancing by increasing statistics and removing lattice artifacts. However, due to signal decay and power-divergent mixing under renormalization, there are no moments beyond the third that exist.

Breakthrough was made about ten years ago when the large-momentum effective theory (LaMET) was introduced [7–9]. For the first time, the x-dependence of the parton distributions could be derived from the boosted matrix elements of equal-time operators, such as $O_\Gamma(z) \equiv \overline{\psi}(z)\Gamma W(z,0)\psi$, which are computable on the lattice. These operators with a spatial separation, can be boosted to approach the light-cone in a large-momentum hadron state. The results at finite momentum define the so-called quasi-PDF and differ from the light-cone PDFs primarily in the ultraviolet (UV) region which can be compensate by the perturbative matching. The LaMET has led enormous progress in the calculation of the parton distributions in the past few years. It meanwhile also motivate the development of some other methods that were earlier or newly proposed. The Ioffe-time pseudo-distributions or pseudo-PDFs (pPDFs) proposed in [10, 11] use the same Wilson-line operators as LaMET, but rely on the short distance factorization (SDF) in the coordinate space to extract either Mellin moments of PDFs or the x-dependent PDFs. Other methods, such as the short-distance expansion of current-current correlators [12, 13], the operator product expansion (OPE) of a Compton amplitude in the unphysical region [14], the hadronic tensor approach [15], and the heavy-quark operator product expansion (HOPE) [16, 17] have also been invistigated. To date, the LaMET and SDF from the Wilson-line operators are the most feasible and explored approaches, making progress in calculations from one-dimensional PDFs to three-dimensional generalized parton distributions (GPDs). What's more, LaMET has also pioneered the extraction of transverse-momentum-dependent distributions (TMDs) from the lattice for the first time which could significantly improve our knowledge of the 3D structure of hadrons. Additionally, in the recent calculations, considerable efforts have been put in towards controlling the lattice artifacts by taking the continuum and physical limit. There have also been significant progress in enhancing the theoretical precision through the higher-order perturbative matching as well as resummation techniques. In this proceeding, we review the latest development mainly in the past year, while ealier work has been extensively reviewed in Ref. [9, 18–20]. All figures are included in this review for illustrative purposes. They come always in their arXiv versions and are reprinted under the arXiv distribution license.

2. Moments from local operators

2.1 Form factors

With many years' efforts, significant advancements have been made in computing moments through the local operators, such as the first moments which are the form factors. The precision determination of electromagnetic form factors of nucleon at low Q^2, followed by the magnetic moments and electromagnetic radii nowadays is a topic of great interest due to their relation to the "proton radius puzzle". In addition, the axial form factor and axial radius are important inputs for the weak process associated with the neutrino-nucleus scattering. A recent calculation of the electromagnetic form factors from Ref. [22] used a set of CLS ensembles at four different lattice spacings with pion masses between 130 and 290 MeV to control the lattice artifacts. Both connected and disconnected diagrams were computed in this study. They did simultaneous fits to the Q^2 dependence with pion-mass, lattice-spacing, and finite-volume extrapolation to the expressions resulting from covariant baryon chiral perturbation theory (BχPT). The results are shown in Fig. 1 which are consistent with the experimental determination of the magnetic moments and suggest a smaller value for both the electric and magnetic radius of the proton. With the remarkable accuracy at few percent level that includes both statistical and systematic errors, these results could make a meaningful contribution to the debate regarding the proton radii. There was another work from the PACS Collaboration [31] that computed the nucleon electric, magnetic and axial form factors using two ensembles with large spatial volumes (exceeding $(10\text{ fm})^3$) and at the physical point. Therefore the quark mass and finite volume effects are negligible in this calculation, while the discretization effect was investigated. It was found the discretization effect on axial charge g_A is negligibly small while that cannot be ignored in the extraction of the isovector radii. Third PACS10 ensemble was planned to take the continuum limit.

There are also progress on computing the form factors at large momentum transfer. The form factors from low to high Q^2 will provide a clearest opportunity to study the transition from non-perturbative to perturbative regime of QCD. In Ref. [32], proton and neutron electromagnetic form factors $G_{E,M}(Q^2)$ up to 8 GeV^2 were calculated with a range of lattice spacings as well as quark masses that approach the physical point. The initial resulting form factors seem to overestimate the ones from experiment by a large factor, but their ratios are in much better agreement with experiment and phenomenology. The calculation will be improved in the near future to further understand the systematics by adding finer lattice spacings.

2.2 Gravitational form factors

The gravitational form factors (GFF) are the second moments of the generalized parton distributions (GPDs), which are also the matrix elements of the energy-momentum tensor (EMT). They provide rich information of the hadron structure, encompassing the distribution of mass and spin,

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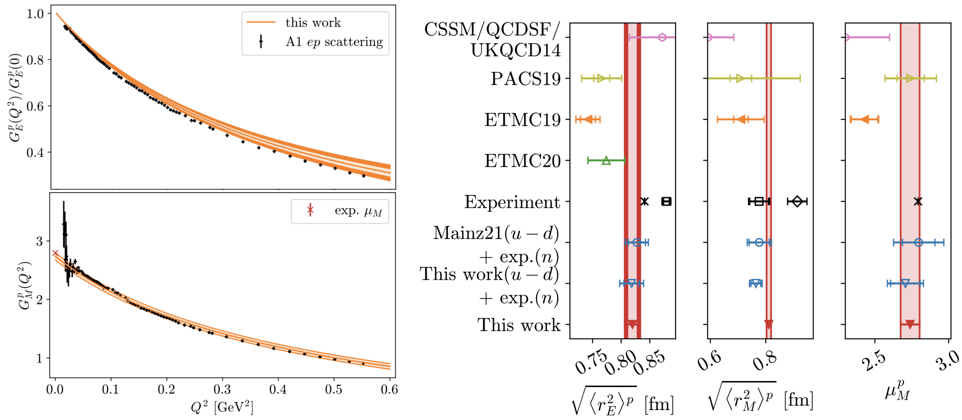

Figure 1: Left: electromagnetic form factors of the proton from Ref. [22] are shown as a function of Q^2. Right:the corresponding electromagnetic radii and the magnetic moment of the proton are shown and compared with other lattice calculations, i.e., Mainz21 [23], ETMC20 [24], ETMC19 [25], PACS19 [26], and CSSM/QCDSF/UKQCD14 [27, 28]. The experimental values from PDG [29] and Mainz/A1 [30] are also shown.

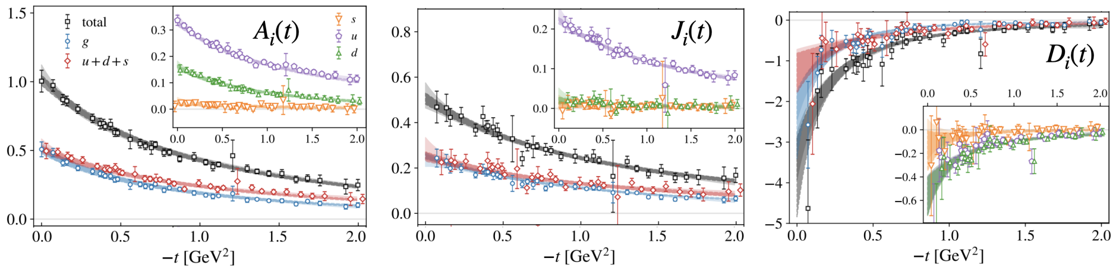

Figure 2: The three GFFs of the proton and their decomposition into gluon and quarks from Ref. [33] are shown as functions of t.

as well as the D-term, which relates to the distributions of energy, angular momentum, and various mechanical properties. In Ref. [33], the flavor decomposition of the proton's $A(t)$, $J(t)$, and $D(t)$ GFFs was reported for $-t \in [0, 2]$ GeV for the first time as shown in Fig. 2. The calculation used one 2+1 flavor ensemble with near to physical pion mass $m_\pi = 170$ MeV. Both connected and disconnected diagrams were computed. The dipole and z-expansion models are used to describe the data and estimate the model dependence. The results reveal that, while the contributions of quarks and gluons to the proton's momentum, spin, and D-term are approximately equal, the gluon contributions act to extend the radial size of the proton over that defined by the quark contributions as quantified through the mass and mechanical radii encoded in the t-dependence of the GFFs. Similarly, the flavor decomposition of the pion GFFs $A^\pi(t)$ and $D^\pi(t)$ with the same lattice setup were reported in Ref. [34].

3. x-dependent parton distributions

As has been mentioned, the moments calculations from local operators are limited due to the signal decay and power divergent mixing under renormalization. Apart from that, the LaMET

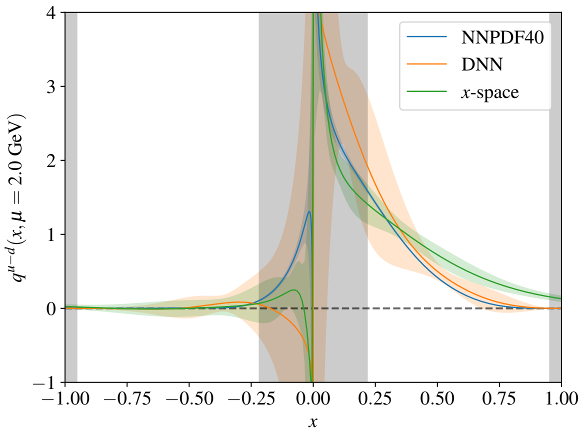

Figure 3: Comparison of the x-dependence of the iso-vector quark PDF between the global analysis from NNPDF4.0 [3], and lattice results from Ref. [36] using DNN and x-space matching.

and SDF of the Wilson-line operators stand as the two most popular and promising methods for extracting light-cone parton distributions from lattice. These two methods, though converging in the infinite-momentum limit, are two distinctive approaches to extract PDFs from coordinate-space correlation functions in high-momentum hadrons. LaMET implements a momentum-space expansion in $\Lambda_{\rm QCD}/[x(1-x)P_z]$ to directly calculate PDFs in a moderate region of Bjorken x. The precision of LaMET predictions is therefore governed by power corrections suppressed by $1/P_z$ as well as the accuracy of the perturbative matching. On the other hand, SDF utilizes a coordinate-space expansion in $z^2\Lambda^2_{\rm QCD}$ to deduce a range of leading-twist correlations, $h(\lambda = zP_z)$, which correspond to the Fourier transformation of PDFs. It enables the model-independent extraction of the moments of PDFs, and the model-based reconstruction of x dependence. The precision of SDF depends on the accuracy of perturbative matching and requires keeping z^2 within a short range to suppress power corrections. Additionally, a large P_z is crucial to extending the range to a higher $\lambda_{\rm max}$, thereby enabling the retrieval of higher moments. In this section, we review the recent development of these two methods including the calculations of various parton distributions as well as progress in the precision control.

3.1 Unpolarized quark PDFs

The unpolarized quark PDF of nucleon has been well determined through the global analysis [1–3] integrating enormous experimental data generated in the past 5 decades and state-of-art theoretical knowledge, which is a great step towards understanding the strong force in nature and the precision determination of the standard model background. The current lattice simulation probably can't provide better constrain then the experimental data in the near future [35]. Instead, the global analysis results can serve as a benchmark to understand the systematics of the lattice methods.

Limited by the poor signal-to-noise ratio of the nucleon state with large momentum boost, most of the lattice simulation in the past few years used heavier than physical quark mass to achieve momentum up to $2 \sim 3$ GeV. A recent calculation performed directly at the physical point with a fine lattice spacing $a = 0.076$ fm was reported in Ref. [36] by the BNL/ANL group. This work concentrated on the iso-vector distribution with nucleon momentum up to 1.53 GeV. The next-to-next-to-leading-order (NNLO) perturbative matching was used in the data analysis for the first time to improve the theoretical accuracy. It was found the difference between matching at NLO and NNLO is small within the errors but non-negligible. The x dependence of the PDF was derived through a deep neural network (DNN) within the SDF approach and additionally through the framework of large-momentum effective theory utilizing a hybrid renormalization scheme as shown in Fig. 3. The DNN results appear to agree with the global analysis from NNPDF4.0 [3] with however large errors because only a small amount of data in the short distance was used in this SDF framework. The x-space LaMET approach shows smaller errors by integriting the long-distance matrix elements in the Fourier transform, which however agree with the NNPDF4.0 and DNN in a limited region of x. Given the fact that the advanced lattice setup as well as the NNLO has been used, it is probably the larger momentum that are needed to suppress the power corrections which should be further investigated in the future.

As the lightest hadrons in nature, pions are the Nambu-Goldstone bosons of QCD and is important to understand the origins of hadron mass and dynamical chiral symmetry breaking. The internal structure of pion is less constraint from experiments, while is easier to explored from theory side including lattice QCD. The recent lattice calculation of pion valence quark PDF at NNLO accuracy from large momentum up to 2.42 GeV and high statistics [37] deliver a pure theoretical prediction that show remarkable agreement with the global analysis in the moderate x region. Various systematics have been well discussed including the discretization effect and theoretical uncertainties. The quark mass dependence was found to be mild in a later work in Ref. [38]. It can be expected that the lattice calculation of pion structure could reach the similar precision to global analysis or even better in the near future and make meaningful contribution to our understanding of the structure of the Nambu-Goldstone bosons.

3.2 Polarized quark PDFs

The polarized PDFs at twist-2 level include the helicity and transversity PDFs. They are essential to understand the nucleon spin structure, which however are not well constraint from limited experimental data. Though measuring of polarized PDFs are is more difficult from experiments, computing the polarized and unpolarized PDFs from lattice are equally difficult. Therefore, it is desired that lattice calculation can provide complementary information.

A recent calculation of the iso-vector helicity quark PDF was reported in Ref. [39] by the HadStruc Collaboration. A single lattice spacing $a = 0.094$ fm and slightly heavier than physical quark mass was used. Matrix elements with a wide range of momentum from 0 to 2.5 GeV was computed and analyzed through NLO pseudo-PDF approach. It was found that the space-like matrix elements contain information on the leading-twist helicity PDFs, as well as an invariant amplitude that induces an additional z^2 contamination of the leading-twist signal. A simultaneous fit was performed to take care of both twist-2 helicity PDF and possible sources of systematic errors from higher-twist and discretization effects. Encouragingly, the final results show good agreement with recent global analyses such as NNPDFpol1.1 [40], JAM17 [41], and JAM22 [42].

The transversity quark PDF was recently calculated in Ref. [43] by the BNL/ANL lattice group directly at the physical point on a fine lattice with $a = 0.076$ fm. The largest momentum used in this calculation is 1.53 GeV. Using the NLO perturbative matching, the x-dependent PDFs were extracted through both SDF and LaMET methods. The latter one, as shown in Fig. 4,

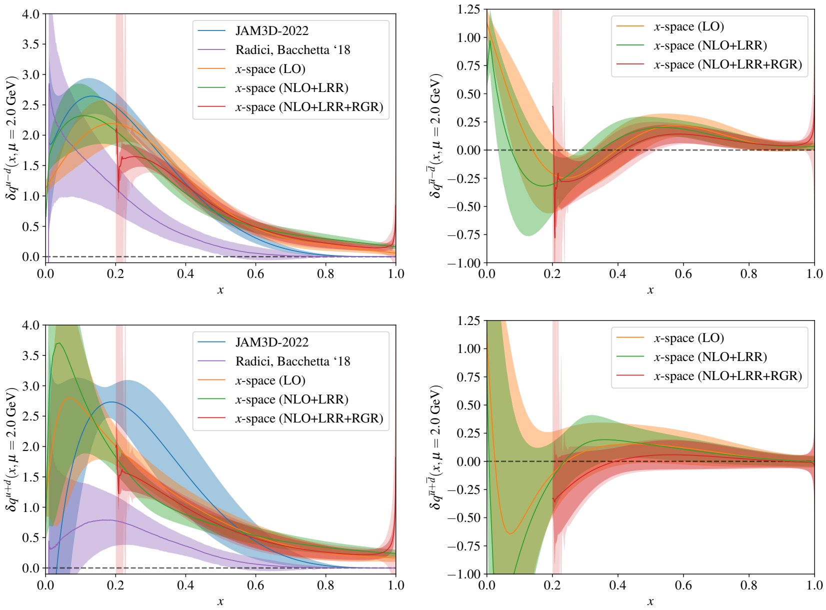

Figure 4: The iso-vector (upper panels) and iso-scalar (lower panels) quark transversity distribution of nucleon from Ref. [43] are shown and compared with the global analyses from JAM3D-22 [44] and Radici, Bacchetta [45].

used the recently developed leading-renormalon resummation (LRR) to remove the leading power correction. The renormalization group resummation (RGR) was also used to improve the accuracy of scale evolution. Reasonably good agreement with global analyses from JAM3D22 [44] in the moderate region of x can be observed, but there is significant tension with the results from Radici, Bacchetta [45].

It is evident that current lattice calculation of the polarized quark PDFs overall shows compatible results and uncertainties with the global analyses. It's worth to mention that the JAM3D22 [44] has already considered the tensor charge from lattice calculation in their analysis. It's reasonable to expect that the lattice calculations could start to provide more complementary information to constrain the nucleon spin structure in the near future.

3.3 Gluon PDFs

The gluons as the mediator bosons of the strong interaction, play a key role in the origin of nucleon mass and spin. However gluon distributions of nucleon are rather difficult to constrain compared to the quarks because they usually start to contribute from the next-to-leading order (NLO) in high-energy scattering. In addition, they are also difficult to be calculated on the lattice due to the poor noise-to-signal ratio.

The MSU lattice group presented the first physical-continuum limit nucleon gluon distribution using the pseudo-PDF approach with 2+1+1 flavors of HISQ ensembles. Three lattice spacing a $\approx$ 0.9, 0.12 and 0.15 fm and three pion masses $m_\pi \approx 220$, 310 and 690 MeV were used to control the discretization effect and quark mass dependence. Large momenta up to 3 GeV were achieved through $O(10^6)$ measurements to suppress the power corrections and provide meaningful information of the

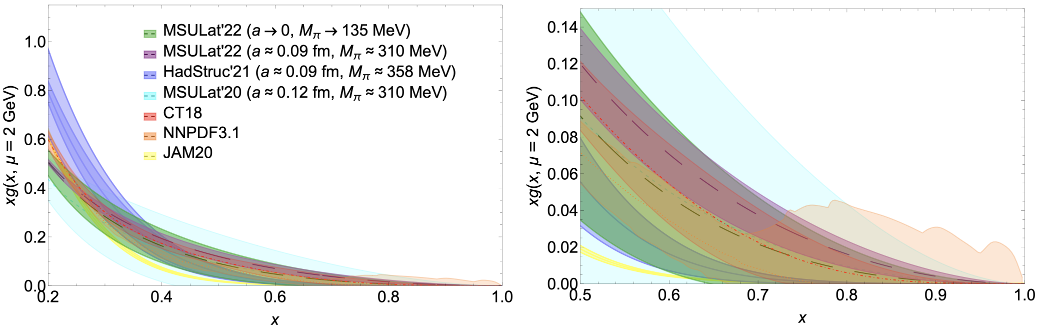

Figure 5: The unpolarized gluon PDF of nucleon from Ref. [46] is shown (MSULat22) and compared with other lattice calculations from HadStruc [47] and MSULat20 [48]. The results from the global fits by CT18 [1] and NNPDF3.1 [50] are also shown. The right panel is a zoomed version of left panel.

gluon PDF $xg(x)/\langle x\rangle_g$. They considered two models to reconstruct the x dependence and investigate the model dependence. Finally, using a momentum fraction $\langle x\rangle_g$ calculated on clover-on-HISQ ensembles [49], the distribution $xg(x)$ was derived as shown in Fig. 5. Overall agreement can be observed comparing to global analysis from CT18 [1] and NNPDF3.1 [50] in the large-x range. Future works are needed to further understand the systematics especially in the theoretical side. Using similar lattice setup, the MSU lattice group also reported the pion gluon distribution as well as the gluon momentum fraction in the continuum-physical limit in Ref. [51]. Their determination of gluon momentum fraction agree with the global analysis as well as previous lattice calculation using ensembles of 2+1+1 flavors but show tension with the calculation with ensembles of 2+1 flavors.

The first exploratory lattice calculation of the gluon helicity distribution of nucleon through the pseudo-PDF approach was reported in Ref. [52]. A single lattice ensemble with lattice spacing a=0.094 fm and unphysical pion masses m_π=358 MeV was used. It was observed that the statistical uncertainties are large especially for the case of large momentum which is essential to suppress the $O(m_p^2/p_z^2)$ corrections. Even though, through a qualitative comparison with global analyses, their result hints at a positive gluon polarization contribution to the nucleon spin budget. Future work with improved statistical and systematical precision could provide a controlled contribution to the determination of the gluon contribution to the proton spin.

3.4 Distribution amplitudes

The distribution amplitudes (DAs) of mesons describe the overlap of the meson state with the leading fock states of collinear valence quarks. They are important inputs to many exclusive processes with large momentum transfer, such as the form factors and B-meson decay, that can be factorized into the non-perturbative DAs and the perturbative hard-scattering kernels.

The first lattice QCD study of the pion DA through the pseudo-PDF approach was reported in Ref. [53] directly at the physical point with a fine lattice spacing $a = 0.076$ fm. The 2nd and 4th Mellin moments were extracted model independently while a flexible parameterization based on Gegenbauer polynomial basis was used to reconstruct the x dependence. The analysis used NLO matching kernel and estimated the systematical errors from the the choice of data range and the parametrization of higher-twist contaminations. The results were used to predict the form factors at moderate range of Q^2, which shows tension with available experimental data and model prediction. This issue could be clarified by the future experimental data of the form factors with larger momentum transfer Q^2.

Recently there is a work from Ref. [54] discussing the precision control in lattice calculation of x-dependent pion DAs with several theoretical improvement. This calculation used three different lattice spacings from 0.06 to 0.12 fm to control the discretization effect at unphysical pion mass $m_\pi = 310$ MeV. The leading-renormalon resummation combined with renormalization group resummation was applied for the first time to remove the leading power correction $O(\Lambda_{\rm QCD}/xP_z)$ in the hybrid renormalization scheme of the LaMET approach and improve the accuracy of scale evolution. The model independent LaMET prediction at moderate x region was combined with the moments extracted through the SDF to deliver a full prediction of $x \in [0, 1]$ dependence. Their final result suggests a broader distribution of the pion DA than the asymptotic one. Most importantly, the theoretical advancements in this work can be used to improve the calculation of other kinds of parton distributions from lattice QCD in the future.

3.5 Generalized parton distributions

The generalized parton distributions (GPDs) are hybrid of the one-dimensional PDFs and form factors. They provide a more comprehensive and nuanced view of the three-dimensional structure of the nucleon, offering insights into the spatial distributions of quarks and gluons. Furthermore, the moments of GPDs are related to the matrix elements of the energy-momentum tensor, from which we can gain valuable insights into the distribution of the hadron's internal energy, momentum, and pressure, as well as the coupling of hadrons to gravity. However extracting the x-dependence of GPDs from experiments is challenging and still in its infancy because of the limited data sets, weak sensitivity to the x dependence as well as the complexity of the multi-dimensional distribution. Encouragingly, the techniques developed for computing PDFs on lattice are also applicable to GPDs. The definition of quasi GPDs matrix elements is very similar to the quasi PDFs but with a momentum transfer $-t = Q^2$ between the initial and final states. However computing GPDs on the lattice appear to be more difficult. Though the light-cone GPDs are Lorentz invariant quantities, the quasi GPDs with finite momentum are frame dependent. Therefore computing the quasi GPDs in the conventional symmetric frame is extremely expensive to consider the $-t$ dependence. In addition, matrix elements with large $-t$ will be noisier due to signal decay.

With years's efforts, encouraging studies have been reported. In the recent calculation of nucleon helicity quark GPDs [55] and pion valence quark GPDs [56], ensembles with physical quark masses were used with lattice spacing $a = 0.09$ fm. Both cases considered the zero-skewness case with four different non-zero momentum transfers $-t \in [0.2, 1]$ GeV^2. The largest hadron momenta achieved were 2.2 GeV and 1.73 GeV for nucleon helicity GPD and pion valence quark GPDs respectively so that meaningful prediction at moderate x region can be derived. The three-dimensional distribution were presented, along with the impact-parameter–dependent distribution which could provide insight on the 3D image for a parton with momentum fraction x to be found in the transverse plane at distance $b_\perp$. In addition, there is a first work of twist-3 axial quark GPDs for the nucleon reported in Ref. [57] with three different momentum transfer of 0.69, 1.38, and 2.76 GeV^2.

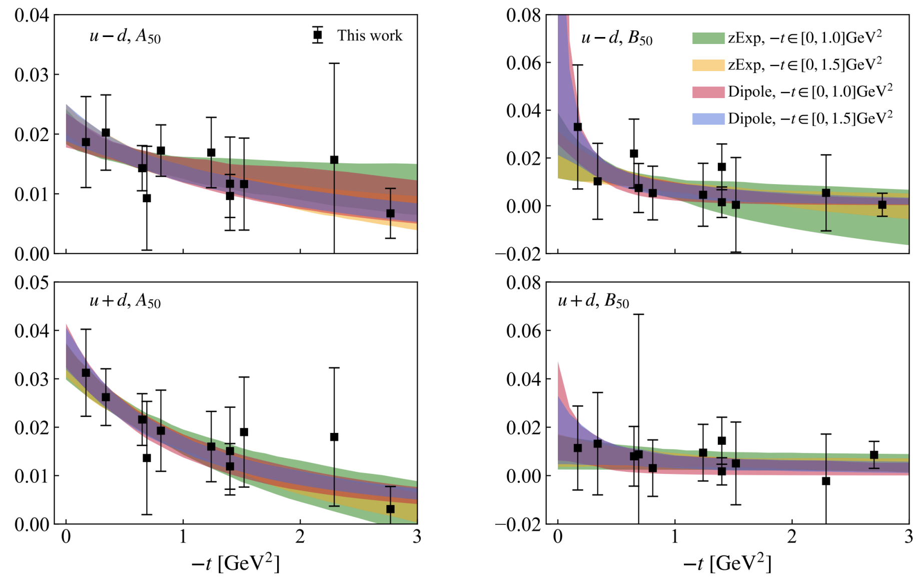

Figure 6: The fifth moments A_{50} and B_{50} derived from SDF approach for iso-vector (upper panels) and iso-scalar (lower panels) nucleon unpolarized GPDs from Ref. [60] are shown as a function of $-t$. The results using traditional local operator methods from ETMC [61] are also shown for comparison.

Though encouraging results were reported, more values of momentum transfer are needed to have a comprehensive understanding of the hadron three-dimensional structure especially in the transverse direction. Recently, progress was made in Ref. [58, 59]. It was shown that through Lorentz covariant parameterization of the matrix elements in terms of Lorentz-invariant amplitudes, matrix elements in different frames can be related to each other. This new development lay the foundation for faster and more effective lattice QCD calculations of GPDs exploiting asymmetric frames. Following the new development, Ref. [60] reported the first results of Mellin moments of unpolarized generalized parton distributions (GPDs) of the proton through the SDF approach with a branch values of momentum transfer. A single lattice with $a = 0.094$ fm and $m_\pi = 260$ MeV was used in this exploratory study. By comparing the first two moments with the results from local operator calculations, the new methods were observed to be valid. What's more, for the first time moments up to the 5th orders were presented as shown in Fig. 6 with reasonable signal and smooth $-t$ dependence. Future work with improved statistical and systematical accuracy has the potential to greatly extend our knowledge of the three-dimensional structure of hadrons.

3.6 Transverse-momentum-dependent distributions

The Transverse-momentum-dependent distributions (TMDs) provide a another view of the three-dimensional structure of hadron in momentum space, including the longitudinal momentum fraction as well as the intrinsic motion in the transverse directions. In addition, we can learn about the spin orbit correlations from the coupling of the quark transverse momentum of with the spin of nucleon. The TMDs consist of two components, namely the beam function and the soft function. The beam function encodes the collinear beam in the fast moving hadron, and is the hadron matrix element of a staple-shaped quark operator along with the light-cone direction. While the soft

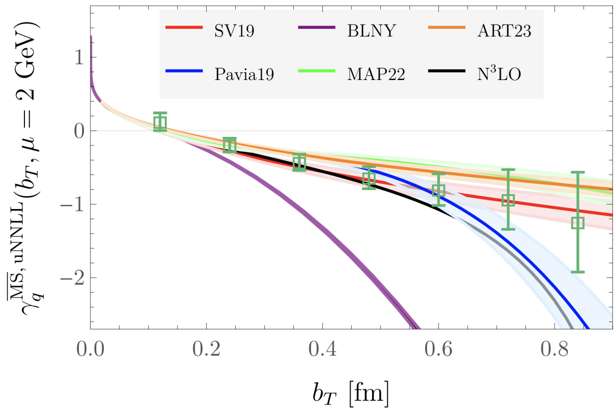

Figure 7: The Collin Soper kernel from Ref. [69] with uNNLL matching in $b_\perp$ space (green squares) are shown and compared with phenomenological parameterizations of experimental data in Ref. [72, 74, 76, 90, 91] labelled BLNY, SV19, Pavia19, MAP22, and ART23, respectively, as well as perturbative results from Ref. [77, 78] labelled N^{3}LO.

function is responsible for summing up the contributions from soft gluons in the process, and is a vacuum matrix element of the bent Wilson line loop along with the two light cone direction. TMDs depend on two scales including the renormalization scale μ and the Collin Soper scale ζ, so that its scale evolution is driven by two renormalization group equations and kernels. In which, the Collin Soper kernel depends on the transverse separation $b_\perp$ and will become non-perturbative when the $1/b_\perp$ reaches Λ_{QCD}. Therefore, it's desired that lattice can calculate the Collin Soper kernel or even the complete TMDs. It turns out that the TMDs can be calculated under the frame work of large momentum effective theory [63, 64, 79]. For quasi beam functions, one can compute the staple-shaped operator along with a spatial direction then boost it to the light cone using a large momentum hadron state. After combined with the with the non-perturbative soft factor as well as the Collin Soper kernel, the quasi-TMDs can be matched to the physical TMDs through the perturbative matching kernel.

The Collin Soper kernel can be extracted by the evolution of quasi TMDs. As a universal kernel, it can be extracted from various TMD distributions such as the quasi TMD PDF [65], quasi TMD wave function [66, 69, 81, 83] as well as the moments of TMDs [70, 71]. It has been extensively explored from lattice calculations in past two years and is in progress towards the precision control. The recent calculation from Ref. [69] has used NNLL matching with inclusion of perturbative power corrections in $1/b_\perp P_z$ to improve the theoretical accuracy. In this work, they used a lattice with $a = 0.12$ fm and quark masses close to physical ones. Various systematic uncertainties have been investigated including well controlled Fourier transform from b_z to x space, and complete analysis of operator mixing. Encouragingly, the kernel was extracted at transverse momentum scales 240 MeV $\lesssim qT \lesssim$ 1.6 GeV with a precision sufficient to begin to discriminate between different phenomenological models in the non-perturbative region as shown in Fig. 7.

The soft factor can be derived from a current-current operators with a transverse separation and highly boosted pion states in opposite direction [79]. With large momentum transfer, these form factors can be factorized into the soft factor and a convolution of the hard kernels with the

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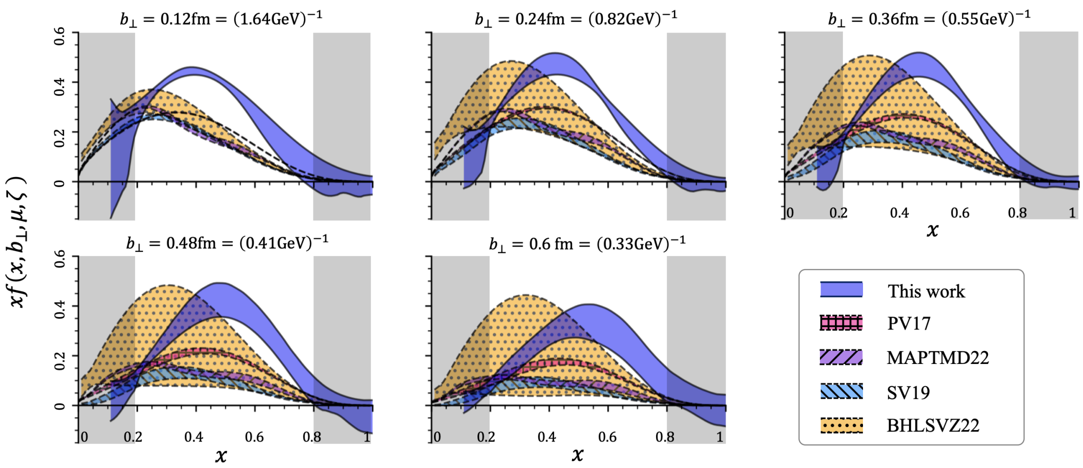

Figure 8: The lattice determination of isovector unpolarized TMDPDFs $xf(x, b_\perp, \mu, \zeta)$ at renormalization scale μ = 2 GeV and rapidity scale $\sqrt{\zeta}$ = 2 GeV from Ref. [88] is shown and compared with global fits from PV17 [89], MAPTMD22 [90], SV19 [91] and BHLSVZ22 [92] global fits (slashed bands). The shaded grey regions imply the endpoint regions where LaMET predictions are not reliable.

quasi TMD wave functions. The calculation of soft factor is very difficult as it is a four point function and require high boosted pion in opposite direction. The pioneering study can be found in Ref. [80, 81]. The recent calculation from Ref. [82, 83] has used NLO matching and is approaching the continuum, physical limit.

With all the critical ingredients ready, one can extract the physical TMDs through a perturbative matching. It's worth to mention that though the TMD calculations are more difficult than the PDFs, the matching appears to be easier as it is a multiplicative matching, independent of spin structure, and there is also no quark gluon mixing [84–87]. However, the power correction of quasi TMD become more complicated coming from the $O(\Lambda_{\rm QCD}/P_z, 1/b_\perp P_z)$ as well as finite Wilson line length η and sub-leading-power TMDs and so on. So it is a highly non-trivial task to compute TMDs from lattice. Recently, by collecting all key ingredients, the pioneering results of the unpolarized TMD PDF of nucleon has established [88] by the Lattice parton collaboration (LPC). The results are shown in Fig. 8 which are qualitatively comparable with phenomenological TMDPDFs. There was also a first presentation of the pion TMD wave functions [93] from lattice calculation. Although many systematic uncertainties were not fully understood so far, the TMDs from first principle lattice calculations can be systematically improved in the future and make meaningful contributions to our understanding of hadron structure and predictions of high-energy scatterings.

3.7 New method for calculating parton physics on the lattice

Apart from the conventional gauge-invariant quasi-PDF approach, a new method for calculating parton physics in the Coulomb gauge was proposed in Ref. [94]. The quasi-PDF defined from the correlations of boosted quarks and gluons in the Coulomb gauge (CG) fall into the same universality class as the gauge-invariant (GI) quasi-PDF, since they both approach the light-cone PDF under an infinite Lorentz boost. The validity of this method was shown by perturbation theory at next-to-leading order and calculation of the pion valence quark PDF on a lattice with spacing a=0.06 fm and pion mass m_π=300 MeV. The results also demonstrate that CG correlations can be multiplicatively renormalized by an overall constant, maintain three-dimensional rotational symmetry and achieve results consistent with the GI case after perturbative matching. Additionally, the CG correlations present several potential benefits, including accessibility to larger off-axis momenta, absence of linear power divergence, and enhanced long-range precision.

Notably, the CG correlations can also be used to calculate broader parton physics such as GPDs and TMDs, which are more computationally demanding than the PDFs. In particular, the quasi-TMDs calculations, whose exact form has been derived using Soft Collinear Effective Theory and verified at one-loop order in Ref. [95], will benefit significantly from the absence of staple-shaped Wilson lines which typically demand extensive memory and computational resources for storage and contractions, as well as from a simplified renormalization process.

3.8 Towards precision control

It is encouraging to see that the lattice simulation are capable of computing many kinds of hadron structures, under the LaMET and SDF framework. It more and more comes to the stage that we need to consider the precision control from the data as well as the theory sides. There have been many progress in increasing the data precision in the recent calculations that were mentioned above, towards the large P_z and physical-continuum limit. Additionally, the theory precision is also important to make a reliable prediction. The up-to-date renormalization technique for SDF is the ratio scheme [11, 35]. As for LaMET, people have used the RI-MOM renormalization [96] in the past few year and now evolve to the hybrid scheme [97] which has smaller systematical errors. For the perturbative matching, the full NLO kernels and part of NNLO kernels [98, 99] are available in the literatures and has been applied in the analysis which significantly improve the matching accuracy. The new sections that people started to look into recently are the large logarithms resummation in the perturbative matching and the power corrections.

In the LaMET approach, the leading power (twist-three) correction appears as $O(\Lambda_{\rm QCD}/P_z)$ due to the linear-divergent self-energy of Wilson line in quasi-PDF operators. For lattice data with hadron momentum P_z of a few GeV, this correction is dominant in matching, as large as 30% or more. Through choosing the mass renormalization parameter consistently with the leading renormalon resummation, it was shown in Ref. [100] that this uncertainty can be eliminated. In addition, there is a DGLAP evolution logarithm that could be large when the nature scale $2xP_z$ is far from the factorization scale μ. These logarithms can be resummed by solving the renormalization group (RG) equation and run the α_s from factorization scale to the nature scale [101]. Then it can be seen that the perturbative matching is unreliable at small x when the $2xP_z$ gets close to the $\Lambda_{\rm QCD}$ but can enhance the matching accuracy in moderate x region by the improved scale evolution. In addition, there are threshold logarithms could become large for the LaMET factorization at large-x region where the hard quark carries most of the hadron momentum and the remained phase space only allows soft gluon radiation that needed to be resummed. The threshold resummation as investigated in Ref. [102] could enhance the accuracy of the lattice calculation of parton distributions at large-x region. Similarly, the SDF in coordinate space also contains the similar large logarithms and has been investigated in Ref. [103] and Ref. [104]. Some new lattice calculations have started to include the new theoretical development to reduce the theoretical uncertainty such as the pion DA [54] and the nucleon transversity PDF [43] that have been mentioned in the previous sections.

4. Summary and prospects

Over the past few years, there has been significant progress in computing the hadron structure from lattice QCD. Remarkable improvements have been made in the calculations of moments. Even more excitingly, there has been significant advancement in the area of x-dependent structure, driven by the development of the LaMET, SDF and other related frameworks. These frameworks have enabled lattice simulations to compute a wide array of hadron structures with increasing precision. This progress encompasses not only data accuracy, particularly in the context of large P_z and as we approach the physical-continuum limit, but also improvements in theoretical precision. In this summary of our review, we offer some remarks about challeges and potential directions for further work.

Exploration of new observables: The scope of research in this field has significantly broadened, transitioning from one-dimensional PDFs to three-dimensional GPDs and TMDs. There have also been groundbreaking studies on distributions beyond twist-2. However, most calculations to date have centered around the valence quark distribution in hadrons. Our understanding of sea quark and gluon distributions remains nascent, particularly evident in the lack of calculations for gluon GPDs and TMDs, hindered by the challenging signal-to-noise ratio. Additionally, the exploration into TMD PDFs is just at its early stage, with a solitary study on the unpolarized TMD PDFs of the proton. The future holds promise for delving into spin-dependent TMDs, including the Sivers and Boer-Mulders functions. Looking ahead, there's a compelling need to delve into x-dependent Wigner distributions or Generalized TMDs (GTMDs), which will deepen our understanding of hadron structure and dynamics by integrating and extending the concepts of GPDs and TMDs. Moreover, recent proposals have been made for the calculation of multi-parton distributions, such as double parton distributions (DPDs) [105, 106], which offer insights into the correlated distribution of multiple partons within a hadron. Furthermore, fragmentation functions, which are crucial to understanding hadron structure and high-energy phenomenology, represent an unexplored potential area in lattice calculations.

Precision control in the lattice extraction: Precision control is becoming increasingly crucial in lattice simulations, particularly as the field advances in computing diverse hadron structures under frameworks like LaMET and SDF. This calls for advancements in both data and theoretical precision. Data precision in general lattice calculations involves addressing issues like discretization effects, physical quark masses, and finite volume effects, which require extensive calculations across multiple ensembles for accurate elimination. Although the finite volume effect appears to be mild, progress has been made in controlling discretization effects and physical quark masses. Theoretical precision is equally vital, as the extraction of parton distributions from quasi or pseudo observables relies on power expansion and perturbative factorizations. Higher-order perturbative QCD matching kernels can enhance accuracy over strong coupling $O(\alpha_s^n)$. The integration of resummation techniques, including renormalization group resummation, threshold resummation, and leading renormalon resummation, should be standardized in future data analyses for their significant contributions. Moreover, controlling power corrections from non-perturbative and higher-twist effects requires matrix elements with large momentum P_z, presenting a challenging issue for lattice simulations due to the more pronounced exponential decay of the signal relative to hadron energy and the narrower energy gap between the ground state and excited states. The

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momentum smearing technique has significantly improved this aspect, but it remains difficult to exceed $P_z \sim 3$ GeV, especially with physical quark masses. Therefore, future breakthroughs in addressing these challenges are eagerly anticipated.

Synergy between lattice QCD and global analysis: Recent lattice QCD calculations, particularly those involving spin-dependent PDFs, GPDs, and the Collins-Soper kernels, as mentioned in this review, have exhibited remarkable alignment with global analyses in certain regions. There is a growing expectation that, in the near future, lattice calculations may achieve similar, or potentially even greater, precision in parton distributions that are currently not well-constrained by experimental data. A collaborative synergy between lattice QCD and global analysis promises a more thorough and integrated understanding of hadron structure, merging theoretical predictions with empirical findings to deepen our knowledge of these fundamental components of matter.

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References

[1] T. J. Hou, J. Gao, T. J. Hobbs, K. Xie, S. Dulat, M. Guzzi, J. Huston, P. Nadolsky, J. Pumplin and C. Schmidt, *et al.* Phys. Rev. D **103**, no.1, 014013 (2021) doi:10.1103/PhysRevD.103.014013 [arXiv:1912.10053 [hep-ph]].

[2] S. Bailey, T. Cridge, L. A. Harland-Lang, A. D. Martin and R. S. Thorne, Eur. Phys. J. C **81**, no.4, 341 (2021) doi:10.1140/epjc/s10052-021-09057-0 [arXiv:2012.04684 [hep-ph]].

[3] R. D. Ball *et al.* [NNPDF], Eur. Phys. J. C **82**, no.5, 428 (2022) doi:10.1140/epjc/s10052-022-10328-7 [arXiv:2109.02653 [hep-ph]].

[4] J. Dudek, R. Ent, R. Essig, K. S. Kumar, C. Meyer, R. D. McKeown, Z. E. Meziani, G. A. Miller, M. Pennington and D. Richards, *et al.* Eur. Phys. J. A **48**, 187 (2012) doi:10.1140/epja/i2012-12187-1 [arXiv:1208.1244 [hep-ex]].

[5] V. D. Burkert, Ann. Rev. Nucl. Part. Sci. **68**, 405-428 (2018) doi:10.1146/annurev-nucl-101917-021129

[6] A. Accardi, J. L. Albacete, M. Anselmino, N. Armesto, E. C. Aschenauer, A. Bacchetta, D. Boer, W. K. Brooks, T. Burton and N. B. Chang, *et al.* Eur. Phys. J. A **52**, no.9, 268 (2016) doi:10.1140/epja/i2016-16268-9 [arXiv:1212.1701 [nucl-ex]].

[7] X. Ji, Phys. Rev. Lett. **110**, 262002 (2013) doi:10.1103/PhysRevLett.110.262002 [arXiv:1305.1539 [hep-ph]].

[8] X. Ji, Sci. China Phys. Mech. Astron. **57**, 1407-1412 (2014) doi:10.1007/s11433-014-5492-3 [arXiv:1404.6680 [hep-ph]].

[9] X. Ji, Y. S. Liu, Y. Liu, J. H. Zhang and Y. Zhao, Rev. Mod. Phys. **93**, no.3, 035005 (2021) doi:10.1103/RevModPhys.93.035005 [arXiv:2004.03543 [hep-ph]].

[10] A. V. Radyushkin, Phys. Rev. D **96**, no.3, 034025 (2017) doi:10.1103/PhysRevD.96.034025 [arXiv:1705.01488 [hep-ph]].

[11] K. Orginos, A. Radyushkin, J. Karpie and S. Zafeiropoulos, Phys. Rev. D **96**, no.9, 094503 (2017) doi:10.1103/PhysRevD.96.094503 [arXiv:1706.05373 [hep-ph]].

[12] V. Braun and D. Müller, Eur. Phys. J. C **55**, 349-361 (2008) doi:10.1140/epjc/s10052-008-0608-4 [arXiv:0709.1348 [hep-ph]].

[13] Y. Q. Ma and J. W. Qiu, Phys. Rev. Lett. **120**, no.2, 022003 (2018) doi:10.1103/PhysRevLett.120.022003 [arXiv:1709.03018 [hep-ph]].

[14] A. J. Chambers, R. Horsley, Y. Nakamura, H. Perlt, P. E. L. Rakow, G. Schierholz, A. Schiller, K. Somfleth, R. D. Young and J. M. Zanotti, Phys. Rev. Lett. **118**, no.24, 242001 (2017) doi:10.1103/PhysRevLett.118.242001 [arXiv:1703.01153 [hep-lat]].

[15] K. F. Liu and S. J. Dong, Phys. Rev. Lett. **72**, 1790-1793 (1994) doi:10.1103/PhysRevLett.72.1790 [arXiv:hep-ph/9306299 [hep-ph]].

[16] W. Detmold and C. J. D. Lin, Phys. Rev. D **73**, 014501 (2006) doi:10.1103/PhysRevD.73.014501 [arXiv:hep-lat/0507007 [hep-lat]].

[17] W. Detmold *et al.* [HOPE], Phys. Rev. D **104**, no.7, 074511 (2021) doi:10.1103/PhysRevD.104.074511 [arXiv:2103.09529 [hep-lat]].

[18] M. Constantinou, Eur. Phys. J. A **57**, no.2, 77 (2021) doi:10.1140/epja/s10050-021-00353-7 [arXiv:2010.02445 [hep-lat]].

[19] M. Constantinou, L. Del Debbio, X. Ji, H. W. Lin, K. F. Liu, C. Monahan, K. Orginos, P. Petreczky, J. W. Qiu and D. Richards, *et al.* [arXiv:2202.07193 [hep-lat]].

[20] H. W. Lin, Few Body Syst. **64**, no.3, 58 (2023) doi:10.1007/s00601-023-01842-9

[21] R. Abdul Khalek, A. Accardi, J. Adam, D. Adamiak, W. Akers, M. Albaladejo, A. Al-bataineh, M. G. Alexeev, F. Ameli and P. Antonioli, *et al.* Nucl. Phys. A **1026**, 122447 (2022) doi:10.1016/j.nuclphysa.2022.122447 [arXiv:2103.05419 [physics.ins-det]].

[22] D. Djukanovic, G. von Hippel, H. B. Meyer, K. Ottnad, M. Salg and H. Wittig, [arXiv:2309.07491 [hep-lat]].

[23] D. Djukanovic, T. Harris, G. von Hippel, P. M. Junnarkar, H. B. Meyer, D. Mohler, K. Ottnad, T. Schulz, J. Wilhelm and H. Wittig, Phys. Rev. D **103**, no.9, 094522 (2021) doi:10.1103/PhysRevD.103.094522 [arXiv:2102.07460 [hep-lat]].

[24] C. Alexandrou, K. Hadjiyiannakou, G. Koutsou, K. Ottnad and M. Petschlies, Phys. Rev. D **101**, no.11, 114504 (2020) doi:10.1103/PhysRevD.101.114504 [arXiv:2002.06984 [hep-lat]].

[25] C. Alexandrou, S. Bacchio, M. Constantinou, J. Finkenrath, K. Hadjiyiannakou, K. Jansen, G. Koutsou and A. Vaquero Aviles-Casco, Phys. Rev. D **100**, no.1, 014509 (2019) doi:10.1103/PhysRevD.100.014509 [arXiv:1812.10311 [hep-lat]].

[26] E. Shintani, K. I. Ishikawa, Y. Kuramashi, S. Sasaki and T. Yamazaki, Phys. Rev. D **99**, no.1, 014510 (2019) [erratum: Phys. Rev. D **102**, no.1, 019902 (2020)] doi:10.1103/PhysRevD.99.014510 [arXiv:1811.07292 [hep-lat]].

[27] P. E. Shanahan *et al.* [CSSM and QCDSF/UKQCD], Phys. Rev. D **89**, 074511 (2014) doi:10.1103/PhysRevD.89.074511 [arXiv:1401.5862 [hep-lat]].

[28] P. E. Shanahan, A. W. Thomas, R. D. Young, J. M. Zanotti, R. Horsley, Y. Nakamura, D. Pleiter, P. E. L. Rakow, G. Schierholz and H. Stüben, Phys. Rev. D **90**, 034502 (2014) doi:10.1103/PhysRevD.90.034502 [arXiv:1403.1965 [hep-lat]].

[29] R. L. Workman *et al.* [Particle Data Group], PTEP **2022**, 083C01 (2022) doi:10.1093/ptep/ptac097

[30] J. C. Bernauer *et al.* [A1], Phys. Rev. C **90**, no.1, 015206 (2014) doi:10.1103/PhysRevC.90.015206 [arXiv:1307.6227 [nucl-ex]]

[31] R. Tsuji *et al.* [PACS], [arXiv:2311.10345 [hep-lat]].

[32] S. Syritsyn, M. Engelhardt, J. Green, S. Krieg, J. Negele and A. Pochinsky, Few Body Syst. **64**, no.3, 72 (2023) doi:10.1007/s00601-023-01839-4

[33] D. C. Hackett, D. A. Pefkou and P. E. Shanahan, [arXiv:2310.08484 [hep-lat]].

[34] D. C. Hackett, P. R. Oare, D. A. Pefkou and P. E. Shanahan, [arXiv:2307.11707 [hep-lat]].

[35] Z. Fan, X. Gao, R. Li, H. W. Lin, N. Karthik, S. Mukherjee, P. Petreczky, S. Syritsyn, Y. B. Yang and R. Zhang, Phys. Rev. D **102**, no.7, 074504 (2020) doi:10.1103/PhysRevD.102.074504 [arXiv:2005.12015 [hep-lat]].

[36] X. Gao, A. D. Hanlon, J. Holligan, N. Karthik, S. Mukherjee, P. Petreczky, S. Syritsyn and Y. Zhao, Phys. Rev. D **107**, no.7, 074509 (2023) doi:10.1103/PhysRevD.107.074509 [arXiv:2212.12569 [hep-lat]].

[37] X. Gao, A. D. Hanlon, S. Mukherjee, P. Petreczky, P. Scior, S. Syritsyn and Y. Zhao, Phys. Rev. Lett. **128**, no.14, 142003 (2022) doi:10.1103/PhysRevLett.128.142003 [arXiv:2112.02208 [hep-lat]].

PoS(LATTICE2023)128

[38] X. Gao, A. D. Hanlon, N. Karthik, S. Mukherjee, P. Petreczky, P. Scior, S. Shi, S. Syritsyn, Y. Zhao and K. Zhou, Phys. Rev. D **106**, no.11, 114510 (2022) doi:10.1103/PhysRevD.106.114510 [arXiv:2208.02297 [hep-lat]].

[39] R. G. Edwards *et al.* [HadStruc], JHEP **03**, 086 (2023) doi:10.1007/JHEP03(2023)086 [arXiv:2211.04434 [hep-lat]].

[40] E. R. Nocera *et al.* [NNPDF], Nucl. Phys. B **887**, 276-308 (2014) doi:10.1016/j.nuclphysb.2014.08.008 [arXiv:1406.5539 [hep-ph]].

[41] J. J. Ethier, N. Sato and W. Melnitchouk, Phys. Rev. Lett. **119**, no.13, 132001 (2017) doi:10.1103/PhysRevLett.119.132001 [arXiv:1705.05889 [hep-ph]].

[42] C. Cocuzza *et al.* [Jefferson Lab Angular Momentum (JAM)], Phys. Rev. D **106**, no.3, L031502 (2022) doi:10.1103/PhysRevD.106.L031502 [arXiv:2202.03372 [hep-ph]].

[43] X. Gao, A. D. Hanlon, S. Mukherjee, P. Petreczky, Q. Shi, S. Syritsyn and Y. Zhao, [arXiv:2310.19047 [hep-lat]].

[44] L. Gamberg *et al.* [Jefferson Lab Angular Momentum (JAM) and Jefferson Lab Angular Momentum], Phys. Rev. D **106**, no.3, 034014 (2022) doi:10.1103/PhysRevD.106.034014 [arXiv:2205.00999 [hep-ph]].

[45] M. Radici and A. Bacchetta, Phys. Rev. Lett. **120**, no.19, 192001 (2018) doi:10.1103/PhysRevLett.120.192001 [arXiv:1802.05212 [hep-ph]].

[46] Z. Fan, W. Good and H. W. Lin, Phys. Rev. D **108**, no.1, 014508 (2023) doi:10.1103/PhysRevD.108.014508 [arXiv:2210.09985 [hep-lat]].

[47] T. Khan *et al.* [HadStruc], Phys. Rev. D **104**, no.9, 094516 (2021) doi:10.1103/PhysRevD.104.094516 [arXiv:2107.08960 [hep-lat]].

[48] Z. Fan, R. Zhang and H. W. Lin, Int. J. Mod. Phys. A **36**, no.13, 2150080 (2021) doi:10.1142/S0217751X21500809 [arXiv:2007.16113 [hep-lat]].

[49] Z. Fan, H. W. Lin and M. Zeilbeck, Phys. Rev. D **107**, no.3, 034505 (2023) doi:10.1103/PhysRevD.107.034505 [arXiv:2208.00980 [hep-lat]].

[50] R. D. Ball *et al.* [NNPDF], Eur. Phys. J. C **77**, no.10, 663 (2017) doi:10.1140/epjc/s10052-017-5199-5 [arXiv:1706.00428 [hep-ph]].

[51] W. Good, K. Hasan, A. Chevis and H. W. Lin, [arXiv:2310.12034 [hep-lat]].

[52] C. Egerer *et al.* [HadStruc], Phys. Rev. D **106**, no.9, 094511 (2022) doi:10.1103/PhysRevD.106.094511 [arXiv:2207.08733 [hep-lat]].

[53] X. Gao, A. D. Hanlon, N. Karthik, S. Mukherjee, P. Petreczky, P. Scior, S. Syritsyn and Y. Zhao, Phys. Rev. D **106**, no.7, 074505 (2022) doi:10.1103/PhysRevD.106.074505 [arXiv:2206.04084 [hep-lat]].

[54] J. Holligan, X. Ji, H. W. Lin, Y. Su and R. Zhang, Nucl. Phys. B **993**, 116282 (2023) doi:10.1016/j.nuclphysb.2023.116282 [arXiv:2301.10372 [hep-lat]].

[55] H. W. Lin, Phys. Lett. B **824**, 136821 (2022) doi:10.1016/j.physletb.2021.136821 [arXiv:2112.07519 [hep-lat]].

[56] H. W. Lin, Phys. Lett. B **846**, 138181 (2023) doi:10.1016/j.physletb.2023.138181 [arXiv:2310.10579 [hep-lat]].

[57] S. Bhattacharya, K. Cichy, M. Constantinou, J. Dodson, A. Metz, A. Scapellato and F. Steffens, Phys. Rev. D **108**, no.5, 054501 (2023) doi:10.1103/PhysRevD.108.054501 [arXiv:2306.05533 [hep-lat]].

[58] S. Bhattacharya, K. Cichy, M. Constantinou, J. Dodson, X. Gao, A. Metz, S. Mukherjee, A. Scapellato, F. Steffens and Y. Zhao, Phys. Rev. D **106**, no.11, 114512 (2022) doi:10.1103/PhysRevD.106.114512 [arXiv:2209.05373 [hep-lat]].

[59] S. Bhattacharya, K. Cichy, M. Constantinou, J. Dodson, X. Gao, A. Metz, J. Miller, S. Mukherjee, P. Petreczky and F. Steffens, *et al.* [arXiv:2310.13114 [hep-lat]].

[60] S. Bhattacharya, K. Cichy, M. Constantinou, X. Gao, A. Metz, J. Miller, S. Mukherjee, P. Petreczky, F. Steffens and Y. Zhao, Phys. Rev. D **108**, no.1, 014507 (2023) doi:10.1103/PhysRevD.108.014507 [arXiv:2305.11117 [hep-lat]].

[61] C. Alexandrou, J. Carbonell, M. Constantinou, P. A. Harraud, P. Guichon, K. Jansen, C. Kallidonis, T. Korzec and M. Papinutto, Phys. Rev. D **83**, 114513 (2011) doi:10.1103/PhysRevD.83.114513 [arXiv:1104.1600 [hep-lat]].

[62] X. Ji, Y. Liu and Y. S. Liu, Nucl. Phys. B **955**, 115054 (2020) doi:10.1016/j.nuclphysb.2020.115054 [arXiv:1910.11415 [hep-ph]].

[63] X. Ji, Y. Liu and Y. S. Liu, Phys. Lett. B **811**, 135946 (2020) doi:10.1016/j.physletb.2020.135946 [arXiv:1911.03840 [hep-ph]].

[64] M. A. Ebert, S. T. Schindler, I. W. Stewart and Y. Zhao, JHEP **04**, 178 (2022) doi:10.1007/JHEP04(2022)178 [arXiv:2201.08401 [hep-ph]].

[65] P. Shanahan, M. Wagman and Y. Zhao, Phys. Rev. D **104**, no.11, 114502 (2021) doi:10.1103/PhysRevD.104.114502 [arXiv:2107.11930 [hep-lat]].

[66] M. H. Chu *et al.* [LPC], Phys. Rev. D **106**, no.3, 034509 (2022) doi:10.1103/PhysRevD.106.034509 [arXiv:2204.00200 [hep-lat]].

[67] Y. Li, S. C. Xia, C. Alexandrou, K. Cichy, M. Constantinou, X. Feng, K. Hadjiyiannakou, K. Jansen, C. Liu and A. Scapellato, *et al.* Phys. Rev. Lett. **128**, no.6, 062002 (2022) doi:10.1103/PhysRevLett.128.062002 [arXiv:2106.13027 [hep-lat]].

[68] M. H. Chu *et al.* [Lattice Parton (LPC)], JHEP **08**, 172 (2023) doi:10.1007/JHEP08(2023)172 [arXiv:2306.06488 [hep-lat]].

[69] A. Avkhadiev, P. Shanahan, M. Wagman and Y. Zhao, [arXiv:2307.12359 [hep-lat]].

[70] M. Schlemmer, A. Vladimirov, C. Zimmermann, M. Engelhardt and A. Schäfer, JHEP **08**, 004 (2021) doi:10.1007/JHEP08(2021)004 [arXiv:2103.16991 [hep-lat]].

[71] H. T. Shu, M. Schlemmer, T. Sizmann, A. Vladimirov, L. Walter, M. Engelhardt, A. Schäfer and Y. B. Yang, Phys. Rev. D **108**, no.7, 074519 (2023) doi:10.1103/PhysRevD.108.074519 [arXiv:2302.06502 [hep-lat]].

[72] F. Landry, R. Brock, P. M. Nadolsky and C. P. Yuan, Phys. Rev. D **67**, 073016 (2003) doi:10.1103/PhysRevD.67.073016 [arXiv:hep-ph/0212159 [hep-ph]].

[73] I. Scimemi and A. Vladimirov, JHEP **06**, 137 (2020) doi:10.1007/JHEP06(2020)137 [arXiv:1912.06532 [hep-ph]].

[74] A. Bacchetta, V. Bertone, C. Bissolotti, G. Bozzi, F. Delcarro, F. Piacenza and M. Radici, JHEP **07**, 117 (2020) doi:10.1007/JHEP07(2020)117 [arXiv:1912.07550 [hep-ph]].

[75] A. Bacchetta *et al.* [MAP (Multi-dimensional Analyses of Partonic distributions)], JHEP **10**, 127 (2022) doi:10.1007/JHEP10(2022)127 [arXiv:2206.07598 [hep-ph]].

[76] V. Moos, I. Scimemi, A. Vladimirov and P. Zurita, [arXiv:2305.07473 [hep-ph]].

[77] Y. Li and H. X. Zhu, Phys. Rev. Lett. **118**, no.2, 022004 (2017) doi:10.1103/PhysRevLett.118.022004 [arXiv:1604.01404 [hep-ph]].

[78] A. A. Vladimirov, Phys. Rev. Lett. **118**, no.6, 062001 (2017) doi:10.1103/PhysRevLett.118.062001 [arXiv:1610.05791 [hep-ph]].

[79] X. Ji, Y. Liu and Y. S. Liu, Nucl. Phys. B **955**, 115054 (2020) doi:10.1016/j.nuclphysb.2020.115054 [arXiv:1910.11415 [hep-ph]].

[80] Q. A. Zhang *et al.* [Lattice Parton], Phys. Rev. Lett. **125**, no.19, 192001 (2020) doi:10.22323/1.396.0477 [arXiv:2005.14572 [hep-lat]].

[81] Y. Li, S. C. Xia, C. Alexandrou, K. Cichy, M. Constantinou, X. Feng, K. Hadjiyiannakou, K. Jansen, C. Liu and A. Scapellato, *et al.* Phys. Rev. Lett. **128**, no.6, 062002 (2022) doi:10.1103/PhysRevLett.128.062002 [arXiv:2106.13027 [hep-lat]].

[82] Z. F. Deng, W. Wang and J. Zeng, JHEP **09**, 046 (2022) doi:10.1007/JHEP09(2022)046 [arXiv:2207.07280 [hep-th]].

[83] M. H. Chu *et al.* [Lattice Parton (LPC)], JHEP **08**, 172 (2023) doi:10.1007/JHEP08(2023)172 [arXiv:2306.06488 [hep-lat]].

[84] A. A. Vladimirov and A. Schäfer, Phys. Rev. D **101**, no.7, 074517 (2020) doi:10.1103/PhysRevD.101.074517 [arXiv:2002.07527 [hep-ph]].

[85] M. A. Ebert, S. T. Schindler, I. W. Stewart and Y. Zhao, JHEP **09**, 099 (2020) doi:10.1007/JHEP09(2020)099 [arXiv:2004.14831 [hep-ph]].

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[86] X. Ji, Y. Liu, A. Schäfer and F. Yuan, Phys. Rev. D **103**, no.7, 074005 (2021) doi:10.1103/PhysRevD.103.074005 [arXiv:2011.13397 [hep-ph]].

[87] S. T. Schindler, I. W. Stewart and Y. Zhao, JHEP **08**, 084 (2022) doi:10.1007/JHEP08(2022)084 [arXiv:2205.12369 [hep-ph]].

[88] J. C. He *et al.* [LPC], [arXiv:2211.02340 [hep-lat]].

[89] A. Bacchetta, F. Delcarro, C. Pisano, M. Radici and A. Signori, JHEP **06**, 081 (2017) [erratum: JHEP **06**, 051 (2019)] doi:10.1007/JHEP06(2017)081 [arXiv:1703.10157 [hep-ph]].

[90] A. Bacchetta *et al.* [MAP (Multi-dimensional Analyses of Partonic distributions)], JHEP **10**, 127 (2022) doi:10.1007/JHEP10(2022)127 [arXiv:2206.07598 [hep-ph]].

[91] I. Scimemi and A. Vladimirov, JHEP **06**, 137 (2020) doi:10.1007/JHEP06(2020)137 [arXiv:1912.06532 [hep-ph]].

[92] M. Bury, F. Hautmann, S. Leal-Gomez, I. Scimemi, A. Vladimirov and P. Zurita, JHEP **10**, 118 (2022) doi:10.1007/JHEP10(2022)118 [arXiv:2201.07114 [hep-ph]].

[93] M. H. Chu, J. C. He, J. Hua, J. Liang, X. Ji, A. Schafer, H. T. Shu, Y. Su, J. H. Wang and W. Wang, *et al.* [arXiv:2302.09961 [hep-lat]].

[94] X. Gao, W. Y. Liu and Y. Zhao, [arXiv:2306.14960 [hep-ph]].

[95] Y. Zhao, [arXiv:2311.01391 [hep-ph]].

[96] J. W. Chen, T. Ishikawa, L. Jin, H. W. Lin, Y. B. Yang, J. H. Zhang and Y. Zhao, Phys. Rev. D **97**, no.1, 014505 (2018) doi:10.1103/PhysRevD.97.014505 [arXiv:1706.01295 [hep-lat]].

[97] X. Ji, Y. Liu, A. Schäfer, W. Wang, Y. B. Yang, J. H. Zhang and Y. Zhao, Nucl. Phys. B **964**, 115311 (2021) doi:10.1016/j.nuclphysb.2021.115311 [arXiv:2008.03886 [hep-ph]].

[98] L. B. Chen, W. Wang and R. Zhu, Phys. Rev. Lett. **126**, no.7, 072002 (2021) doi:10.1103/PhysRevLett.126.072002 [arXiv:2006.14825 [hep-ph]].

[99] Z. Y. Li, Y. Q. Ma and J. W. Qiu, Phys. Rev. Lett. **126**, no.7, 072001 (2021) doi:10.1103/PhysRevLett.126.072001 [arXiv:2006.12370 [hep-ph]].

[100] R. Zhang, J. Holligan, X. Ji and Y. Su, Phys. Lett. B **844**, 138081 (2023) doi:10.1016/j.physletb.2023.138081 [arXiv:2305.05212 [hep-lat]].

[101] Y. Su, J. Holligan, X. Ji, F. Yao, J. H. Zhang and R. Zhang, Nucl. Phys. B **991**, 116201 (2023) doi:10.1016/j.nuclphysb.2023.116201 [arXiv:2209.01236 [hep-ph]].

[102] X. Ji, Y. Liu and Y. Su, JHEP **08**, 037 (2023) doi:10.1007/JHEP08(2023)037 [arXiv:2305.04416 [hep-ph]].

[103] X. Gao, K. Lee, S. Mukherjee, C. Shugert and Y. Zhao, Phys. Rev. D **103**, no.9, 094504 (2021) doi:10.1103/PhysRevD.103.094504 [arXiv:2102.01101 [hep-ph]].

[104] H. Dutrieux, J. Karpie, C. Monahan, K. Orginos and S. Zafeiropoulos, [arXiv:2310.19926 [hep-lat]].

[105] J. H. Zhang, [arXiv:2304.12481 [hep-ph]].

[106] M. Jaarsma, R. Rahn and W. J. Waalewijn, JHEP **12**, 014 (2023) doi:10.1007/JHEP12(2023)014 [arXiv:2305.09716 [hep-ph]].