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The magnetized Gross-Neveu model at finite chemical potential

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We study the (2+1)-dimensional Gross-Neveu model at non-zero chemical potential and subjected to a homogeneous background magnetic field. We do so both analytically, in the limit of an infinite number of fermion flavors in which mean-field approaches become exact, as well as on the lattice for a single flavor. The rich and exotic phase structure observed in the mean-field limit is found to be destroyed when strong quantum fluctuations are present in the system. Instead, in the phase of spontaneously broken chiral symmetry the magnetic field enhances this breaking for all choices of parameters. As a byproduct, we find indications for a first-order phase transition in the chemical potential for vanishing magnetic field but also provide hints that this could rather be a finite-size than a finite-flavor-number effect.

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1. Introduction

The study of strongly-interacting matter under extreme conditions has widespread applications, ranging from the description of heavy-ion collisions or compact stellar objects to models of the early universe. In this context, ‘extreme conditions’ refers to exceedingly high temperatures or densities, strong magnetic fields and combinations thereof. The strong interactions are described by the theory of Quantum Chromodynamics (QCD), whose behavior at zero and non-zero temperature and within background magnetic fields is well understood by now, owing to extensive numerical studies in the framework of lattice quantum field theory [1]. However, little is known about the finite-density regime of QCD due to a strong complex-action problem preventing the straightforward application of conventional lattice methods based on importance sampling.

In order to nonetheless gain insight into the behavior of strongly-interacting matter at finite density, one commonly reverts to the study of effective models expected to reproduce QCD phenomenology within their range of validity. One particular class of such model theories is constituted by four-Fermi theories, i.e., models of relativistic fermions with local four-point self-interactions. Apart from various applications in condensed-matter physics (see, e.g., [2]), they have contributed substantially to our understanding of strongly interacting matter at finite temperature and density [3]. Moreover, they have been shown to reproduce the inverse magnetic catalysis phenomenon observed in lattice simulations of QCD within background magnetic fields [1], provided that the four-Fermi coupling runs appropriately with the magnetic field [4]. Most model approaches, however, employ the mean-field approximation, which, despite its success, is not guaranteed to describe real physical systems faithfully due to the suppression of quantum fluctuations. In order to assess the effect of said fluctuations, one may, e.g., perform lattice simulations beyond the mean-field limit and this is, in fact, the approach we pursue in this work. Many comparable previous studies found that mean-field approaches generally provide a solid understanding of the qualitative features of a theory, with corrections arising only on a quantitative level when going beyond [5–7]. In this work, however, we present a counter-example to this assertion.

2. The Gross-Neveu model

We study the simplest four-Fermi theory, the so-called Gross-Neveu model [8],

$$\mathcal{L} = \bar{\psi}(x) \left(\partial\!\!\!/ + \sigma(x) + \mu\gamma_0 + \mathrm{i}e A\!\!\!/(x) \right) \psi(x) + \frac{N_\mathrm{f}}{2g^2}\sigma^2(x) \;, \tag{1}$$

where the auxiliary scalar field $\sigma(x)$ was introduced in exchange for the scalar-scalar interaction term $\left(\bar{\psi}\psi\right)^2$ via a Hubbard-Stratonovich transformation as usual. In (1), $\psi(x)$ is used to denote N_f flavors of massless fermion fields, μ is the fermion number chemical potential, e denotes the elementary electric charge, A_μ controls an external magnetic field and g^2 denotes the four-Fermi coupling. The Lagrangian (1) is invariant under discrete chiral transformations of the form

$$\psi(x) \longrightarrow \mathrm{i}\gamma_5\psi(x) \;, \quad \bar{\psi}(x) \longrightarrow \mathrm{i}\bar{\psi}(x)\gamma_5 \;, \quad \sigma(x) \longrightarrow -\sigma(x) \;, \tag{2}$$

and this symmetry is broken by a fermionic mass term. A breaking pattern like this entails that one may employ the chiral condensate $\langle\bar{\psi}\psi\rangle$, which is related to the expectation value of σ via

$$\langle\bar{\psi}\psi\rangle = \frac{\mathrm{i}N_\mathrm{f}}{g^2}\langle\sigma\rangle \;, \tag{3}$$

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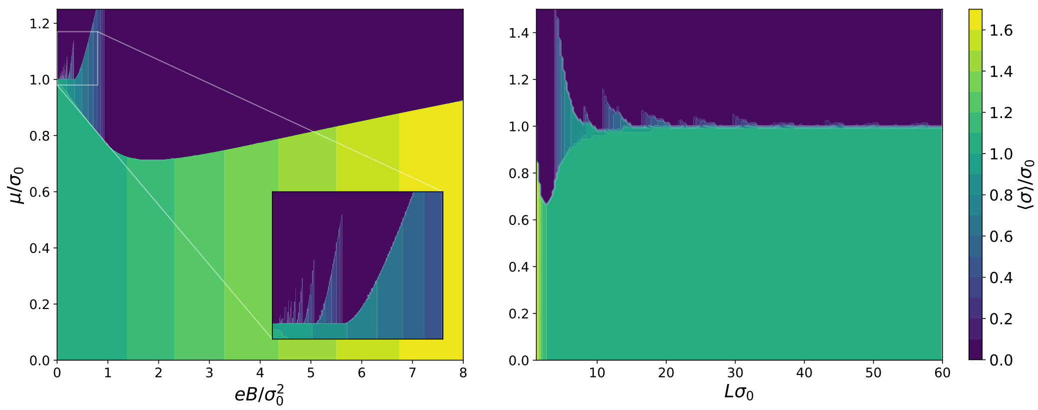

Figure 1: Phase diagrams in the mean-field limit at zero temperature. Left: $L = \infty$. Right: $B = 0$. The scale σ_0 is set by the value of $\langle\sigma\rangle$ at vanishing B, T, and μ and in an infinite volume.

as an order parameter for chiral symmetry breaking. Throughout, we shall work in $2+1$ (Euclidean) space-time dimensions and with four-component spinors. The magnetic field $B = F_{12} := \partial_1 A_2(x) - \partial_2 A_1(x)$ is chosen to be constant and homogeneous.

3. The large-N_f limit

A first approach to studying the Gross-Neveu and similar models is to let the number of fermionic flavors tend to infinity, $N_\mathrm{f} \to \infty$. In fact, in this limit the mean-field approximation becomes exact and the problem of determining the phase structure in (B, T, μ) space is reduced to a minimization problem by means of a saddle point expansion terminated at the lowest order. More concretely, if we assume the scalar field to be independent of space and time, $\sigma(x) = \sigma = const.$, then $\langle\sigma\rangle$ is given by the minimum position of the effective potential

$$V_\mathrm{eff}(\sigma) = \frac{\sigma^2}{2g^2} - \frac{1}{V} \ln\det\left(\partial\!\!\!/ + \sigma + \mu\gamma_0 + \mathrm{i}eA\!\!\!/\right) , \tag{4}$$

where $V = \beta L^2$ denotes the space-time volume, $\beta = 1/T$ is the inverse temperature and L denotes the extent of space in each direction.

This effective potential can be computed in closed form, see, e.g., [9] and its minimization allows for the study of the model's phase structure. The situation at vanishing chemical potential was treated exhaustively in [7], whereas we are concerned with non-zero μ but zero temperature here. In the following, g^2 is assumed to be larger than a critical value, in such a way that chiral symmetry is spontaneously broken at vanishing B, T, and μ. We show the phase diagram of the model in (B, μ)-space in the infinite-volume limit in Fig. 1 (left). One observes that for low μ the magnetic field enhances chiral symmetry breaking, a phenomenon referred to as magnetic catalysis [10]. At larger μ, however, a small region emerges in which the magnetic field instead has the opposite effect and one finds so-called inverse magnetic catalysis [11]. Perhaps most interestingly, close to the region of inverse magnetic catalysis, the formation of Landau levels in the model gives rise to a pattern of multiple (first-order) phase transitions in μ between the phase of broken chiral symmetry and the symmetric phase. It is interesting to note that a non-zero magnetic field can induce first-order transitions which are not present at $B = 0$ in the infinite-volume limit.

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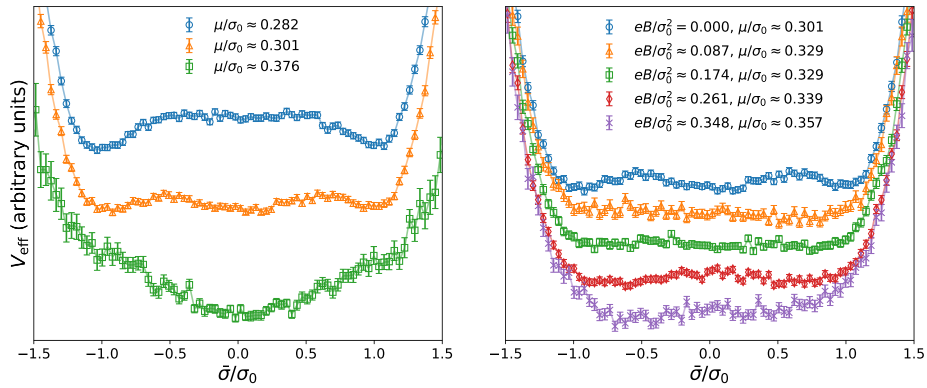

Figure 2: Constraint effective potential on an 8^3 lattice at $T/\sigma_0 \approx 0.118$ with $a\sigma_0 \approx 1.063$ (a denotes the lattice spacing). Left: $B = 0$. Right: $B \neq 0$ and μ close to the phase transition. The potentials are plotted in arbitrary units and shifted vertically for visual clarity.

For $L < \infty$, on the other hand, the discretization of spatial momenta also gives rise to a fully discrete energy spectrum, in close analogy to the Landau quantization. This analogy can be appreciated from Fig. 1 (right), where we plot the phase diagram in the (L, μ) plane for $B = 0$. Similar to the situation at non-zero magnetic field, one thus observes multiple phase transitions which can be of first order and the same is also found for low non-zero temperatures [9]. Notice that, despite the finite spatial volume, these transitions are still proper phase transitions since we work in the large-$N_{\rm f}$ limit and $N_{\rm f}$ and V enter the path integral in an analogous way. In what follows, we study what happens to the mean-field phase structure presented in this section when considering a finite number of fermion flavors, i.e., going beyond the mean-field limit.

4. Lattice results

In order to address this question, we study the model on the lattice; in particular, we perform simulations, employing Neuberger's overlap Dirac operator [12], at $N_{\rm f} = 1$ in order to deviate from the large-$N_{\rm f}$ limit as much as possible. For details on our simulation setup, measured observables and scale-setting, as well as for a list of all parameter values we have performed simulations for, we refer to [9]. This reference also outlines how the complex-action problem arising in our simulations is avoided.

We first consider the case of vanishing magnetic field but non-zero μ, for which previous lattice simulations employing staggered fermions have conjectured the existence of a first-order phase transition at low non-zero temperatures [13]. This observation was later explained to be a consequence of going beyond the mean-field limit [14]. However, given our remarks in the previous section and the fact that simulations can be performed only in finite volumes, it is not entirely clear what the true origin of this first-order phase transition is, since a similar phenomenon can also be observed in the large-$N_{\rm f}$ limit on a finite volume. After all, one typically expects quantum fluctuations to weaken phase transitions rather than strengthen them.

In order to study the possible existence of a first-order transition in the model at $B = 0$ we show in Fig. 2 (left) the (appropriately normalized) logarithm of the probability distribution of $\bar{\sigma} := \frac{1}{V}\sum_x \sigma(x)$, corresponding to the (constraint) effective potential, for different values of μ.

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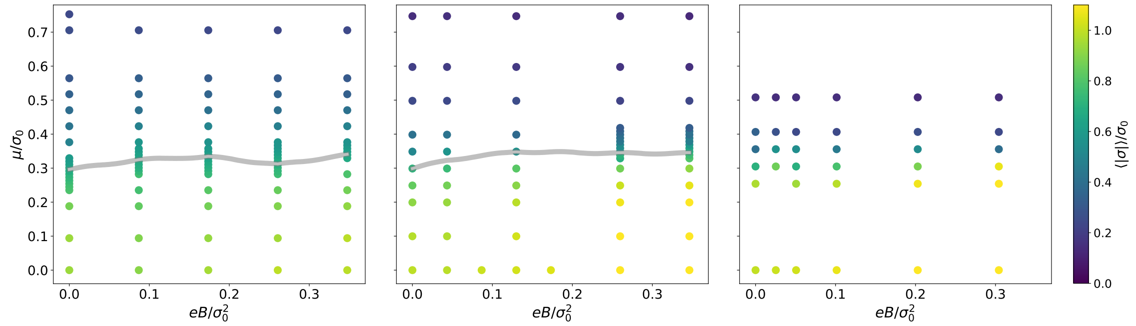

Figure 3: (B, μ) phase diagram for increasing physical volume and fixed lattice spacing. Left: 8^3 lattice, $a\sigma_0 \approx 1.063$. Center: 12^3 lattice, $a\sigma_0 \approx 1.004$. Right: 16^3 lattice, $a\sigma_0 \approx 0.984$. The thick lines show a rough estimate of the critical chemical potential.

As one can see, the effective potential has two degenerate minima at low μ but develops a third one around $\mu/\sigma_0 \approx 0.301$, while for large enough μ it has only a single minimum at zero. This behavior is reminiscent of the mean-field limit and could hint at a first-order transition. However, since both N_f and V are small in our simulations, this finding is not fully conclusive. The situation is somewhat different when considering non-zero magnetic fields, as seen in Fig. 2 (right), where we plot the effective potential for various values of B and for μ close to the phase transition. We no longer see clear evidence for a third minimum in the potential, which rather suggests a second-order phase transition in the thermodynamic limit.

Let us now study the phase structure of the model in (B, μ) space. The analog of Fig. 1 (left) for $N_\mathrm{f} = 1$ is presented in Figs. 3 and 4; the former shows increasing volumes at fixed lattice spacing a, while the approach to the continuum limit at constant volume is depicted in the latter. For the smaller lattices, where the data allowed for a rough estimate of the critical chemical potential μ_c of the transition, we have indicated the dependence of this estimate on B as thick gray lines. One observes that, apart from finite-size effects the magnetic field enhances chiral symmetry breaking for all values of μ below the transition and the critical chemical potential increases with B. Thus, our data are consistent with magnetic catalysis for all $\mu < \mu_c$. Moreover, we do not observe any trace of multiple phase transitions in μ for finite B. These findings are in stark contrast to the large-N_f expectations and also contradict the analytical study [15] working at finite N_f. We believe that this discrepancy arises due to strong fluctuations of $\sigma(x)$ present at $N_\mathrm{f} = 1$ but absent in the mean-field limit and in [15]. Further investigations along this line of research are currently underway. It could, however, also be possible that our current approach simply does not allow for a sufficiently fine parameter scan with sufficiently high statistics to resolve these delicate features. Estimates on this are presented in [9].

5. Summary & Outlook

We have investigated the Gross-Neveu model (1) in $2 + 1$ dimensions, both analytically in the limit of a large number of fermion flavors, as well as on the lattice using $N_\mathrm{f} = 1$ flavor of overlap fermions. In the mean-field limit, we find a rich phase structure in the parameter space spanned by the temperature, the chemical potential, a homogeneous background magnetic field, and the

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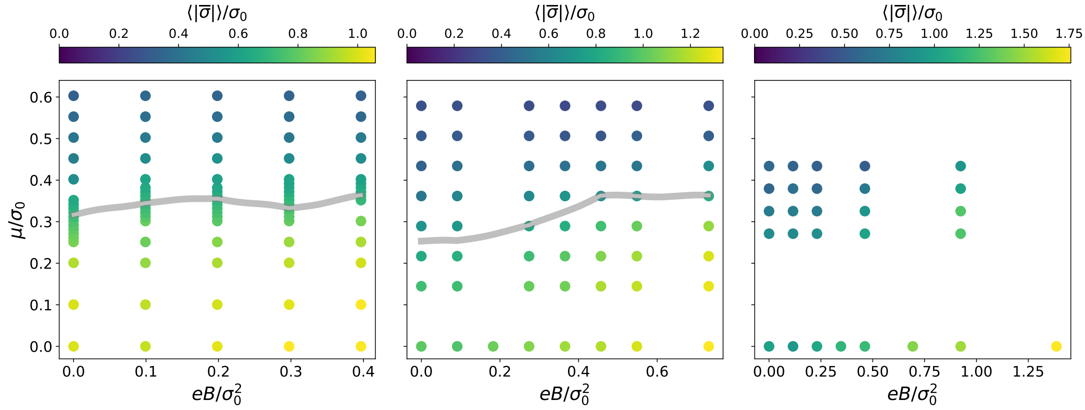

Figure 4: (B, μ) phase diagram for decreasing lattice spacing and fixed physical volume. Left: 8^3 lattice, $a\sigma_0 \approx 0.995$. Center: 12^3 lattice, $a\sigma_0 \approx 0.691$. Right: 16^3 lattice, $a\sigma_0 \approx 0.451$. The thick lines show a rough estimate of the critical chemical potential.

spatial volume. In particular, at non-zero magnetic field this phase structure features both magnetic catalysis and inverse magnetic catalysis, as well as multiple phase transition patterns. Our lattice simulations, on the other hand, indicate that for $N_\mathrm{f} = 1$ the situation is drastically different and, in fact, much simpler: We find only magnetic catalysis to be present for all chemical potentials below some critical value, the latter increasing with B, and no evidence for multiple transitions. This investigation thus provides a counter-example for the common lore that the large-N_f limit usually serves as a reliable guiding principle for the study of the qualitative behavior of a four-Fermi theory. It would be interesting to test this observation also in more realistic models of QCD, which up to now have been studied mostly from the mean-field point of view.

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