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Renormalization of Karsten-Wilczek Quarks on a Staggered Background

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The Karsten-Wilczek action is a formulation of minimally doubled fermions on the lattice. It explicitly breaks hypercubic symmetry and introduces three counterterms with respective bare parameters. We present a tuning of the bare parameters of the Karsten-Wilczek action on staggered configurations at the physical point. We study the magnitude of the taste-splitting as a function of the lattice spacing.

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1. Introduction

The Karsten-Wilczek (KW) action [1, 2] is the simplest implementation of so-called minimally doubled fermions in lattice field theory. The action eliminates most of the spurious fermionic degrees of freedom known as "doublers." It reduces the number of doublers from fifteen to one, leaving two mass-degenerate quark species, while explicitly preserving an ultra-local chiral symmetry. In the case of degenerate up and down quarks this action allows the study of the chiral transition without a rooting, which one is forced to employ with staggered quarks.

However, minimally doubled fermions come at a price. The action breaks the hypercubic symmetry of the lattice and introduces three counterterms to the naïve theory [3]. While the KW action is highly appealing for the aforementioned features, renormalization constitutes a multidimensional tuning problem for the selection of appropriate values for the bare parameters.

We present a mixed action study of a method of tuning these bare parameters nonperturbatively, exploring the hierarchy of the tuning parameters and how accurately one needs to tune. We take measurements with the KW action on stored gauge configurations computed with the 4-stout staggered fermion action at the physical point. We conclude with an exploration of the scaling behavior of the mass-splitting of the ground states of two mesonic channels with tuned parameters.

2. The Karsten-Wilczek action

The Karsten-Wilczek action [1, 2]

$$S_F^{KW} = S_F^N + \sum_x \sum_{j=1}^{3} \bar{\psi}(x) \frac{i\zeta}{2} \gamma^\alpha \left(2\psi(x) - U_j(x)\psi(x+\hat{j}) - U_j^\dagger(x)\psi(x-\hat{j}) \right), \tag{1}$$

where ζ is the Wilczek parameter, $U_\mu(x)$ is the link variable at site x in the μ direction, and α is an arbitrary spacetime direction, adds to the naïve lattice fermion action

$$S_F^N = \sum_x \sum_{\mu=0}^{3} \bar{\psi}(x)\gamma_\mu \frac{1}{2} \left[U_\mu(x)\psi(x+\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu}) \right] + m_0 \sum_x \bar{\psi}(x)\psi(x) \tag{2}$$

additional terms which break hypercubic symmetry. These terms are only added along the spacetime directions where $\mu \neq \alpha$, making α a special axis and explicitly breaking the hypercubic symmetry of S_F^N. In this work, we take $\alpha = 0$, the time direction. The action requires the following three counterterms, for which one-loop perturbative results exist [4]

$$S^{3f} = c \sum_x \bar{\psi}(x) i\gamma^\alpha \psi(x), \tag{3}$$

$$S^{4f} = d \sum_x \bar{\psi}(x) \frac{1}{2} \gamma^\alpha \left(U_\alpha(x)\psi(x+\hat{\alpha}) - U_\alpha^\dagger(x)\psi(x-\hat{\alpha}) \right), \tag{4}$$

$$S^{4g} = d_G \sum_x \sum_{\mu\neq\alpha} \mathrm{Re\,Tr}(1 - \mathcal{P}_{\mu\alpha}(x)) \,. \tag{5}$$

Here $\mathcal{P}_{\mu\nu}(x)$ is the plaquette at x along the μ and ν directions. S^{3f} and S^{4f} are fermionic counterterms of dimension 3 and 4 respectively. The gluonic counterterm S^{4g} plays no role in our mixed action study. It will be relevant for dynamical KW simulations.

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These terms are manifestations of the anisotropy of the lattice breaking hypercubic symmetry [3]. The counterterms S^{4f} and S^{4g} act as anisotropy parameters of the fermion and gauge terms, respectively. To emphasize the role of S^{4f}, we express its bare parameter d in terms of the anisotropy $\xi_0 = 1 + d$ in our numerical work. Following the preferred axis $\alpha = 0$, we target a renormalization anisotropy $\xi_f = 1$ and find the bare anisotropy ξ_0 that restores isotropy.

The multidimensional nonpertrubative tuning of (c, ξ_0) can be made tractable with certain considerations. Boosted perturbation theory estimates of the nonperturbative parameters and quenched simulations using the KW action indicate that the anisotropy parameter has mild effects compared to those of c and that c can be considered almost independent of ξ_0 [3]. We quantify this statement and explore a method for tuning the relevant parameter c and the bare anisotropy ξ_0. Having an efficient method will be critical eventually when tuning dynamical KW simulations [5].

We use the KW action with $\alpha = 0$, $\zeta = +1$. We use the w_0 scale setting to convert from lattice units into physical units [6]. The 4-stout action's parameters were introduced in Ref. [7].

3. Nonperturbative tuning of bare parameters

A suitable method for tuning the bare parameters of the KW action nonperturbatively uses the existence of oscillating contributions to the correlation functions

$$C_\mu(x, y) \sim \left\langle \bar{\psi}(x)\gamma_\mu\psi(x)\bar{\psi}(y)\gamma_\mu\psi(y) \right\rangle \tag{6}$$

of some KW fermions. These oscillations are related to fermion doubling [3]. Importantly, their frequency is sensitive to the c parameter. Fermion partners can be identified by the spin-taste structure of the KW action [8]. A relevant fermion pair is the γ_0 and γ_5 channels. The correlators for γ_0 and γ_5 are taken *parallel* to the preferred axis of the KW action, $\alpha = 0$ in this case. The correlator of γ_5 *perpendicular* to $\alpha = 0$ in a spatial direction is needed for tuning ξ_0. The renormalized fermionic anisotropy is defined as the ratio of the perpendicular (spatial) mass of γ_5 to the parallel (temporal) mass: $\xi_f = m_\perp / m_\parallel$.

The parallel correlator for γ_0 exhibits oscillations while that of γ_5 does not. The tuning criterion for the c parameter is where the frequency spectrum of the oscillations of the γ_0 channel recovers its tree-level form [3]. At the tuned c, the oscillation of the γ_0 correlator is described by $(-1)^n$, where n is the position along the temporal extent of the lattice. The frequency of the oscillation at a given c value can be described by $\omega = \omega_c + \pi$, where π is the frequency of the rapid oscillations at tuned c and ω_c is a beat frequency that appears in the rapid oscillations when c is detuned. Thus, the tuning criterion is equivalently stated as where $\omega_c = 0$ and the beat vanishes. Fig. 1 shows an example of correlators in the γ_0 and γ_5 channels at detuned c where both the rapid oscillation and the beat oscillation can be observed for γ_0. When we average over the symmetric halves of the γ_0 correlator $C_0(x, y)$ and eliminate the rapid oscillation with a factor of $(-1)^n$,

$$C(n) = (-1)^n \frac{1}{2}(C_0(0, n) + C_0(0, N_t - n)) \quad \text{for } 0 \leq n \leq N_t/2, \tag{7}$$

where N_t is the temporal extent of the lattice, the correlator is well described by the model

$$C(n) \approx A \cosh(m(n - N_t/2)) \cos(\omega_c n - \phi) \tag{8}$$

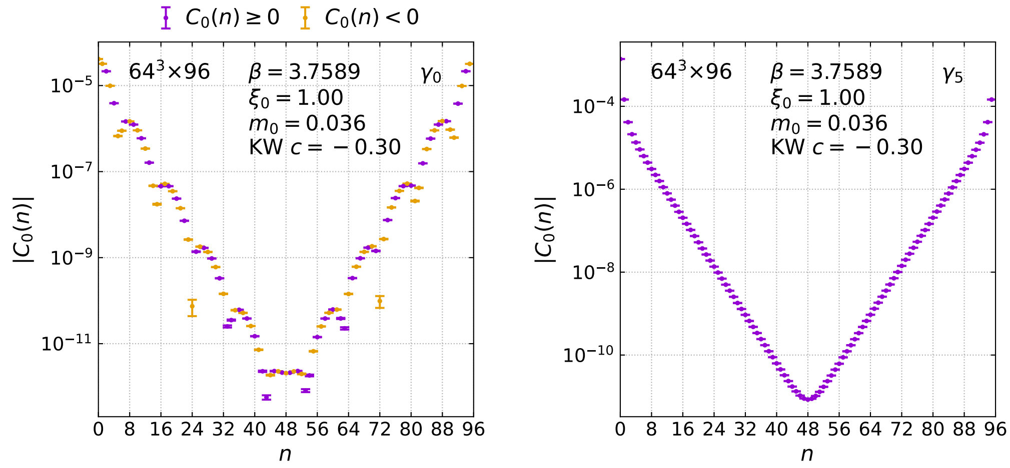

Figure 1: Correlators for the γ_0 (left) and γ_5 (right) fermion channels for the Karsten-Wilczek action. The frequency of the oscillating γ_0 correlator depends on the c parameter of the action. A beat oscillation on top of the $(-1)^n$ oscillation (visible in the left plot) occurs when the c parameter is detuned.

where m is the mass of γ_0 in lattice units and ϕ is a phase factor. The beat frequency and the mass decouple in this way. We use various fitting methods for extracting m and ω_c because of the exotic shape of the correlator. We tune c for a given ξ_0 by scanning through c and extrapolating with a linear fit to where $\omega_c = 0$.

An immediate difficulty that arises in tuning c is the finite length of the lattice, which makes fitting the frequency unreliable when the wavelength of the beat is greater than N_t in the relative vicinity of tuned c. This can be ameliorated through the use of tiling gauge configurations. Tiling means the gauge configurations are doubled (or quadrupled) to form a $N_x^3 \times (2N_t)$ or $N_x^3 \times (4N_t)$ lattice on which the propagator is studied. This allows to study the propagator in a longer range than that of the dynamical simulation. Tiling extends the idea of a mixed action study where the quarks live on an extended lattice without maintaining the long wave-length gauge fluctuations, which would anyway be irrelevant for the divergent parts of the diagrams. Thus, when used properly, tiling the stored gauge configurations increases the precision with which ω_c is measured. Fig. 2 shows an example of $C(n)$ at detuned c for three different amounts of tiling in the left plot, as well as $\omega_c(c)$ around tuned c for each level of tiling for comparison in the right plot. We use $4\times$ and occasionally $1\times$ tiling throughout this study.

An example of finding the tuned c value is shown in the left plot of Fig. 3, where the result is combined from linear fits on either side of the tuned c. An additional parameter for our measurements is the number of stout smearing steps we apply. For any value of ξ_0, the effect of applying more smearing steps is a power law decrease in the magnitude of the tuned c value. An example of this is shown in the right plot of Fig. 3. We consistently use four steps of stout smearing throughout this study, the same smearing level that was used in the staggered simulation to create the ensembles.

In this way, we tune c for any value of ξ_0. ξ_0 can then be tuned according to the desired renormalized anisotropy ξ_f. Since we perform measurements on isotropic staggered configurations, we tune by interpolating $\xi_f(\xi_0)$ at tuned c to $\xi_f(\xi_0) = 1$. An example of this is shown in the left

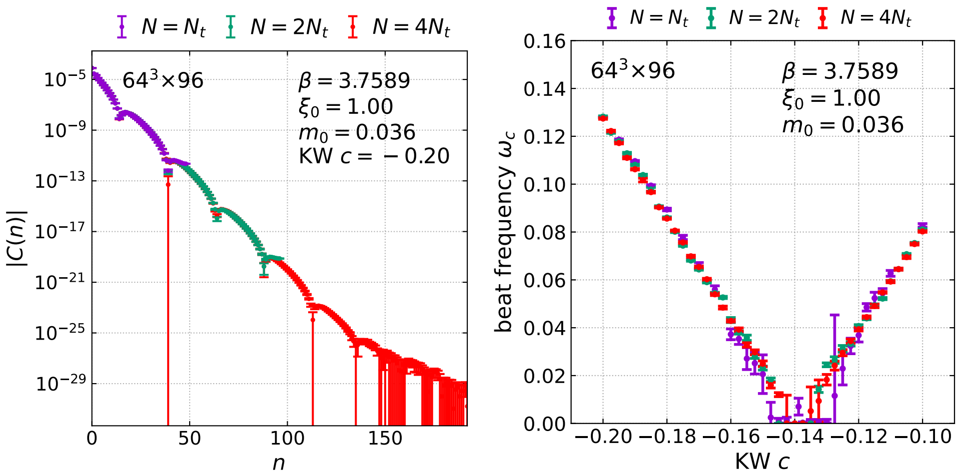

Figure 2: (Left) The symmetrized γ_0 correlator measured with different amounts of tiling with stored gauge configurations. The proper use of tiling permits longer correlation lengths to be measured and increases the precision of the measurement of the beat frequency ω_c. (Right) Measurements of ω_c as a function of the c parameter at the same three tilings.

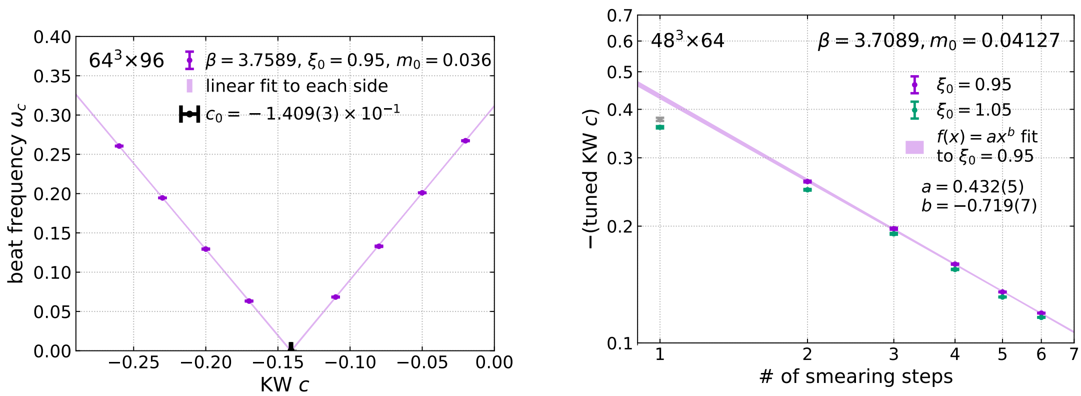

Figure 3: (Left) ω_c as a function of the c parameter. The c parameter is tuned at the value where the beat frequency vanishes. (Right) The variation of the tuned value of c with the number of stout smearing steps applied. We used 4 stout smearing steps throughout the rest of the analysis.

plot of Fig. 4. One could proceed to find tuned c at tuned ξ_0 using either a final scan in c at tuned ξ_0, or an interpolation of $c(\xi_0)$ to the tuned ξ_0. We use the latter, keeping in mind the computational efficiency of the method. The right plot in Fig. 4 shows an example of this interpolation. The question naturally arises as to how accurately one needs to tune c and ξ_0, or how stable the result is to slight mistunings. Thankfully, the results are quite stable, as the $\xi_f(c)$ function has vanishing derivative (a maximum) near the tuned value.

Until this point, we have not mentioned the bare mass parameter of the KW action m_0, which determines the ground state masses of the pseudoscalar γ_0 and γ_5 channels at tuned c and ξ_0. The left plot of Fig. 5 shows the physical γ_0 and γ_5 masses, M_0 and M_5 respectively, as functions of the c parameter for fixed ξ_0 and m_0. We may also speak of the stability of the tuning in regard to M_0 and

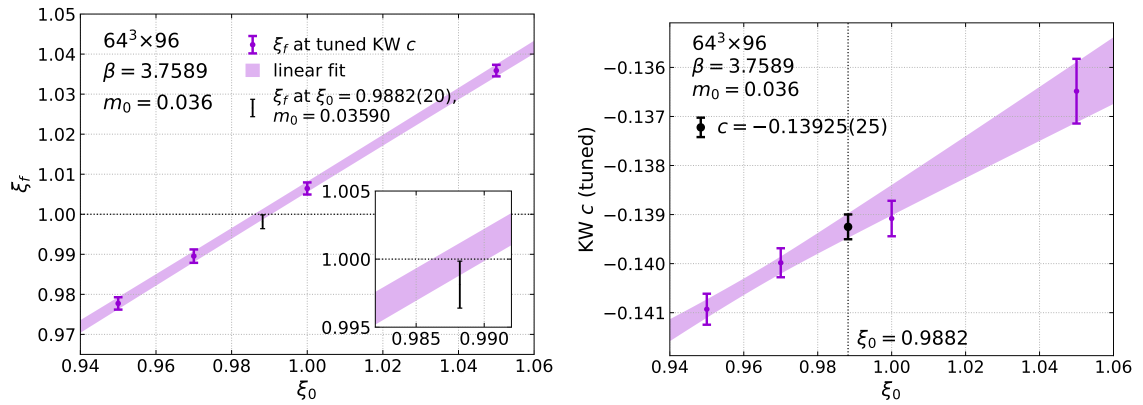

Figure 4: (Left) The renormalized anisotropy ξ_f of the mass of the pseudoscalar γ_5 channel as a function the bare anisotropy ξ_0. The c parameter is individually tuned at each ξ_0 value. $\xi_f = 1$ was the tuning criterion for ξ_0. (Right) Interpolating to the tuned value of the c parameter at the tuned value of ξ_0.

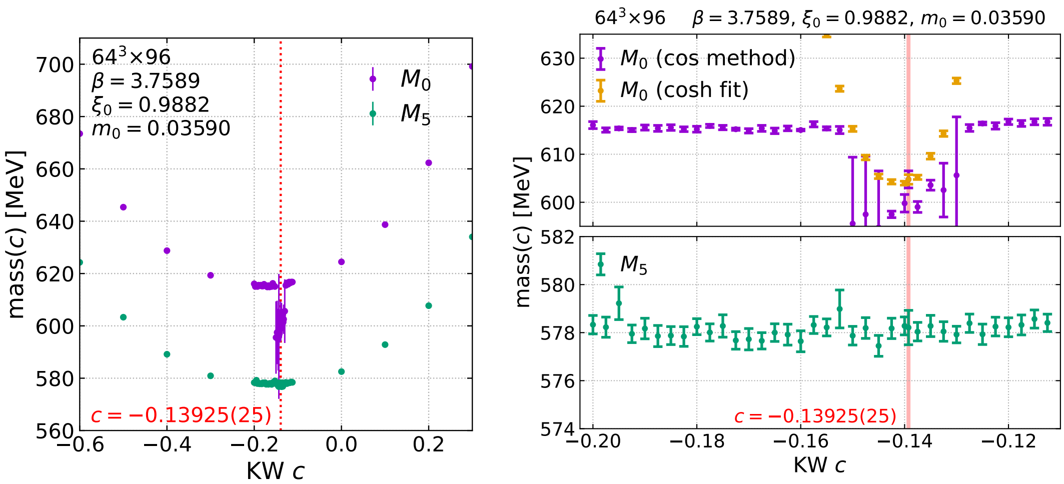

Figure 5: (Left) Physical masses of the γ_0 and γ_5 channels as functions of the c parameter at fixed ξ_0 and bare mass m_0. $M_0(c)$ and $M_5(c)$ are concave up, with minima near the tuned value of c. (Right) Zoom in near tuned c. While M_5 is very stable around tuned c, M_0 exhibits a sudden dip in the immediate vicinity of tuned c, which can be observed across multiple fitting methods (explained in the text).

M_5. Both masses are concave up functions of c with minima near the tuned value of c, indicating the stability of both quantities to small mistunings of c. An interesting effect is observed, however, in the behavior of $M_0(c)$ very close to the tuned c. While $M_5(c)$ is practically flat in the immediate vicinity of the tuned c, $M_0(c)$ experiences a sudden dip over a small interval around tuned c. This region of the plot is enlarged in the right plot of Fig. 5. We observe this effect at all β values at which we take measurements. Very close to the tuned c, the determination of ω_c is very difficult because of the finite size of the lattice.

This dip around tuned c in the regime where ω_c can no longer be reliably determined would seem to pose a great challenge to tuning c, since a small mistuning appears to cause a significant difference in M_0. However, the accuracy and the precision of the method of tuning with ω_c is such

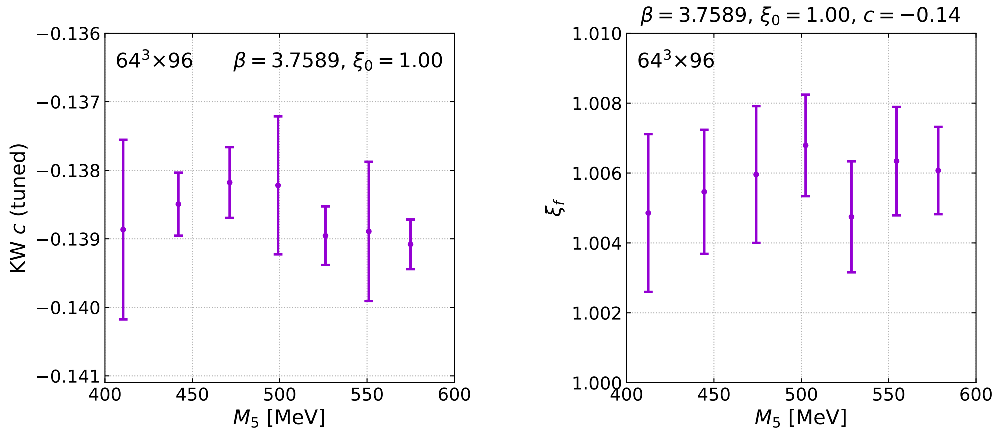

Figure 6: Stability of the tuned c value at fixed ξ_0 (left) and the renormalized anisotropy ξ_f at fixed ξ_0 and c (right) with respect to the pseudoscalar mass M_5.

that we are able to reliably penetrate this difficult region. The right plot of Fig. 5 shows the value of M_0, extracted using two different methods: one method utilizing a cosine fit to the beat frequency component of the correlator, the other being a direct cosh fit to the correlator. The latter method is only viable when the wavelength of the beat is large compared to the lattice, hence no beat is apparently visible. The values of M_0 extracted from both methods are in agreement at the tuned value of c determined from where $\omega_c = 0$. We can then conclude that the correct determination of the value of M_0 is robust, in that the same value is obtained with different fit methods, but is subordinate to a precise tuning of c as shown in Fig. 3.

Thus, the c parameter determined from the condition of vanishing beat frequency in the γ_0 propagator is very close to the extrema of various renormalized quantities (e.g. ξ_f, meson mass). The derivative of C-even quantities with respect to c are odd, and should vanish with a restored symmetry. However, the use of e.g $dM_5/dc = 0$ for tuning of c is hindered by noise (see Fig. 5).

We find that the tuned value of c for a given ξ_0 is very stable with respect to changes in the physical mass of the γ_5 channel M_5. Further, for fixed c and ξ_0, the renormalized fermionic anisotropy ξ_f is very stable with respect to changes in M_5 as well. These results are shown in Fig. 6. In Fig. 7, we show the results of tuning c and ξ_0 at several lattice spacings with the γ_5 mass held constant at $M_5 = 578.4$ MeV.

A hierarchy of the bare parameters is thus established. Most critical in the tuning procedure is the c parameter, as the physical masses of oscillating fermion channels are highly dependent on it. The bare anisotropy ξ_0 follows in importance. Finally, the bare mass m_0 (alternatively the physical mass M_5) is last, as it has the mildest effect.

4. Taste-splitting of γ_0 and γ_5 channels

Finally, we present an investigation of the mass-splitting of the ground states of the γ_0 an γ_5 channels, which are parity partners in the spin-taste structure of the KW action [8]. The quadratic mass difference $\Delta M^2 = M_0^2 - M_5^2$ is a quantity which is stable against changes in physical mass M_5 for fixed c and ξ_0, as shown in the left plot of Fig. 8. In the right plot of the same figure,

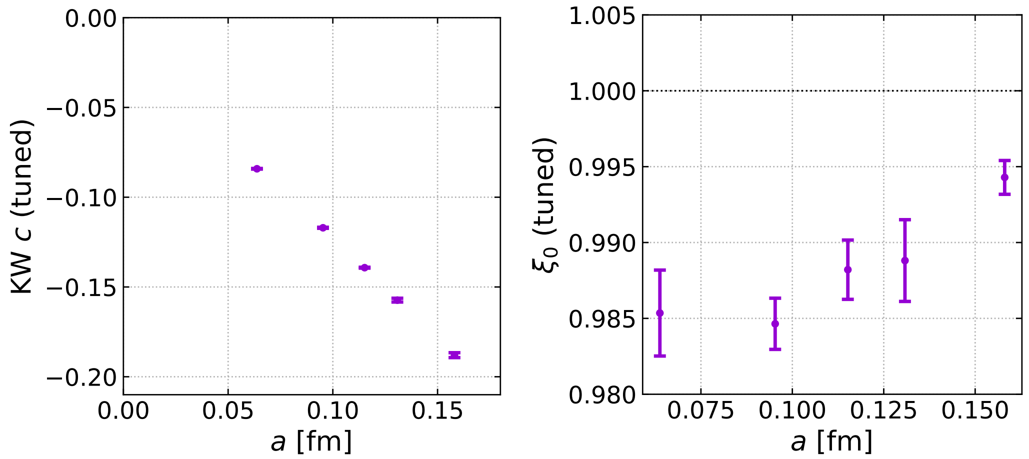

Figure 7: Tuned values of the c parameter (left) and ξ_0 (right) at various lattice spacings for constant pseudoscalar mass M_5.

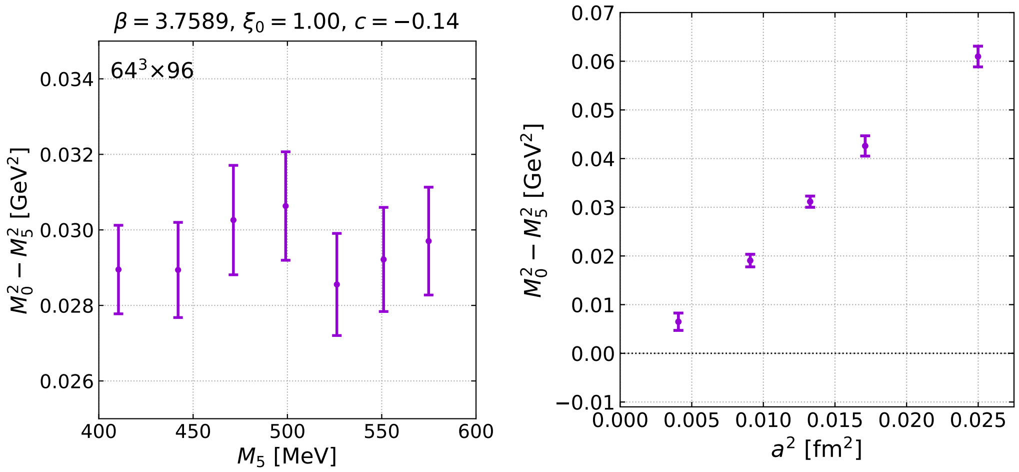

Figure 8: (Left) Stability of the taste-splitting $\Delta M^2 = M_0^2 - M_5^2$ of the γ_0 and γ_5 channels with respect to the pseudoscalar mass M_5. (Right) Taste-splitting vs. lattice spacing squared.

we show ΔM^2 at tuned c and ξ_0 with fixed $M_5 = 578.4$ MeV, as a function of the lattice spacing squared. A naïve linear extrapolation to the continuum limit excluding the coarsest lattice would yield $\Delta M^2 < 0$. However, these lattice spacings may fall outside the linear scaling regime.

5. Conclusion

We presented a mixed action study of tuning the bare parameters of the Karsten-Wilczek action using gauge configurations generated with the staggered 4-stout fermion action. We observed the dominance of the dimension-3 KW c parameter, which can be tuned precisely using the frequencies of oscillating fermionic correlation functions of the KW action. The bare anisotropy ξ_0 follows in importance, and must be tuned – like with other anisotropic discretizations – even if the target anisotropy is 1. The physical mass of the ground state of the γ_5 channel, dependent on the bare mass m_0, does not significantly effect the tuned values of c and ξ_0, hence tuning can be performed at a fixed pseudoscalar mass. Lastly, we showed the mass-splitting of the parity partners γ_0 and γ_5 at constant physical mass with tuned parameters as a function of the lattice spacing squared.

Acknowledgments

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