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High temperature $U(1)_A$ breaking in the chiral limit

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We solve the long-standing problem concerning the fate of the chiral $U(1)_A$ symmetry in QCD-like theories at high temperature in the chiral limit. We introduce a simple instanton based random matrix model that precisely reproduces the properties of the lowest part of the lattice overlap Dirac spectrum. We show that in the chiral limit the instanton gas splits into a free gas component with a density proportional to m^{N_f} and a gas of instanton-antiinstanton molecules. While the latter do not influence the chiral properties, for any nonzero quark mass the free gas component produces a singular spectral peak at zero that dominates Banks-Casher type spectral sums. By calculating these we show that the difference of the pion and delta susceptibility vanishes only for three or more massless flavors, however, the chiral condensate is zero already for two massless flavors.

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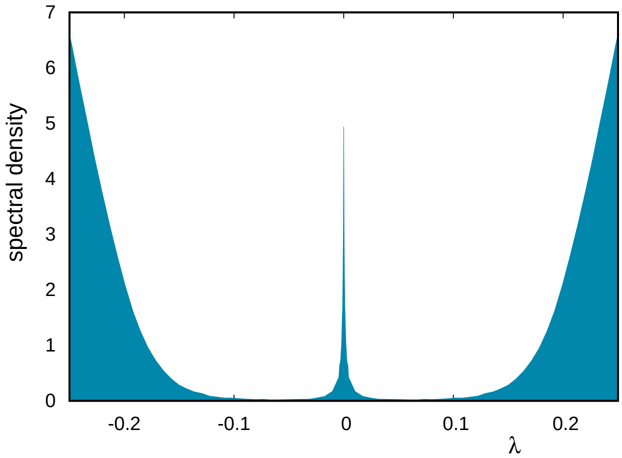

Figure 1: The distribution of the lowest Dirac eigenvalue on configurations containing exactly one instanton-antiinstanton pair on the lattice and in the random matrix model in a spatial volume of 32^3 in lattice units (left panel). The same comparison for the second lowest eigenvalue on a larger volume of 40^3 (right panel). For comparison we also show the distribution used for the fit, the one in the previous figure.

1. Introduction

The approximate $SU(2)_A \times U(1)_A$ chiral symmetry of light quarks is an essential feature of quantum-chromodynamics (QCD). While the $SU(2)_A$ part, spontaneously broken at low temperatures, is expected to be restored above the crossover temperature T_c, the flavor singlet $U(1)_A$ part is anomalous due to the presence of instantons. The fate of the flavor singlet part is the subject of a long debate that was started by the seminal paper of Pisarski and Wilczek, who discussed possible scenarios for the finite temperature chiral transition in the chiral limit [1]. Their results were based on the ϵ-expansion, and there has been an ongoing effort to check it using lattice QCD.

In the chiral limit the order parameter of the spontaneously broken chiral symmetry, $\langle\bar{\psi}\psi\rangle$ can be written in terms of the spectral density $\rho(\lambda)$ of the quark Dirac operator as

$$\lim_{m\to 0} \langle\bar{\psi}\psi\rangle \propto \rho(0).$$

This is the Banks-Casher relation [2], showing that the low lying spectrum of the Dirac operator is intimately related to how chiral symmetry is realized. Indeed, for some time the standard lore was that at the transition temperature to the quark-gluon plasma, the spontaneously broken chiral symmetry is restored, and $\rho(0)$ vanishes, up to small explicit breaking effects due to the finite, but small light quark masses.

However, this view was challenged when instead of a small $\rho(0)$, a sharp rise was found in the spectral density of the overlap Dirac operator at zero [3]. The reason this spike in the spectral density went unnoticed before, was that unlike the overlap, the staggered and Wilson lattice Dirac operators, used exclusively previously, did not respect chiral symmetry, and could not properly resolve the smallest Dirac eigenvalues, the ones that make up the spike of the density. For some time this finding remained largely ignored, mostly because it was considered to be a quenched artifact, the result of ignoring the quark determinant in the path integral. Indeed, the determinant disfavors eigenvalues of small magnitude, and is expected to suppress the spike in the spectral density at zero. Later on, the spike was found to persist even in the presence of light dynamical quarks [4–7]. However, some doubts could still remain, as the lattice fermions used in these works were not chirally symmetric, and their poor resolution of the small Dirac eigenvalues might have lead to an improper suppression of the spike. Indeed, results by the JLQCD collaboration using chiral lattice fermions suggest that toward the chiral limit the spectral spike completely disappears already at a nonzero quark mass [8, 9]. Another possibility, the one we advocate in the present paper, is that indeed, chiral fermions are needed for a proper suppression of the spike, but the spike is still present at any finite quark mass, however, it becomes undetectably small with the currently used lattice volumes and statistics.

It seems to us that using the currently available lattice technology it is not possible to explore the fate of the spike in the spectral density and also the fate of the $U(1)_A$ symmetry for light dynamical quarks. In the present work we suggest a different approach, based on the finding that in the quenched theory the statistics of the eigenvalues in the spectral spike are consistent with mixing zero modes of a free instanton gas, a proposal already put forward in Ref. [3] and recently confirmed in more detail [10].

2. Random matrix model

We propose a random matrix model for the description of the mixing zero modes, the zero mode zone (ZMZ) of the free instanton gas in quenched QCD. In a free instanton gas the number distribution of instantons and antiinstantons are independent and identical Poisson distributions with mean $\chi_0 V/2$, where χ_0 is the topological susceptibility. A random matrix is constructed by first drawing the number of instantons n_i and antiinstantons n_a from the Poisson distributions. The size of the matrix is $(n_\mathrm{i}+n_\mathrm{a})\times(n_\mathrm{i}+n_\mathrm{a})$. Since zero modes of like charge objects are protected from mixing by the index theorem, the matrix has two diagonal blocks of zeros of size $n_\mathrm{i}\times n_\mathrm{i}$ and $n_\mathrm{a}\times n_\mathrm{a}$. At high temperatures the instanton zero mode wave functions decay exponentially with exponent πT [11], and we assume that the mixing matrix elements of instanton and antiinstanton zero modes also follow this exponential decay with the distance of the given instanton and antiinstanton. In a noninteracting gas, the location of the topological objects are chosen randomly within a three-dimensional[1] box of size L. In this way the remaining off-diagonal blocks of the matrix are filled with elements of the form $w_{kl} = A\cdot\exp(-\pi T r_{kl})$, where r_{kl} is the distance of the randomly placed instanton k and antiinstanton l. This completes the construction of one random matrix, and we can easily generate ensembles of such random matrices.

The model has two parameters, the topological susceptibility χ_0 and the prefactor A. To determine these parameters we consider an ensemble of 20k quenched lattice configurations generated with the Wilson action at $T = 1.11T_c$ on lattices of size $32^3\times 8$. Computing the lowest eigenvalues of the overlap Dirac operator on this ensemble and counting the exact zero eigenvalues yields the topological susceptibility. For fitting the single remaining parameter A, we consider the distribution of the lowest nonzero Dirac eigenvalue on those configurations that have exactly two complex conjugate eigenvalues in the ZMZ, corresponding to an instanton-antiinstanton pair. In Fig. 2 (left

[1]At high temperature the instantons typically occupy the whole available space in the temporal direction, so we ignore the temporal location of the instantons.

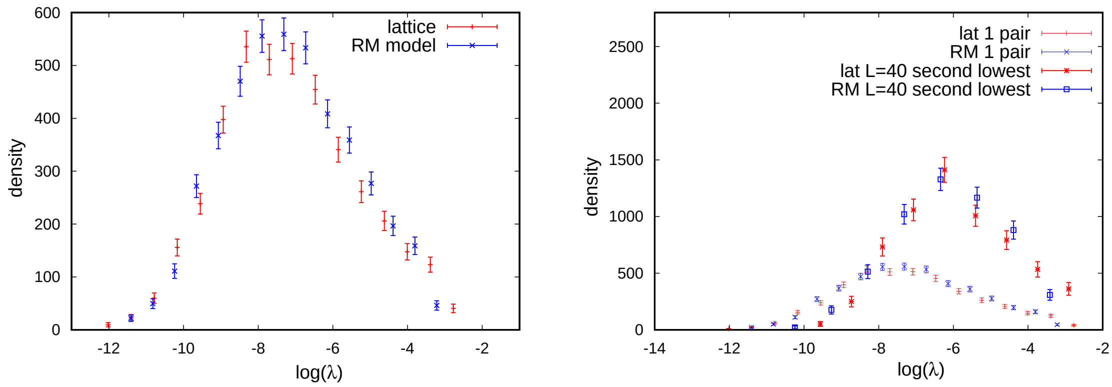

Figure 2: The distribution of the lowest Dirac eigenvalue on the lattice ensemble described in the text, compared to that in the matrix model with the best fit parameter A (left panel). In both cases we used only the configurations with exactly one instanton-antiinstanton pair , and to better resolve the spike in the spectral density, we plotted the distribution of the natural log of the eigenvalues. The right panel shows a similar comparison, but with the second lowest eigenvalue, and on a larger lattice with $L = 40$.

panel) we compare the distribution of the lattice eigenvalues with that of the random matrix model with the best fit parameter A. It is already remarkable that the whole distribution can be fitted with just this one parameter, but now that the model is fixed we can compare different properties of the Dirac spectrum on the lattice and in the matrix model. The comparison can be made also for the distribution of the second lowest eigenvalue or the full spectral density, and also on different volumes. As an illustration, in Fig. 2 (right panel) we show the comparison for the second lowest eigenvalue on a larger volume. These tests demonstrate that the random matrix model properly describes the details of the lattice overlap spectrum. Simulations of the random matrix model on larger volumes, not accessible to direct lattice simulations, indicate the in the thermodynamic limit the spectral density is singular at zero [12].

3. Including dynamical quarks

On the lattice, including dynamical quarks means that in addition to the quenched Boltzmann weights, each configuration gets another weight factor proportional to the quark Dirac determinant $\det(D+m)^{N_f}$. For simplicity we assume N_f degenerate quark flavors. The determinant of the lattice Dirac operator can be split into the contribution of the zero mode zone and that of the bulk as

$$\det(D+m) = \prod_{i \in ZMZ} (\lambda_i + m) \cdot \prod_{i \in \text{bulk}} (\lambda_i + m). \tag{1}$$

As can be seen in Fig. 1, at high temperatures the ZMZ and the bulk are well separated, therefore eigenvalues in the bulk are not expected to be correlated with the ones in the ZMZ. Our main concern here is to study how the determinant suppresses the eigenvalues in the ZMZ. It is thus a good approximation to ignore the bulk contribution to the determinant, as that will only give a trivial factor in the path integral for quantities depending on the ZMZ. Especially for small quark mass, the suppression of the eigenvalues in the ZMZ will be driven by the contribution of those small eigenvalues to the determinant. This is exactly the part of the Dirac spectrum that is faithfully

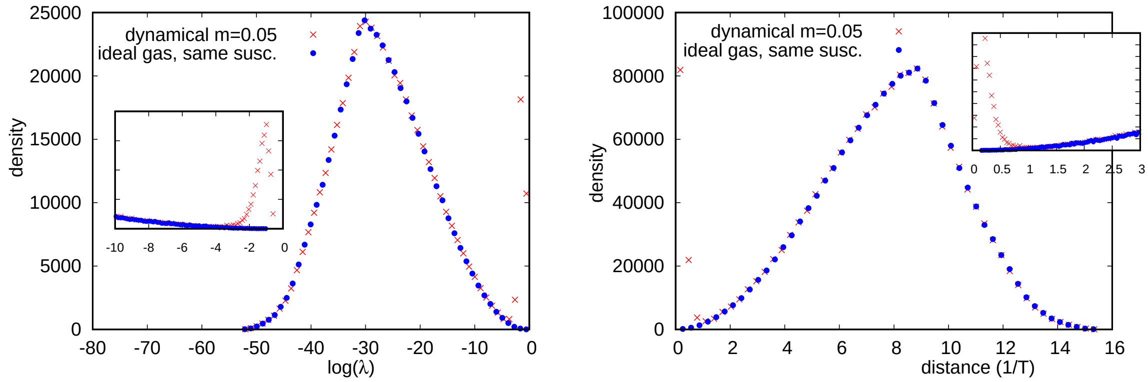

Figure 3: The spectral density of the random matrix model with two degenerate dynamical quark flavors of mass $m = 0.05$ (in lattice units) compared to the spectral density of the matrix model of the free instanton gas with the same topological susceptibility (left panel). A comparison of the density of nearest opposite charged topological objects between the quenched and the dynamical random matrix ensemble (right panel).

represented by the random matrix model, so including the determinant of the random matrices in the statistical weight will properly describe the suppression of the spectral spike by dynamical quarks. Thus the random matrix model we propose for the ZMZ of QCD with dynamical quarks is the one described in the previous section, with the additional weight $\det(D+m)^{N_f}$ for each instanton configuration, where D is the random matrix corresponding to the given instanton configuration.

The numerical simulation of this model reveals that the topological susceptibility is suppressed by light dynamical quarks as

$$\chi(m) = \chi_0 m^{N_f}, \tag{2}$$

where χ_0 is the quenched susceptibility, the one obtained without including the quark determinant. To understand this behavior we note that throughout the simulations we found that the eigenvalues in the spectral spike always satisfied $|\lambda| \ll m$, even for the smallest quark mass of $m = 0.01$ (in lattice units). The reason for this is that as the quarks become lighter, the determinant suppresses the number of instantons, the typical instanton-antiinstanton distance grows, and the matrix elements become exponentially small, resulting in ever smaller eigenvalues. As a result, in the chiral limit the magnitude of the eigenvalues in the ZMZ decreases much faster than the quark mass, and they will always remain smaller than the quark mass. If the eigenvalues are much smaller than the quark mass then to a very good approximation the determinant of a matrix with n_i instantons and n_a antiinstantons can be written as

$$\det(D+m) = \prod_i (\lambda_i + m) \approx m^{n_\mathrm{i}+n_\mathrm{a}}. \tag{3}$$

With N_f quark flavors, each (anti)instanton contributes a suppression factor m^{N_f} to the determinant, and the distribution of (anti)instanton numbers is still Poissonian, but with a susceptibility suppressed as $\chi_0 \to \chi_0 m^{N_f}$. This explains the quark mass dependence of the susceptibility in Eq. (2). The fact that the Poisson distribution of the number of topological objects is preserved also implies that even in the presence of light dynamical quarks, the lowest part of the Dirac spectrum can still be understood as the zero mode zone of a free instanton gas. To demonstrate this, in the left panel of Fig. 3 we compare the spectral density of the random matrix model with two degenerate dynamical quark flavors of mass $m = 0.05$ (in lattice units) with that of the matrix model of the free instanton gas (without the determinant) with the same topological susceptibility. The two curves exactly agree, except for the largest eigenvalues, where the model with dynamical quarks shows an excess of eigenvalues. Large eigenvalues in the matrix model indicate that there might be large matrix elements which in turn would imply that in the dynamical case there are nearby instanton-antiinstanton pairs. To check that, in the right panel of Fig. 3 we compare the density of the nearest opposite charged objects as a function of their distance for the dynamical and quenched matrix ensembles of the left panel of the same figure. Indeed, in the dynamical case we find an excess of tightly bound instanton-antiinstanton pairs at a distance smaller than the instanton size $1/T$. These instanton-antiinstanton "molecules" are held together by the attractive force due to light dynamical quarks.

4. Chiral condensate and $U(1)_A$ breaking susceptibility

We have seen that with light dynamical quarks the instanton gas has two components. There is a dilute gas of free instantons, responsible for the small Dirac eigenvalues, i.e. the spectral spike. Besides that, there is a component of tightly bound instanton-antiinstanton molecules. Our simulations reveal that in the chiral limit the eigenvalues corresponding to these two components behave differently. While the free instanton eigenvalues in the spike become smaller as the gas becomes more dilute, the eigenvalues corresponding to the molecules remain at a fixed scale in the spectrum. This is because the size of the molecules does not change with the quark mass. An important consequence of this is that in the chiral limit any nonvanishing contribution to Banks-Casher type sums can only come from the free instanton generated part of the spectrum.

The simplest of these sums is the one providing the chiral condensate. In the chiral limit the chiral condensate can be written in terms of the Dirac spectrum as

$$\langle\bar{\psi}\psi\rangle \approx \langle\sum_i \frac{m}{m^2+\lambda_i^2}\rangle \approx \underbrace{\begin{pmatrix}\text{avg. number of in-}\\ \text{stantons in free gas}\end{pmatrix}}_{\chi_0 m^{N_\mathrm{f}} V} \cdot \frac{1}{m} = m^{N_\mathrm{f}-1}\chi_0 V. \tag{4}$$

Here we used the fact that the eigenvalues corresponding to the molecular component of the instanton gas remain at a fixed scale, and in the chiral limit they do not contribute to the sum. In contrast, the magnitude of eigenvalues generated by the free gas component becomes smaller in the chiral limit so rapidly that $\lim_{m\to 0}\frac{\lambda}{m} = 0$, and all these eigenvalues will contribute a term $1/m$ to the sum. So the chiral limit of the condensate is given by the contribution of the free instanton gas. This result shows that for two and more flavors of vanishing mass, the chiral condensate goes to zero, and the spontaneously broken chiral symmetry is restored, as expected. We also note that Eq. (4) is consistent with the expansion of the QCD free energy density in terms of the quark mass around the chiral point [13], and also consistent with the resulting quasi-instanton picture of [14]. In fact, our instanton-based matrix model provides a physical explanation for the quasi-instanton picture, and shows that it is valid not only in the small quark mass limit.

Another quantity of interest in the chiral limit is the susceptibility $\chi_\pi - \chi_\delta$, a nonzero value of which signals the breaking of the $U(1)_A$ symmetry [5, 6, 9]. Similarly to the chiral condensate, in the chiral limit this quantity can also be written as a sum over the Dirac spectrum as

$$\chi_\pi - \chi_\delta \approx \langle \sum_i \frac{m^2}{(m^2+\lambda_i^2)^2} \rangle \approx m^{N_\mathrm{f}} \chi_0 V \cdot \frac{1}{m^2} \;=\; m^{N_\mathrm{f}-2} \chi_0 V, \tag{5}$$

showing that even for two flavors of vanishing mass, the symmetry remains effectively broken.

Acknowledgments

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References

[1] R. D. Pisarski and F. Wilczek, Phys. Rev. D **29**, 338-341 (1984) doi:10.1103/PhysRevD.29.338

[2] T. Banks and A. Casher, Nucl. Phys. B **169**, 103-125 (1980) doi:10.1016/0550-3213(80)90255-2

[3] R. G. Edwards, U. M. Heller, J. E. Kiskis and R. Narayanan, Phys. Rev. D **61**, 074504 (2000) doi:10.1103/PhysRevD.61.074504.

[4] A. Alexandru and I. Horváth, Phys. Rev. D **92**, no.4, 045038 (2015) doi:10.1103/PhysRevD.92.045038.

[5] H. T. Ding, S. T. Li, S. Mukherjee, A. Tomiya, X. D. Wang and Y. Zhang, Phys. Rev. Lett. **126**, no.8, 082001 (2021) doi:10.1103/PhysRevLett.126.082001.

[6] O. Kaczmarek, L. Mazur and S. Sharma, Phys. Rev. D **104**, no.9, 094518 (2021) doi:10.1103/PhysRevD.104.094518.

[7] X. L. Meng *et al.* [χQCD and CLQCD], [arXiv:2305.09459 [hep-lat]]. [8]

[8] S. Aoki *et al.* [JLQCD], PTEP **2022**, no.2, 023B05 (2022) doi:10.1093/ptep/ptac001

[9] S. Aoki *et al.* [JLQCD], Phys. Rev. D **103**, no.7, 074506 (2021) doi:10.1103/PhysRevD.103.074506.

[10] R. A. Vig and T. G. Kovacs, Phys. Rev. D **103**, no.11, 114510 (2021) doi:10.1103/PhysRevD.103.114510.

[11] D. J. Gross, R. D. Pisarski and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981) doi:10.1103/RevModPhys.53.43.

[12] T. G. Kovacs, [arXiv:2311.04208 [hep-lat]].

[13] T. Kanazawa and N. Yamamoto, JHEP **01**, 141 (2016) doi:10.1007/JHEP01(2016)141.

[14] T. Kanazawa and N. Yamamoto, Phys. Rev. D **91**, 105015 (2015) doi:10.1103/PhysRevD.91.105015.