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Structure-dependent electromagnetic finite-volume effects through order $1/L^3$

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We consider electromagnetic finite-volume effects through order $1/L^3$ in different formulations of QED, where L is the periodicity of the spatial volume. An inherent problem at this order is the appearance of structure-dependent quantities related to form factors and the analytical structure of the correlation functions. The non-local constraint of the widely used $\mathrm{QED_L}$ regularization gives rise to structure-dependent effects that are difficult to evaluate analytically and can act as a precision bottleneck in lattice calculations. For this reason, we consider general volume expansions relevant for the mass spectrum as well as leptonic decay rates in $\mathrm{QED_C}$, $\mathrm{QED_L}$ and $\mathrm{QED_L^{IR}}$, the latter being a class of non-local formulations generalising $\mathrm{QED_L}$. One choice within this class is $\mathrm{QED_r}$, first introduced at this conference, and we show that the effects of non-locality for the $1/L^3$ term in the expansion can be removed. We observe that there are still $1/L^3$ contributions unrelated to the (non-)locality of the studied QED formulations, but rather to collinear singularities in the physical amplitudes.

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1. Introduction

Lattice quantum chromodynamics (QCD) offers a systematically improvable approach for computing observables where non-perturbative dynamics play a crucial role. With relative precision in lattice calculations approaching the (sub-)percent level [1, 2], one has to include effects from the up- and down-quark mass difference as well as electromagnetism. While the strong isospin-breaking from quark masses poses computational complications, including electromagnetic effects in the simulations is fundamentally more challenging [3]. The underlying issue is constraining the long-range nature of the electromagnetic force into finite-volume spacetimes. Gauss' law in fact forbids charged states in finite volumes with periodic boundary conditions due to zero-momentum modes of photons, but there are several ways to circumvent the problem through dedicated finite-volume formulations of quantum electrodynamics (QED) [3–7]. Different prescriptions affect the volume-dependence of calculated observables in different ways. In this talk we analytically study finite-volume effects in formulations where the dependence scales as a polynomial of inverse powers of the spatial volume-extent L, along with potential logarithmic divergences $\log L$.

It is very valuable to analytically subtract the leading volume scaling [1, 2, 8–10]. The isospin-breaking corrections to leptonic decays $P \to \ell\nu_\ell$, where P is a pion or kaon and ℓ a lepton with corresponding neutrino ν_ℓ, have been calculated in Refs. [1, 2] in lattice QCD with electromagnetic corrections in the $\mathrm{QED_L}$ formulation [3]. It was recently observed that the $1/L^3$ correction in $\mathrm{QED_L}$ potentially is very sizeable [2, 10], thus motivating an analytical evaluation of the $1/L^3$ coefficient in the expansion. The dependence on the internal structure of the decaying meson in this coefficient can be handled using the method of Ref. [10] building on form factor decompositions and electromagnetic Ward identities.

In this talk, we present a general study of the evaluation of structure-dependent finite-volume effects through order $1/L^3$ in QED formulations with power-law scaling. In particular we consider $\mathrm{QED_L^{IR}}$, introduced in Ref. [7]. In this approach, the photon propagator is modified by reweighting some number of low momentum modes, generally to achieve some desired behaviour in the large-volume expansion of a given observable. $\mathrm{QED_L}$ is a special case of this in which only the zero-momentum mode is modified by weighting it to zero, i.e. quenching it from the theory. $\mathrm{QED_r}$ is another particular alternative designed to remove the $1/L^3$ terms from a variety of observables, first introduced at this conference [11] and discussed in more detail below. We apply the results to leptonic decay rates in the framework of $\mathrm{QED_L^{IR}}$, and for the first time determine the $1/L^3$ contribution. We compare the values of finite-volume coefficients in $\mathrm{QED_L}$, $\mathrm{QED_r}$ as well as $\mathrm{QED_C}$ [5]. We observe that one finite-volume coefficient can lead to order $1/L^3$ effects in $\mathrm{QED_C}$.

2. Generalities on the finite-volume expansion

Let us consider an observable O in QCD+QED in a spacetime of geometry $V = \mathbb{R} \times L^3$. We will only be interested in leading-order QED corrections, i.e. order $\alpha = e^2/(4\pi)$ in the fine-structure constant α. A determination of the observable in lattice simulations induces a volume-dependence, $O(L)$, and the finite-volume effects are given by the difference $\Delta O(L) = O(L) - O(\infty)$. We restrict ourselves to the case where $O(L)$ at most is logarithmically infrared divergent, as is relevant for leptonic decays. In QED formulations with a massless finite-volume photon propagator, $\Delta O(L)$ will in addition to the infrared divergence $\kappa_{\log} \log L$ contain a series of terms κ_n/L^n. The coefficients κ_n depend on the chosen QED formulation, on the physical process and in general on internal structure

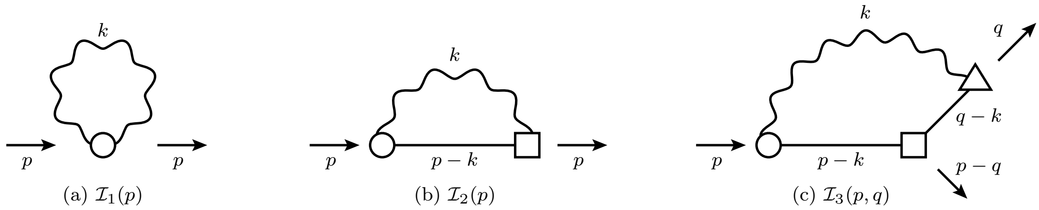

Figure 1: Three diagram topologies arising for the mass spectrum and leptonic decays, (a) the tadpole, (b) the sunset and (c) the triangle diagram. The momenta p and q are on-shell and correspond to massive particles, whereas k is the virtual photon momentum. The circles, squares and triangles represent structure- and formulation-dependent vertex functions.

of the interacting particles. The formulation dependence stems from the different ways of handling photon zero-momentum modes. In $\mathrm{QED}_\mathrm{L}^\mathrm{IR}$ (in particular $\mathrm{QED_L}$ and $\mathrm{QED_r}$) the problematic modes are removed on each time-slice, rendering the theory non-local [7]. In $\mathrm{QED_C}$, charge-conjugated boundary conditions are introduced, which in a local way forbids zero-momentum modes but introduces flavour mixing and charge non-conservation over the boundary [5].

It is well-known that the κ_n for the mass spectrum are independent of structure through order $1/L^2$ [5, 7, 8], and for leptonic decays through order $1/L$ [9, 10]. The leading structure-dependence for these observables was determined for $\mathrm{QED_L}$ in Ref. [10], but it was observed that there are particular structure-dependent contributions at order $1/L^3$ due to the non-locality of $\mathrm{QED_L}$. These contributions correspond to branch-cuts in the underlying correlation functions defining the physical process, e.g. the Compton scattering tensor in the case of the mass spectrum [10].

3. Finite-volume expansion of the sunset topology

To obtain the volume-expansion one can use the method developed in Refs. [8–10], where the correlation function is skeleton expanded into a set of diagrams built from structure-dependent irreducible vertex functions. For our purposes we simply need to assume that such a decomposition has been made. Example topologies that appear for the mass spectrum and leptonic decays are shown in Fig. 1. In the following, we will be interested in finite-volume effects in the sunset diagram in Fig. 1(b), $\Delta\mathcal{I}_2$.

The choice of QED formulation enters through the photon propagator for momentum $k = (k_0, \mathbf{k})$ and the set of allowed momenta $\mathbf{k} \in \Pi$. The sets are the same for $\mathrm{QED}_\mathrm{L}^\mathrm{IR}$, $\mathrm{QED_L}$ and $\mathrm{QED_r}$, but different for $\mathrm{QED_C}$ due to the charge-conjugated boundary conditions. Explicitly the sets are

$$\begin{aligned} \mathrm{QED}_\mathrm{L}^\mathrm{IR}: \quad & \Pi = \Pi_\mathrm{L} = \left\{ \frac{2\pi\,\mathbf{n}}{L} \,\middle|\, \mathbf{n} \in \mathbb{Z}^3 \setminus \{0,0,0\} \right\} , \\ \mathrm{QED_C}: \quad & \Pi = \Pi_\mathrm{C} = \left\{ \frac{2\pi}{L}\left(\mathbf{n} + \frac{\bar{\mathbf{n}}}{2}\right) \,\middle|\, \mathbf{n} \in \mathbb{Z}^3 \,,\ \bar{\mathbf{n}} = (1,1,1) \right\} . \end{aligned} \tag{1}$$

We write the Feynman-gauge propagator as

$$D_{\mu\nu}(k) = \delta_{\mu\nu} \frac{1 + w_{|\mathbf{n}|^2}}{k_0^2 + \mathbf{k}^2} , \qquad \mathbf{k} \in \Pi . \tag{2}$$

Here $w_{|\mathbf{n}|^2}$ is a discrete set of weights on photon-momentum modes $\mathbf{k} \in \Pi$, as was first introduced to define $\mathrm{QED}_\mathrm{L}^\mathrm{IR}$ [7]. However, we can use this form to consider also other formulations, in particular $w_{|\mathbf{n}|^2} = 0$ for $\mathrm{QED_L}$ and $\mathrm{QED_C}$, and $w_{|\mathbf{n}|^2} = \delta_{\mathbf{n}^2,1}/6$ for $\mathrm{QED_r}$.

In this way the propagator $D_{\mu\nu}(k)$ in (2) allows for a common treatment of finite-volume effects of the QED formulations. For the sunset topology, we thus have

$$\Delta\mathcal{I}_2 = \left(\frac{1}{L^3}\sum_{\mathbf{k}\in\Pi} - \int\frac{d^3\mathbf{k}}{(2\pi)^3}\right)\int\frac{dk_0}{2\pi}\,\frac{f\,(k_0,|\mathbf{k}|,\mathbf{p}\cdot\mathbf{k})}{\left[(p-k)^2+m^2\right]}\,\frac{1+w_{|\mathbf{n}|^2}}{k^2}\,. \tag{3}$$

Here m is the mass of the particle with momentum $p-k$, with $p^2=-m^2$. We assume that the we are in a boosted frame with $p=(i\omega(\mathbf{p}),\mathbf{p})$. The numerator contains the observable-, structure and formulation-dependent function $f\,(k_0,|\mathbf{k}|,\mathbf{p}\cdot\mathbf{k})$, which is free of poles in k_0 but can contain branch-cuts. The arguments of f show the possible ways the photon momentum can appear.

The first step in obtaining a volume expansion in inverse powers of L is to perform the k_0-integral in (3). As can be seen from the integrand, there are two poles in the upper half-plane, one from the photon propagator and one from the massive propagator. All the remaining analytic structure from potential branch-cuts resides in f and we have

$$\int\frac{dk_0}{2\pi}\,\frac{f\,(k_0,|\mathbf{k}|,\mathbf{p}\cdot\mathbf{k})}{\left[(p-k)^2+m^2\right]}\,\frac{1}{k^2} = g^{\text{poles}}(\mathbf{k}) + g^{\text{rem}}(\mathbf{k})\,. \tag{4}$$

Here g^{rem} corresponds to the non-pole contributions. Consequently, the finite-volume effects are

$$\Delta\mathcal{I}_2 = \left(\frac{1}{L^3}\sum_{\mathbf{k}\in\Pi} - \int\frac{d^3\mathbf{k}}{(2\pi)^3}\right)\left(1+w_{|\mathbf{n}|^2}\right)\left[g^{\text{poles}}(\mathbf{k}) + g^{\text{rem}}(\mathbf{k})\right] = \Delta\mathcal{I}_2^{\text{poles}} + \Delta\mathcal{I}_2^{\text{rem}}\,. \tag{5}$$

From (1) we see that the allowed momenta $\mathbf{k}$ in both Π_{L} and Π_{C} scale as $1/L$, which means that the integrand/summand can be expanded systematically in L. The coefficients in the expansion will contain derivatives of the function f which depend on structure and the QED formulation, as well as a set of finite-volume coefficients defined by the sum-integral differences

$$c_j^w(\mathbf{v}) = \left(\sum_{\mathbf{n}\in\Pi} - \int d^3\mathbf{n}\right)\frac{1+w_{|\mathbf{n}|^2}}{|\mathbf{n}|^j\,(1-\mathbf{v}\cdot\hat{\mathbf{n}})}\,. \tag{6}$$

Here $\mathbf{v}=\mathbf{p}/\omega(\mathbf{p})$ is the velocity of the massive particle. Note that the factors $1/|\mathbf{n}|$ and $1/(1-\mathbf{v}\cdot\hat{\mathbf{n}})$ appear due to underlying soft and collinear singularities in the physical process. For a deeper discussion of the properties of these finite-volume coefficients, see Ref. [12]. We also define $c_j^w(\mathbf{0}) = c_j^w$.

The cut-contribution $g^{\text{rem}}(\mathbf{k})$ is by definition a smooth function at $|\mathbf{k}|=0$. With $g^{\text{rem}}(\mathbf{0}) = C^{\text{rem}}$ the Poisson summation formula then immediately allows us to write (through order $1/L^3$)

$$\Delta\mathcal{I}_2^{\text{rem}} = \frac{c_0^w\,C^{\text{rem}}}{L^3}\,. \tag{7}$$

On the other hand, the pole contribution $g^{\text{poles}}(\mathbf{k})$ is not a smooth function, which in turn yields

$$\begin{aligned}\Delta\mathcal{I}_2^{\text{poles}} = {} & \frac{f\,c_2^w\,(\mathbf{v})}{16\pi^2 L\omega(\mathbf{p})}\\ & + \frac{f^{(0,0,1)}\omega(\mathbf{p})\left(c_1^w\,(\mathbf{v}) - c_1^w\right) + \left(f^{(0,1,0)} + i f^{(1,0,0)}\right)c_1^w\,(\mathbf{v})}{8\pi L^2\omega(\mathbf{p})}\\ & + \frac{1}{8L^3}\left\{\frac{c_0^w\,(\mathbf{v})}{\omega(\mathbf{p})}\left[f^{(0,2,0)} + \ldots\right] - c_0^w\left[f^{(0,0,2)}\omega(\mathbf{p}) + \ldots\right]\right\}.\end{aligned} \tag{8}$$

In this expressions the function f and its derivatives are evaluated at $(k_0 = 0,\ |\mathbf{k}| = 0,\ \mathbf{k}\cdot\mathbf{p} = 0)$. The ellipses contain other contributions from the function f, and have been introduced to avoid lengthy expressions. The crucial observation here is that branch-cut term $C^{\rm rem}$ only multiplies the coefficient c_0^w, while the poles contribute both to c_0^w and $c_0^w(\mathbf{v})$. Also, we emphasise that both $c_j^w(\mathbf{v})$ and the function f depend on the chosen QED prescription. Assuming that f is known, the finite-volume effects for the sunset topology are given by (7)–(8), where the coefficients $c_j^w(\mathbf{v})$ from (6) are defined by the allowed set of momenta in (1) as well as the coefficients $w_{|\mathbf{n}|^2}$. We define $c_j^w(\mathbf{v}) = c_j(\mathbf{v})$ for QED_L, $c_j^w(\mathbf{v}) = \bar{c}_j(\mathbf{v})$ for QED_r and $c_j^w(\mathbf{v}) = c_j^\star(\mathbf{v})$ for QED_C.

As a final remark, the coefficient c_0^w is non-zero in QED_L due to the non-locality of the theory, $c_0 = -1$. On the contrary, $c_0^\star = 0$ in QED_C, meaning that the branch-cut does not contribute in this case. The particular choice of parameters $w_{|\mathbf{n}|^2}$ in QED_r gives $\bar{c}_0 = 0$, even though the theory has removed the zero-modes in a non-local fashion. This property makes QED_r an interesting possible formulation to use in future lattice calculations, and we are currently performing a dedicated volume study in this framework to validate its suitability for future work [11]. In Section 5 we numerically study the coefficients appearing in (7)–(8) for QED_L, QED_r and QED_C.

4. Leptonic decay rates in $\text{QED}_\text{L}^\text{IR}$

We now apply the above formalism to derive the finite-volume effects for the leptonic decay $P \to \ell\nu_\ell$ through order $1/L^3$ in $\text{QED}_\text{L}^\text{IR}$. This extends the results in Refs. [2, 9, 10], and requires expanding also the diagrams in Figs. (a) and (c). Although we here limit the order to $1/L^3$ we have derived all structure-dependent vertex functions needed for leptonic decays, which would contribute also beyond order $1/L^3$. These will be presented in future work. Leaving out details about leptonic decays that can be found in Ref. [10], the relevant function here is $Y^{(3)}(L)$ that in the rest frame of the decaying meson is found to be

$$\begin{aligned}
Y^{(3)}(L) = &\ \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + \frac{c_3^{\rm IR} - 2\,c_3^{\rm IR}(\mathbf{v}_\ell)}{2\pi} - 2\,A_1(\mathbf{v}_\ell) + 2\log\left(\frac{m_W L}{4\pi}\right) \\
&- 2\,A_1(\mathbf{v}_\ell)\left[\log\left(\frac{m_P L}{4\pi}\right) + \log\left(\frac{m_\ell L}{4\pi}\right)\right] - \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\, c_2^{\rm IR} - 4\, r_\ell^2\, c_2^{\rm IR}(\mathbf{v}_\ell)}{1 - r_\ell^4}\right] \\
&+ \frac{1}{(m_P L)^2}\left[-\frac{F_A^P}{f_P}\,\frac{4\pi\, m_P\,\left[(1+r_\ell^2)^2\, c_1^{\rm IR} - 4\, r_\ell^2\, c_1^{\rm IR}(\mathbf{v}_\ell)\right]}{1 - r_\ell^4} + \frac{8\pi\,\left[(1+r_\ell^2)\, c_1^{\rm IR} - 2\, c_1^{\rm IR}(\mathbf{v}_\ell)\right]}{(1 - r_\ell^4)}\right] \\
&+ \frac{32\pi^2\, m_P}{f_P(1 - r_\ell^4)\,(m_P L)^3}\left\{c_0^{\rm IR}(\mathbf{v}_\ell)\,\left[F_V^P - F_A^P + 2m_P^2 r_\ell^2\, A^{(0,1)}\left(0, -m_P^2\right)\right] + c_0^{\rm IR}\, C_\ell\right\}.
\end{aligned} \tag{9}$$

Here the $c_j^{\rm IR}(\mathbf{v}_\ell)$ are $\text{QED}_\text{L}^\text{IR}$ coefficients depending on the lepton velocity $\mathbf{v}_\ell$, m_ℓ the lepton mass, m_W the W-boson mass, m_P the meson mass, $r_\ell = m_\ell/m_P$ and $A_1(\mathbf{v}_\ell)$ a known function [10]. The branch-cuts of the correlation function appear in C_ℓ together with structure-dependent form factors and point-like contributions. It is at the moment unknown how one would calculate the branch-cut. The coefficient $c_0^{\rm IR}(\mathbf{v}_\ell)$ multiplies only structure-dependent form factors, F_V^P, $F_A^P = A(0, -m_P^2)$ and its derivative $A^{(0,1)}(0, -m_P^2)$ (where A is a function of k^2 and $p\cdot k$). These are related to the real radiative leptonic decay $P \to \ell\nu_\ell\gamma$ and can be determined on the lattice [13]. We emphasise that without knowledge of C_ℓ the above expression cannot be used in practice. However, in QED_r we have $\bar{c}_0 = 0$ which circumvents the problem and allows for full control through order $1/L^3$.

j	$c_j(\mathbf{v})$	$\bar{c}_j(\mathbf{v})$	$c_j^*(\mathbf{v})$	c_j	$\bar{c}_j$	c_j^*
2	-16.3454	-14.9613	-3.20674	-8.91363	-7.91363	-5.49014
1	-5.73018	-4.3461	3.51224	-2.8373	-1.8373	-0.80194
0	-2.12369	-0.7396	3.69273	-1	0	0

Table 1: Finite-volume coefficients $c_j(\mathbf{v})$, $\bar{c}_j(\mathbf{v})$ and $c_j^*(\mathbf{v})$, in $\mathrm{QED_L}$, $\mathrm{QED_r}$ and $\mathrm{QED_C}$, respectively.

5. Numerical comparison of finite-volume coefficients

We now perform a numerical calculation of the finite-volume coefficients $c_j^w(\mathbf{v})$ and c_j^w appearing in (7)–(8), for $\mathrm{QED_L}$, $\mathrm{QED_r}$ and $\mathrm{QED_C}$. The velocity is chosen to be $\mathbf{v} = |\mathbf{v}|(1,1,1)/\sqrt{3}$, with $|\mathbf{v}| = 0.912401$ corresponding to the muon velocity in $K \to \mu\nu_\mu$. The values of the coefficients for $j = 0,1,2$ are shown in Table 1. We immediately see the non-locality of $\mathrm{QED_L}$ from $c_0 = -1$, the locality of $\mathrm{QED_C}$ from $c_0^\star = 0$, and the effects of non-locality being removed from the choice of action parameters in $\mathrm{QED_r}$ with $\bar{c}_0 = 0$. The velocity-dependent coefficients $c_0(\mathbf{v})$, $\bar{c}_0(\mathbf{v})$ and $c_0^\star(\mathbf{v})$ are in fact all non-zero. This shows that despite the locality of $\mathrm{QED_C}$ there can be finite-volume effects at order $1/L^3$. An interesting observation is that if one were to do an angular averaging over all directions of the lepton velocity, the averaged finite-volume coefficients reduce to those at zero-velocity according to $\int d\Omega\, c_j^w(\mathbf{v}) \propto c_j^w$ [7]. This means that if c_0^w is zero in a formulation, an angular averaging would remove all volume-effects related also to $c_0^w(\mathbf{v})$ [11, 12].

6. Conclusions

In this talk we have considered structure-dependent finite-volume effects decaying as inverse powers of L. The study is relevant for $\mathrm{QED_L^{IR}}$, $\mathrm{QED_L}$, $\mathrm{QED_C}$ and $\mathrm{QED_r}$, the last of which was first introduced at this conference. We derived formulae through order $1/L^3$, highlighting the impact of (non-)locality in the QED formulations. A particularly important finding was the possibility to eliminate the full $1/L^3$ contribution for leptonic decays in $\mathrm{QED_r}$, which previously was found to be a precision bottleneck in the literature. With a dedicated volume-study for leptonic decays in $\mathrm{QED_r}$, we are currently investigating the suitability of this formulation in future lattice calculations.

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