

QCD equation of state in the presence of magnetic fields at low density

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Peripheral heavy-ion collisions are expected to exhibit magnetic fields with magnitudes comparable to the QCD scale, as well as non-zero baryon densities. Whereas QCD at finite magnetic fields can be simulated directly with standard lattice algorithms, the implementation of real chemical potentials is hindered by the infamous sign problem. Aiming to shed light on the QCD transition and on the equation of state in that regime, we carry out lattice QCD simulations with 2+1+1 flavors of staggered quarks with physical masses at finite magnetic fields and employ a Taylor expansion scheme to circumvent the sign problem. We present the leading-order coefficient of the expansion calculated at non-zero magnetic fields and discuss the impact of the field on the strangeness neutrality condition.

*The 40th International Symposium on Lattice Field Theory, LATTICE2023 July 31 to August 4, 2023
Fermilab, Illinois*

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1. Introduction

The impact of high baryon densities on strongly interacting matter is relevant for the phenomenology of future heavy-ion collision experiments, such as FAIR and NICA, as well as for the description of the inner core of compact neutron stars. However, simulating the underlying theory, QCD at high baryon densities, using lattice techniques remains a formidable challenge and subject of ongoing research (for a recent review, see [1]). The biggest caveat is the emergence of a sign problem and its increasing severity at high density, which hinder the exploration of the QCD phase diagram deep into the low temperature (T) and high baryon chemical potential (μ_B) region, necessary for the description of the above-mentioned systems. Therefore, due to the prohibitive nature of the sign problem, alternative techniques are necessary to circumvent it, such as reweighting, Taylor expansion, or analytical continuation from imaginary chemical potentials. However, these techniques are restricted to low baryon densities and hence a thorough understanding of hot and dense QCD from first principles remains elusive. Furthermore, an ongoing quest for the conjectured critical endpoint in the phase diagram, where the QCD crossover would become a first-order transition, has received increasing efforts in the lattice community (for a review, see [2]), as well as via functional methods, see e.g. the recent review [3].

Additionally, strong magnetic fields are also expected in peripheral heavy-ion collisions and certain types of neutron stars (magnetars). In contrast to a baryon chemical potential, magnetic fields are free of a sign problem and can be directly simulated using lattice QCD. Moreover, the magnetic field is known to strengthen the QCD crossover [4] and, at large enough B , evidence was found that the nature of the QCD transition changes to a first-order phase transition, with a predicted critical endpoint in the range 4 - 9 GeV² [5]. It is tempting to conjecture that the critical endpoints in the T - μ_B and T - B planes might even be connected by a single critical line, raising interesting questions on the relationship between magnetic fields and baryon densities as probes of the QCD phase diagram. In Figure 1 we sketch the conjectured phase diagram in the T - μ_B - B space. In the context of heavy-ion collisions, the QCD equation of state at nonzero density is an important ingredient for model simulations and it has been computed for both zero [6] and non-zero [7] magnetic fields. Also phenomenologically relevant for the heavy-ion setup is the strangeness neutrality condition. During the chemical freeze-out stage, the strangeness content of the fireball averages zero, and the parameters for which this happens can be determined from fluctuations of conserved charges [8]. This property has been previously studied on the lattice for a vanishing magnetic field [9]. In this work, we extend our previous studies and investigate, for the first time, the joint impact of a chemical potential and a magnetic field on the strangeness neutral point using a Taylor expansion framework. We also compute the equation of state in the presence of μ_B and B to shed light on their role in QCD thermodynamics. While here we assume the magnetic field to be constant in space, the case of inhomogeneous magnetic fields – expected to be more relevant for the heavy-ion collision setting – can also be simulated directly on the lattice [10] and the impact of a non-zero chemical potential included in the same way.

This proceedings article is organized as follows: in Section 2 we describe our magnetic field background and its implementation on the lattice. This is followed by Section 3, where we give the details of our simulation setup. In Section 4 we discuss our results on the impact of a non-zero magnetic field on the strangeness neutrality condition. In Section 5 we present our determination

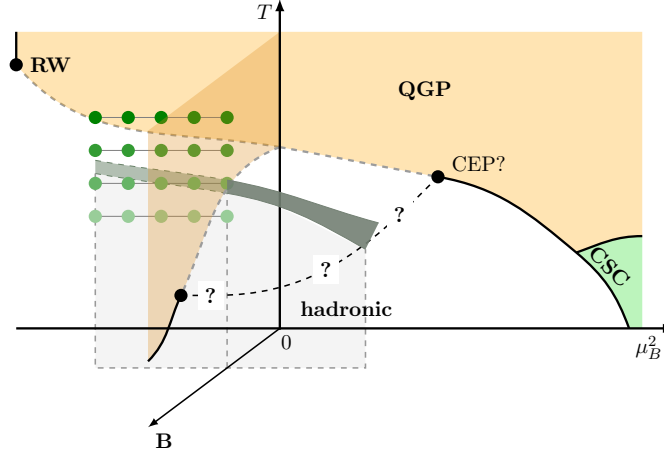


Figure 1: Conjectured three-dimensional QCD phase diagram in the temperature – magnetic field – baryon chemical potential space. The left side of the figure corresponds to imaginary, while the right side to real baryon chemical potentials. The green dots indicate our current simulations and the gray band is a possible outcome for the transition temperature.

of the leading-order (LO) coefficient in the Taylor expansion of the pressure for three magnetic field strengths, and in Section 6 we summarize our findings and state our conclusions.

2. Background magnetic field on the lattice

In a finite periodic volume, the magnetic field flux is quantized. On the lattice, the presence of such a field is implemented by multiplying the SU(3) links by complex phases associated with the magnetic field. For multiple flavors, these phases are defined as $u_\mu^f = e^{iaq_f A_\mu}$, where q_f denotes the charge of the quark flavor f and A_μ is the U(1) background gauge field. For convenience, A_μ is defined so that the phases satisfy periodic boundary conditions. In particular, in the case of a uniform magnetic field pointing in the z direction the links are as follows [11]

$$\begin{aligned}
 u_x^f(x, y, z, t) &= \begin{cases} e^{-iq_f B L_x y} & \text{if } x = L_x - a \\ 1 & \text{if } x \neq L_x - a \end{cases} \\
 u_y^f(x, y, z, t) &= e^{iaq_f B x} \\
 u_z^f(x, y, z, t) &= 1 \\
 u_t^f(x, y, z, t) &= 1
 \end{aligned}
 \quad \text{with} \quad B = \frac{2\pi N_b}{q L_x L_y}, \quad N_b \in \mathbb{Z}. \quad (1)$$

where q is a reference charge given by the modulus of the greatest common divisor of the quark charges, *i.e.* for QCD $q = e/3$, such that $q_f/q \in \mathbb{Z}$, and e is the elementary electric charge. Throughout this text, we express the magnetic field dependence of observables in terms of the renormalization-group-invariant quantity eB .

3. Simulations setup and observables

We simulate 2+1+1 flavors of staggered quarks with masses at the physical point using the Symanzik improved gauge action with four stout smearing steps. We carry out the simulations on a $32^3 \times 8$ lattice for three magnetic field strengths, namely $eB = 0, 0.5, 0.8 \text{ GeV}^2$, and scan a range of temperatures from 125 to 300 MeV in the $B = 0$ case, and from 135 to 200 MeV in the $B > 0$ cases. We change the temperature by changing the lattice spacing a via the inverse gauge coupling β . This implies that for a given N_b , the physical value of eB changes for different temperatures. Therefore, to achieve the desired values, for each $eB \neq 0$ we do simulations using two values of N_b , corresponding to the closest neighboring integers.

To circumvent the sign problem, we Taylor-expand the pressure (normalized by T^4) in the quark chemical potential basis as follows

$$\frac{P}{T^4} = \sum_{ijk} \chi_{ijk} \hat{\mu}_u^i \hat{\mu}_d^j \hat{\mu}_s^k, \quad \text{where} \quad \chi_{ijk} = \frac{1}{VT^3} \left(\frac{\partial}{\partial \hat{\mu}_u} \right)^i \left(\frac{\partial}{\partial \hat{\mu}_d} \right)^j \left(\frac{\partial}{\partial \hat{\mu}_s} \right)^k \log Z \Big|_{\mu_u=\mu_d=\mu_s=0}, \quad (2)$$

$\hat{\mu}_X \equiv \mu_X/T$, with $X = u, d, s$, and Z is the grand canonical partition function. We compute the coefficients χ_{ijk} appearing in the sum above and relate them to the susceptibilities in the physical basis $\{\mu_B, \mu_Q, \mu_S\}$ using the transformations

$$\mu_u = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q, \quad \mu_d = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q, \quad \mu_s = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S. \quad (3)$$

On the physical basis, the susceptibilities are related to the ones in the quark basis by

$$\chi_{BB} = \frac{1}{9}(\chi_{200} + \chi_{020} + \chi_{002} + 2\chi_{110} + 2\chi_{101} + 2\chi_{011}) \quad (4)$$

$$\chi_{QQ} = \frac{1}{9}(4\chi_{200} + \chi_{020} + \chi_{002} - 4\chi_{110} - 4\chi_{101} + 2\chi_{011}) \quad (5)$$

$$\chi_{SS} = \chi_{002} \quad (6)$$

Inserting these relations in Eq. (2) for the pressure we obtain to quadratic order in the chemical potentials,

$$\frac{P}{T^4} = c_0 + \frac{\chi_{BB}}{2} \hat{\mu}_B^2 + \frac{\chi_{QQ}}{2} \hat{\mu}_Q^2 + \frac{\chi_{SS}}{2} \hat{\mu}_S^2 + \chi_{BQ} \hat{\mu}_B \hat{\mu}_Q + \chi_{BS} \hat{\mu}_B \hat{\mu}_S + \chi_{QS} \hat{\mu}_Q \hat{\mu}_S. \quad (7)$$

To make closer contact with the heavy-ion collision setup, Eq. (7) can be further constrained in terms of the chemical potentials. In the next section, we discuss these constraints and compute the necessary and sufficient conditions for them to be satisfied.

4. Strangeness neutrality and isospin asymmetry

In a heavy-ion collision experiment, the number of valence quarks present in the colliding ions constrains the conserved quantities in the baryon, charge, and strangeness basis. For Pb atoms, for instance, the fractions of u , d , and s quarks are 53.5%, 46.5% and 0%, respectively. Therefore, we expect strangeness neutrality $\langle n_S \rangle = 0$, as well as $\langle n_Q \rangle / \langle n_B \rangle \approx 0.4$, which gives a slight

deviation from the isospin-symmetric point ($\langle n_Q \rangle / \langle n_B \rangle = 0.5$). To achieve these conditions in our lattice simulations, we express μ_Q and μ_S in terms of μ_B perturbatively

$$\mu_Q = q_1 \mu_B + \mathcal{O}(\mu_B^3), \quad \mu_S = s_1 \mu_B + \mathcal{O}(\mu_B^3), \quad (8)$$

and determine the leading-order coefficients q_1 and s_1 such that the desired constraints are fulfilled. In terms of the susceptibilities of conserved charges, the coefficients are given by [8]

$$q_1 = \frac{0.4(\chi_{BB}\chi_{SS} - \chi_{BS}^2) - (\chi_{BQ}\chi_{SS} - \chi_{BS}\chi_{QS})}{\chi_{QQ}\chi_{SS} - \chi_{QS}^2 - 0.4(\chi_{BQ}\chi_{SS} - \chi_{BS}\chi_{QS})}, \quad s_1 = -\frac{(\chi_{BS} + q_1\chi_{QS})}{\chi_{SS}}. \quad (9)$$

In Figure 2 we show the coefficients q_1 and s_1 as a function of temperature for different magnetic fields. We now discuss the physical interpretation of this behavior in terms of the hadronic content

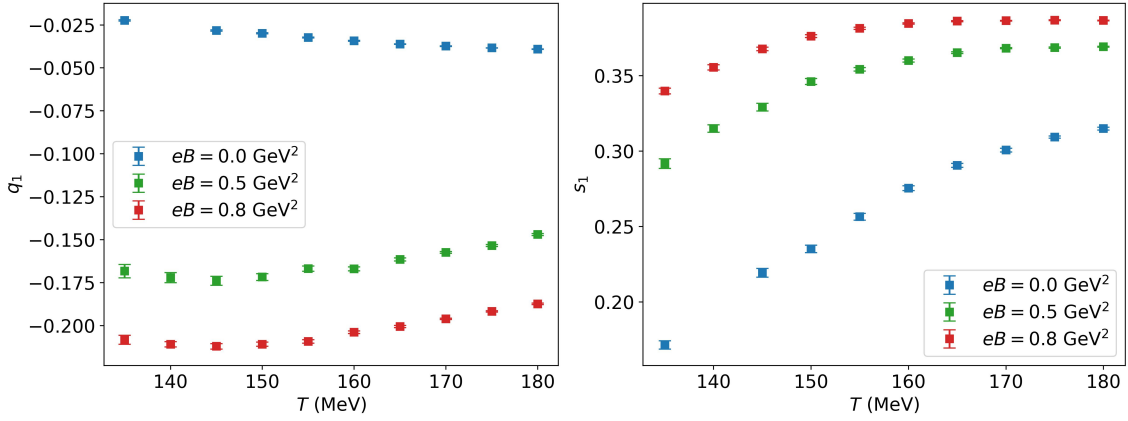


Figure 2: Linear coefficients of the charge (left) and strange (right) chemical potentials in terms of the baryon chemical potential for the strangeness neutrality condition as a function of T for different magnetic fields.

of the system, inspired by the hadron resonance gas (HRG) model.

In the isospin-symmetric case, and with $eB = 0$, we would have $\langle n_Q \rangle / \langle n_B \rangle = 0.5$, and the system would consist of an equal number of protons and neutrons. However, to have a ratio slightly below the isospin point, the positive charges have to be suppressed, producing an imbalance between protons and neutrons. This suppression is encoded in the fact that q_1 is negative, as shown on the left side of Figure 2. At $eB \neq 0$, both protons and neutrons (more precisely, their spin states with lowest energy) become lighter [12] due to their nontrivial magnetic moments. This enhances the effect further, and in order to keep the ratio $\langle n_Q \rangle / \langle n_B \rangle$ fixed, given the aforementioned u , d , and s quark abundances, the protons have to be suppressed even more. Therefore, q_1 increases in magnitude and moves towards more negative values. We can formulate a similar argument for the behavior of s_1 , on the right plot, in terms of the Λ^0 baryon (the lightest hadron with non-zero strangeness, which also features a nonzero magnetic moment), but since the strange quark has negative strangeness by definition, the behavior is the opposite and s_1 is enhanced by the magnetic field. Next, we apply these coefficients for the computation of the equation of state in leading order.

5. Leading-order Taylor coefficient

Using the computation of the susceptibilities and the coefficients q_1 and s_1 done in the previous section, we can express the pressure in terms of the baryon chemical potential

$$\frac{P}{T^4} = c_0 + \left(\frac{1}{2}\chi_{BB} + \frac{1}{2}\chi_{QQ}q_1^2 + \frac{1}{2}\chi_{SS}s_1^2 + \chi_{BQ}q_1 + \chi_{BS}s_1 + \chi_{QS}q_1s_1 \right) \hat{\mu}_B^2. \quad (10)$$

The coefficient c_0 represents the pressure at zero density, whose B -dependence is characterized for weak magnetic fields by the magnetic susceptibility of QCD matter (see e.g. Ref. [13]). In turn, the coefficient of $\hat{\mu}_B^2$ is the leading-order contribution to the pressure due to nonzero density and we define it as c_2 . In Figure 3 we show three susceptibilities that contribute to c_2 . In Figure 4, we show

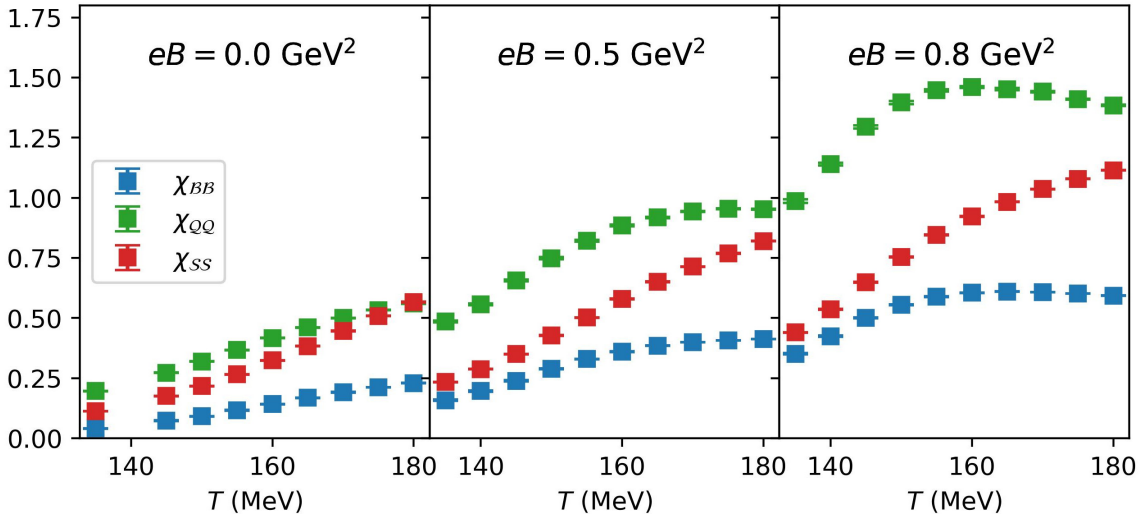


Figure 3: χ_{BB} , χ_{QQ} and χ_{SS} susceptibilities as functions of T for different magnetic field strengths.

our results for the behavior of this coefficient as a function of T and B . Due to the parity symmetry of the partition function, $Z(B) = Z(-B)$, the c_2 coefficient can only contain even powers of the magnetic field. Therefore, we fit our data using a quartic function to account for the non-quadratic behavior at high B . If we associate the inflection point of c_2 with the pseudo-critical temperature (T_c) of the QCD crossover, we observe that the transition is shifted to lower values of T_c as the magnetic field grows, as expected from the known behavior of other observables.

6. Summary and Outlook

In this work, we carried out lattice simulations to study the QCD transition and the equation of state in the presence of both magnetic fields as well as baryon chemical potentials, employing a Taylor expansion framework for the latter. We determined, for the first time in the literature, the impact of the magnetic field on the strangeness neutral line in the presence of an isospin imbalance at low densities by computing the linear coefficients q_1 and s_1 as functions of eB and T . We also provided a physical picture of the behavior of these coefficients with the magnetic field in terms of the hadronic content of the system. We observed that q_1 is suppressed to negative values, while

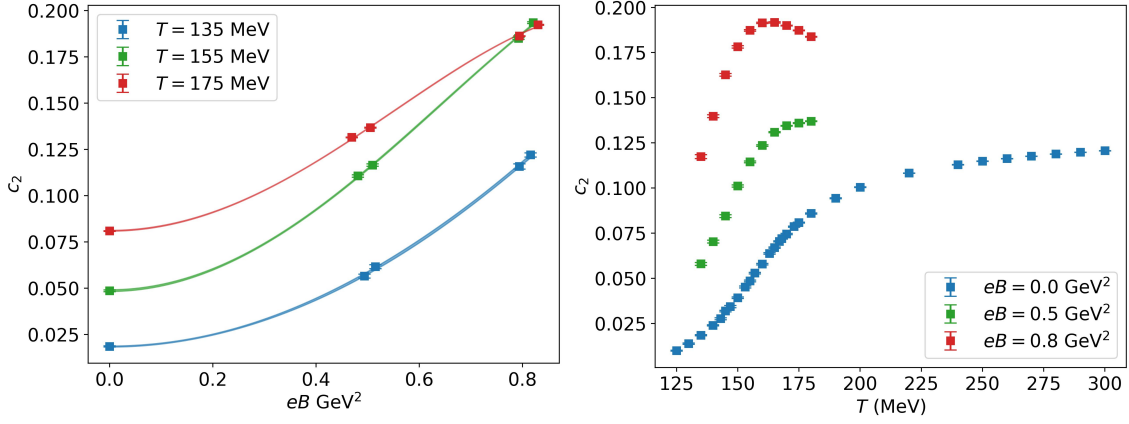


Figure 4: Left plot: leading-order contribution to the equation of state, c_2 , as a function of the magnetic field for different temperatures. Right: c_2 as a function of temperature for different magnetic fields.

s_1 is enhanced by the magnetic field, and noted that this can be explained by the suppression of protons and Λ^0 baryons. Using the strangeness neutral line defined by these coefficients, we also computed the leading-order contribution to the equation of state for several values of the magnetic field.

In the future, we aim to extend this study to higher-order susceptibilities and compute the equation of state at higher μ_B , as well as using finer lattices and finally computing the continuum limit of our observables.

Acknowledgments

This research has received funding from the programme "Netzwerke 2021", an initiative of the Ministry of Culture and Science of the State of Northrhine Westphalia. The authors gratefully acknowledge the Gauss Centre for Supercomputing e.V. (www.gauss-centre.eu) for funding this project by providing computing time on the GCS Supercomputer SuperMUC-NG at Leibniz Supercomputing Centre (www.lrz.de).

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