PROCEEDINGS OF SCIENCE

Non-perturbative mixing and renormalisation of $\Delta F = 2$ Four-Fermion Operators

Isabel Campos Plasencia,[a] **Mattia Dalla Brida,**[b] **Giulia Maria de Divitiis,**[c,d] **Andrew Lytle,**[e] **Riccardo Marinelli,**[f,g,*] **Mauro Papinutto**[f,g] **and Anastassios Vladikas**[d]

[a] *Instituto de Física de Cantabria IFCA-CSIC, Avda. de los Castros s/n, 39005, Santander, Spain*
[b] *Theoretical Physics Department, CERN, CH-1211, Geneva 23, Switzerland*
[c] *Dipartimento di Fisica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, 00133 Roma, Italy*
[d] *INFN, Sezione di Tor Vergata, Via della Ricerca Scientifica 1, 00133 Roma, Italy*
[e] *Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois, 61801, USA*
[f] *Dipartimento di Fisica, Università di Roma La Sapienza, Piazzale A. Moro 2, 00185 Roma, Italy*
[g] *INFN, Sezione di Roma, Piazzale A. Moro 2, 00185 Roma, Italy*

E-mail: isabel.campos@csic.es, mattia.dalla.brida@cern.ch, giulia.dedivitiis@roma2.infn.it, atlytle@illinois.edu, riccardo.marinelli@uniroma1.it, mauro.papinutto@uniroma1.it, tassos.vladikas@roma2.infn.it

We present preliminary results for the Renormalization Group (RG) running of the complete basis of $\Delta F = 2$ four-fermion operators in QCD with $N_f = 3$ dynamical massless flavors. We use O(a)-improved Wilson fermions in a mixed action setup, with chirally rotated Schrödinger functional (χSF) boundary conditions for the valence quarks and Schrödinger functional (SF) boundary conditions for the sea quarks. The RG evolution operators are evaluated non-perturbatively via the matrix step-scaling functions (matrix SSF) while the perturbative running is computed using a new approach [1, 2] relying on the Poincaré-Dulac theorem.

The 40th International Symposium on Lattice Field Theory (Lattice2023), 31 July 2023 - 4 August, 2023 Batavia, Illinois, USA

*Speaker

PoS(LATTICE2023)270

1. Four-quark operators for $\Delta F = 2$ transitions

The $\Delta F = 2$ transitions, among flavour physics processes, provide some of the most stringent constraints on New Physics (NP) that can be searched for in particle accelerators. The existence of new particles can be indeed tested by looking for their possible effects on low-energy processes. The most general $\Delta F = 2$ weak effective Hamiltonian can be constructed in terms of a complete set of parity even (PE) and parity odd (PO) 4-quark operators, viz.

$$\begin{aligned} \text{PE}: \quad & \mathcal{Q}_k^{\pm} \in \left\{ \mathcal{O}^{\pm}_{[VV+AA]},\, \mathcal{O}^{\pm}_{[VV-AA]},\, \mathcal{O}^{\pm}_{[SS-PP]},\, \mathcal{O}^{\pm}_{[SS+PP]},\, 2\mathcal{O}^{\pm}_{[TT]}\cdot \right\}, \\ \text{PO}: \quad & \mathcal{Q}_k^{\pm} \in \left\{ \mathcal{O}^{\pm}_{[VA+AV]},\, \mathcal{O}^{\pm}_{[VA-AV]},\, \mathcal{O}^{\pm}_{[PS-SP]},\, \mathcal{O}^{\pm}_{[PS+SP]},\, -2\mathcal{O}^{\pm}_{[T\tilde{T}]} \right\}, \end{aligned} \tag{1}$$

where we understand $\mathcal{O}^{\pm}_{[\Gamma_1\Gamma_2]} := \frac{1}{2}\left[(\bar\psi_1\Gamma_1\psi_2)(\bar\psi_3\Gamma_2\psi_4) \pm (\bar\psi_1\Gamma_1\psi_4)(\bar\psi_3\Gamma_2\psi_2) \right]$.

When considering Wilson-fermions, chiral symmetry is broken by the regularisation, this results in a complicated mixing of the PE operators under renormalisation, while the PO ones still renormalise as in chirally preserving regularizations, namely [3]

$$[\mathcal{Q}_1]_R = \mathcal{Z}_{11}\mathcal{Q}_1, \quad \begin{bmatrix} \mathcal{Q}_2 \\ \mathcal{Q}_3 \end{bmatrix}_R = \begin{bmatrix} \mathcal{Z}_{22} & \mathcal{Z}_{23} \\ \mathcal{Z}_{32} & \mathcal{Z}_{33} \end{bmatrix} \begin{bmatrix} \mathcal{Q}_2 \\ \mathcal{Q}_3 \end{bmatrix}, \quad \begin{bmatrix} \mathcal{Q}_4 \\ \mathcal{Q}_5 \end{bmatrix}_R = \begin{bmatrix} \mathcal{Z}_{44} & \mathcal{Z}_{45} \\ \mathcal{Z}_{54} & \mathcal{Z}_{55} \end{bmatrix} \begin{bmatrix} \mathcal{Q}_4 \\ \mathcal{Q}_5 \end{bmatrix}. \tag{2}$$

Using the four-quark operators, the renormalised effective Hamiltonian can be expressed as

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_i V^i_{CKM} C_i(\mu) O_i(\mu) , \tag{3}$$

the coefficients $C_i(\mu)$ are the Wilson coefficients, whose RG running is studied non-perturbatively in this work.

2. Perturbative running for $N_\text{f} = 3$ QCD

In this work we present the procedure followed to evaluate the RG running of the Wilson coefficients C_i that is encoded in the RG *evolution operator* $\hat{\mathbf{U}}(\mu)$. The usual derivation to obtain such operators can be found in [4], but it is not well-defined for $N_\text{f} = 3$. The problem has been solved as suggested in [1, 2]: the Poincaré-Dulac theorem guarantees the existence of a basis transformation

$$\bar{\mathcal{Q}}'(\mu) = \mathbf{S}(g)\bar{\mathcal{Q}}(\mu), \quad \mathbf{S}(g) = \left(\mathbb{1} + \sum_{k=1}^{\infty} \mathbf{H}_{2k} g^{2k} \right) \mathbf{S}_\text{D} \tag{4}$$

that puts the matrix $\mathbf{A}(g) := \dfrac{\gamma(g)}{\beta(g)} = \dfrac{1}{g}\left(\sum_{k=0}^{\infty} A_{2k} g^{2k} \right)$ in the so-called *canonical form*, i.e. $\mathbf{A}^{\text{can}}(g) = \frac{1}{g}\left(\mathbf{\Lambda} + g^2\mathbf{N}_2\right)$, where $\mathbf{\Lambda}$ is the diagonal matrix of the eigenvalues, $\mathbf{N}_2$ is an upper-diagonal matrix and $\mathbf{H}_{2(k+1)}$ can be obtained recursively from $\mathbf{H}_2, \ldots, \mathbf{H}_{2k}$, $b_0, \ldots, b_{k+1}$ and $\boldsymbol{\gamma}_0, \ldots, \boldsymbol{\gamma}_{k+1}$. For example, we have

$$\begin{aligned} \mathbf{A}_2^\text{D} + 2\mathbf{H}_2 - [\mathbf{\Lambda}, \mathbf{H}_2] &= \mathbf{N}_2 , \\ \mathbf{A}_4^\text{D} + 4\mathbf{H}_4 - [\mathbf{\Lambda}, \mathbf{H}_4] &= \mathbf{N}_2\mathbf{H}_2 - \mathbf{H}_2\mathbf{A}_2^\text{D} , \end{aligned} \tag{5}$$

where we called $\mathbf{A}^{\mathrm{D}}_{2k} \equiv \mathbf{S}_{\mathrm{D}}\mathbf{A}_{2k}S^{-1}_{\mathbf{D}}$. In this operator basis, the RG evolution operator $\hat{\mathbf{U}}_{\mathrm{can}}(g(\mu))$ is the solution of a differential equation

$$\frac{\mathrm{d}}{\mathrm{d}g}\hat{\mathbf{U}}^{-1}_{\mathrm{can}}(g) = \mathbf{A}^{\mathrm{can}}(g)\hat{\mathbf{U}}^{-1}_{\mathrm{can}}(g) \tag{6}$$

that gives

$$\hat{\mathbf{U}}_{\mathrm{can}}(\mu) = g(\mu)^{-\mathbf{\Lambda}}g(\mu)^{-\mathbf{N}} . \tag{7}$$

Going back to the original operator basis, we get $\hat{\mathbf{U}}(\mu) = \mathbf{S}^{-1}_{\mathrm{D}}\hat{\mathbf{U}}_{\mathrm{can}}(\mu)\mathbf{S}(\mu)$.

3. Non-perturbative running

The non-perturbative part of the operator running is obtained as explained in [5] considering the matrix step-scaling functions (SSFs)

$$\boldsymbol{\sigma}(u) := \mathbf{U}(\mu/2,\mu)\Big|_{\bar{g}^2(\mu)=u} = \lim_{a\to 0}\mathbf{\Sigma}\left(g_0^2, a/L\right)\Big|_{\bar{g}^2(1/L)=u} \tag{8}$$

where $\mathbf{\Sigma}\left(g_0^2, a/L\right)$ is the matrix step-scaling function obtained from the renormalisation constants:

$$\mathbf{\Sigma}\left(g_0^2, a/L\right) = \mathcal{Z}\left(g_0^2, a/2L\right)\left[\mathcal{Z}\left(g_0^2, a/L\right)\right]^{-1} . \tag{9}$$

Moreover, $\mathcal{O}(a^2\bar{g}^2)$ lattice artefacts can be removed by considering *subtracted* step-scaling functions

$$\tilde{\mathbf{\Sigma}}(u, a/L) := \mathbf{\Sigma}(u, a/L)\left[\mathbf{1} + u\log(2)\boldsymbol{\delta}_k(a/L)\boldsymbol{\gamma}_0\right]^{-1}\Big|_{u=\bar{g}^2(L)} . \tag{10}$$

where the $\delta_k(a/L)$s have been computed in [6]. The continuum extrapolation of $\tilde{\mathbf{\Sigma}}(u, a/L)$ has been done simultaneously on data corresponding to the eight couplings $u \in \{1.1100,\ 1.1844,\ 1.2656,\ 1.3627,\ 1.4808,\ 1.6173,\ 1.7943,\ 2.0120\}$, with lattice sizes $L/a = 8,\ 12$ (plus $L/a = 16$ for $u = 2.0120$) using the ansatz

$$\left[\tilde{\mathbf{\Sigma}}\left(u_n, \frac{a}{L}\right)\right]_{ij} = [\boldsymbol{\sigma}(u_n)]_{ij} + \left(\frac{a}{L}\right)^2\sum_{m=0}^{2}\left[\boldsymbol{\rho}_m\right]_{ij}u_n^m , \tag{11}$$

with $n = 1,\ldots,8$. The fit parameters have been found by minimising the χ^2 function, and two examples of fit for a SSF matrix element can be found in figure 1.

Having performed the continuum extrapolations at the different couplings available, the functional dependence of $\boldsymbol{\sigma}(u)$ is obtained by fitting the latter with

$$\boldsymbol{\sigma}(u) = \mathbf{1} + \mathbf{r}_1 u + \mathbf{r}_2 u^2 + \mathbf{r}_3 u^3 + \mathbf{r}_4 u^4 , \tag{12}$$

where the coefficients $\mathbf{r}_1$ and $\mathbf{r}_2$ are fixed by using perturbation theory

$$\mathbf{r}_1 = \boldsymbol{\gamma}_0 \ln 2 , \quad \mathbf{r}_2 = \boldsymbol{\gamma}_1 \ln 2 + b_0\boldsymbol{\gamma}_0 \ln^2 2 + \frac{1}{2}(\boldsymbol{\gamma}_0)^2 \ln^2 2 . \tag{13}$$

The NLO anomalous dimension matrix $\boldsymbol{\gamma}_1$ in the χSF scheme has been evaluated in [6], analogously to what done for the SF scheme in [4]. Two examples for these fits are shown in figure 2.

A series of N squared-couplings $u_1 \ldots u_{\mathrm{N}}$ is then generated through the coupling step-scaling function [7], i.e. evaluating $u_n = \sigma^{-1}(u_{n-1})$, in order to compute non-perturbatively the quantity

$$\mathbf{U}(\mu_{\mathrm{had}}, \mu_{\mathrm{pt}}) = \boldsymbol{\sigma}(\mu_1)\cdots\boldsymbol{\sigma}(\mu_N) , \quad \text{with} \quad \boldsymbol{\sigma}(\mu) \equiv \boldsymbol{\sigma}(u(\mu)). \tag{14}$$

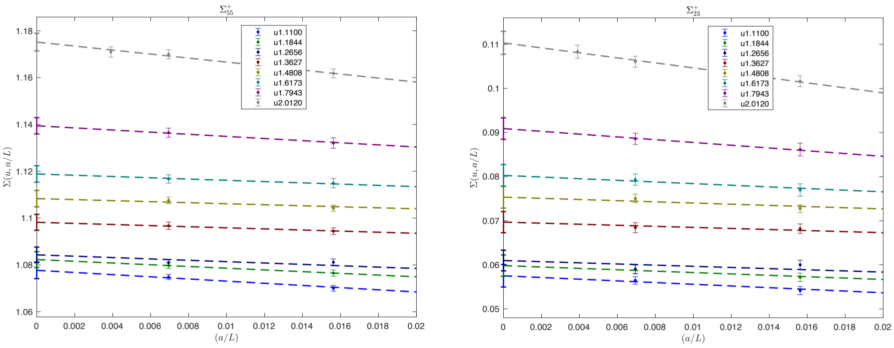

Figure 1: Continuum extrapolations of the matrix elements $\left[\tilde{\mathbf{\Sigma}}(u, a/L)\right]_{55}$, $\left[\tilde{\mathbf{\Sigma}}(u, a/L)\right]_{23}$. The extrapolated continuum values $[\boldsymbol{\sigma}(u)]_{55}$, $[\boldsymbol{\sigma}(u)]_{23}$ at every coupling, along with their uncertainties, are depicted as vertical bands at $a/L = 0$.

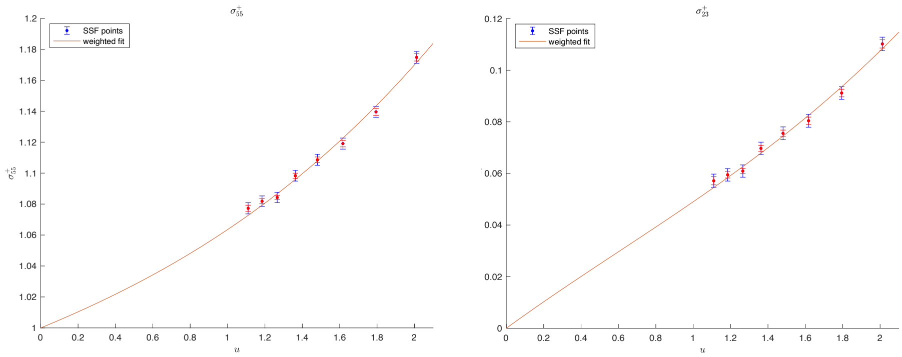

Figure 2: Fit of the continumm extrapolations $[\sigma]_{55}$ and $[\boldsymbol{\sigma}]_{23}$ as functions of the squared coupling u.

4. Results

Using the factorisation properties of the evolution operators [4], we obtain the RG evolution operator as function of the scale $\mu \leq \mu_{\rm pt}$:

$$\hat{\mathbf{U}}(\mu) = \mathbf{S}_{\rm D}^{-1} u(\mu_{\rm pt})^{-\mathbf{\Lambda}/2} u(\mu_{\rm pt})^{-\mathbf{N}_2/2}\Big(\mathbb{1} + u(\mu_{\rm pt})\mathbf{H}_2 + u^2(\mu_{\rm pt})\mathbf{H}_4 + u^3(\mu_{\rm pt})\mathbf{H}_6\Big)\mathbf{S}_{\rm D}\mathbf{U}(\mu_{\rm pt}, \mu) \tag{15}$$

where μ is related to SF coupling u through the relation [7]

$$\frac{\mu}{\Lambda} = (b_0 u)^{\frac{b_1}{2b_0^2}} \exp\left(\frac{1}{2b_0 u}\right) \exp\left[\int_0^{\sqrt{u}} \mathrm{d}x \left(\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right)\right], \tag{16}$$

and Λ has been computed in [8].

We associate three kinds of uncertainties to the running that we computed:

- statistical error computed propagating the error on $\boldsymbol{\sigma}(u)$ in Eq. (14);
- systematic error due to the point chosen to match the perturbative running with the non-perturbative one;

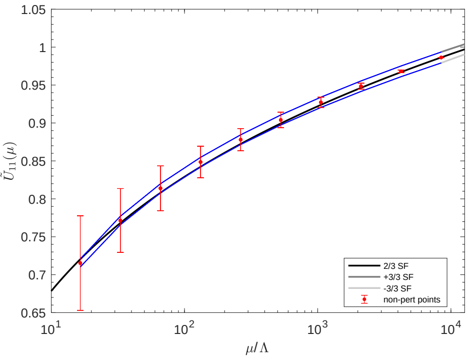

Figure 3: Non-perturbative running $\hat{U}_{11}(\mu)$ (points in red, $N = 9$) compared to the NLO prediction (curve in black). The curves in blue are built evolving non-perturbatively the $\pm(3/3)\chi$SF runnings and quantify the systematic error.

- systematic error due to the lack of knowledge of the NNLO and NNNLO matrices $\boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}_3$.

Having no theoretical hint on the entity of $\boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}_3$, we only try to give a rough estimate of the systematic error at the last point. First we neglect $\boldsymbol{\gamma}_3$ which only enters in the evaluation of $\mathbf{H}_6$. For what concerns $\boldsymbol{\gamma}_2$ we consider two rough guesstimates: $\boldsymbol{\gamma}_2^{\rm L} = -\boldsymbol{\gamma}_1/4$, $\boldsymbol{\gamma}_2^{\rm R} = \boldsymbol{\gamma}_1/4$, whose rough choice seems reasonable by requiring the ratio between the elements of $\boldsymbol{\gamma}_2$ and those of $\boldsymbol{\gamma}_1$ to be of the same order of magnitude of the ratio between the elements of $\boldsymbol{\gamma}_1$ and those of $\boldsymbol{\gamma}_0$. Using the guesses for $\boldsymbol{\gamma}_2$ we evaluated guesstimates of the matrices $\mathbf{H}_4^{\rm L/R}$, $\mathbf{H}_6^{\rm L/R}$ and consequently the guessed perturbative RG evolution $\hat{\mathbf{U}}^{\rm L/R}(u_{\rm pt})$ that we will address as $\pm(3/3)\chi$SF. We were then able to evaluate the complete running defined in (15), where we imposed the matching at the same scale used for the $(2/3)\chi$SF results. Eventually, we defined this kind of systematic error as the difference between the complete $(2/3)\chi$SF running and the guessed $\pm(3/3)\chi$SF running.

The first two errors in the above list depend on the scale $\mu_{\rm pt}$ where the matching with perturbation theory is performed and thus on the number of steps N in Eq. (14). The results for the complete running are shown in Fig. (3), (4) and (5), where the non-perturbative points are compared to the perturbative $(2/3)\chi$SF running and both statistical and systematic uncertainties are separately displayed. We found that $N = 9$ resulted in limited statistical uncertainties and very small systematic ones, as it is possible to notice from the fact that the non-perturbative points approach the perturbative running with the same slope at $\mu_{\rm pt}$. The only exception to the generally observed behaviour is the matrix element $\hat{U}_{22}$, and the cause of this disagreement could be due to the very large NLO anomalous dimension or to some problem in the fit of the SSF $\sigma_{22}(u)$ which we are going to investigate in the near future.

5. Conclusions

In this work we computed the non-perturbative renormalisation of the $\Delta F = 2$ four-fermion operators introduced in Eq. (1), investigating the RG running and mixing of the operator basis in the χSF scheme in $N_{\rm f} = 3$ massless QCD, between the low energy scale $\mu_0 \sim \mathcal{O}(4)$ GeV and the high energy scale $\mu_{\rm pt} \sim \mathcal{O}(10^3)$ GeV. At the latter scale we performed the matching with the NLO perturbative running by following the strategy explained in [1, 2].

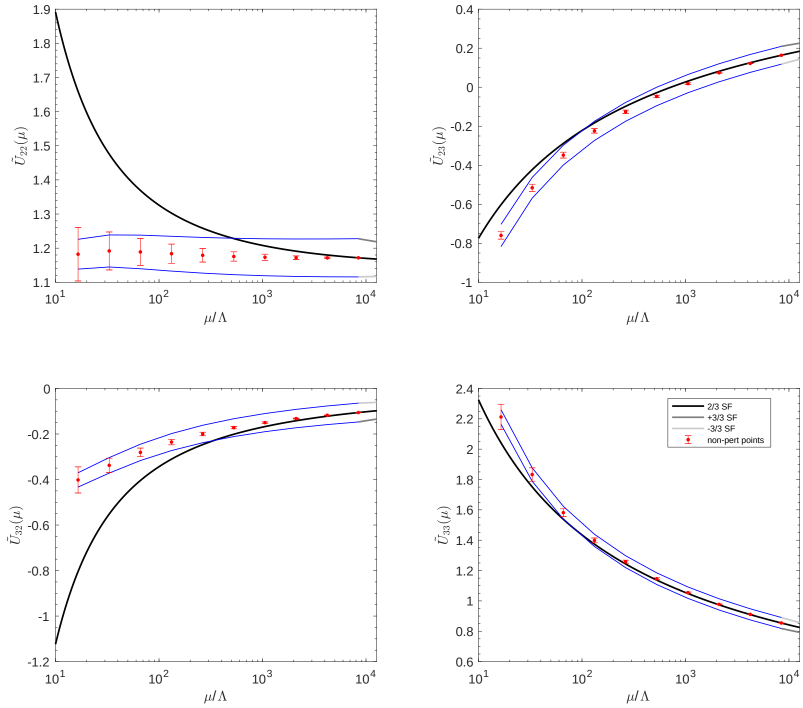

Figure 4: Non-perturbative running for the matrix elements 2|3 of the evolution operator $\hat{\mathbf{U}}(\mu)$ (points in red, $N = 9$). The results are compared to the NLO prediction (black curve). The curves in blue are built evolving non-perturbatively the $\pm(3/3)\chi$SF runnings and quantify the systematic error.

The statistical uncertainties have been evaluated through the bootstrap analysis of the simulated data and have been propagated in the various computations. The systematic uncertainties due to the choice of the scale μ_{pt}, at which non-perturbative and perturbative results are matched, depend on the number of steps N in Eq. (14). We have minimised such uncertainties by increasing N up to a value for which the non-perturbative points approach the perturbative running with the same slope at the corresponding scale μ_{pt}, while trying to keep simultaneously the statistical error under control. Finally, the systematic uncertainties related to the lack of knowledge of the higher perturbative orders have been roughly estimated by choosing an order of magnitude guess for the NNLO anomalous dimension matrix $\boldsymbol{\gamma}_2$. This question deserves however further studies.

Our preliminary results seem to suggest that the use of NLO perturbation theory down to scales of $\mathcal{O}(4)$ GeV may be problematic, as it can be seen in the plots of $\hat{\mathbf{U}}$. We have still to finish assessing the systematics but, if confirmed, these results would pose a serious question about the opportunity of using Wilson coefficients evaluated perturbatively down to a scale of $\mathcal{O}(3)$ GeV, at which presently computed hadronic matrix elements are renormalized.

Acknowledgements: AL acknowledges support by the U.S. Department of Energy under grant number DE-SC0015655.

References

[1] Marco Bochicchio. *Eur. Phys. J. C*, 81(8):749, 2021.

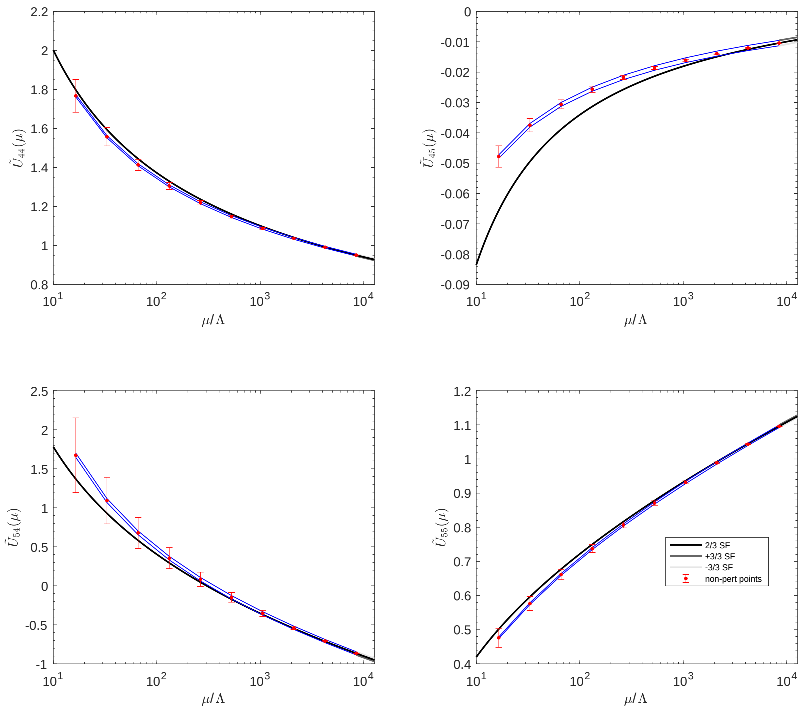

Figure 5: Non-perturbative running for the matrix elements 4|5 of the evolution operator $\hat{\mathbf{U}}(\mu)$ (points in red, $N = 9$). The results are compared to the NLO prediction (black curve). The curves in blue are built evolving non-perturbatively the $\pm(3/3)\chi$SF runnings and quantify the systematic error.

[2] Matteo Becchetti and Marco Bochicchio. *Eur. Phys. J. C*, 82(10):866, 2022.

[3] A. Donini, V. Giménez, G. Martinelli, M. Talevi, and A. Vladikas. *The European Physical Journal C*, 10(1):121–142, aug 1999.

[4] Mauro Papinutto, Carlos Pena, and David Preti. *Eur. Phys. J. C*, 77(6):376, 2017. [Erratum: Eur.Phys.J.C 78, 21 (2018)].

[5] P. Dimopoulos, G. Herdoíza, M. Papinutto, C. Pena, D. Preti, and A. Vladikas. *The European Physical Journal C*, 78(7), jul 2018.

[6] M. Dalla Brida, A. Mellini, M. Papinutto, F. Scardino, and P. Vilaseca. Renormalization of $\Delta F = 2$ operators in the chirally rotated Schrödinger functional. In preparation.

[7] Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, Stefan Sint, and Rainer Sommer. *The European Physical Journal C*, 78(5), may 2018.

[8] Mattia Bruno, Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, Stefan Schaefer, Hubert Simma, Stefan Sint, and Rainer Sommer. *Physical Review Letters*, 119(10), sep 2017.